# Tricritical behavior in two dimensions. II. Universal quantities from the $\epsilon$ expansion

A. L. Lewis
*Physics Department, Natural Sciences II, University of California, Santa Cruz, California 95064*

F. W. Adams
*Physics Department, University of California, Berkeley, California 94720*
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The tricirical exponents $\eta_t$ and $\phi_t$ are calculated through order $\epsilon^3$ and $\epsilon^2$, respectively, in the isotropic $n$-component model, where $\epsilon = 3 - d$. The estimate for $\eta_t$, in two dimensions for the Ising case, is 0.027; the series for $\phi_t$ is quite ill behaved, producing a negative estimate at this order. Beginning at this order in $\epsilon$, the spherical-model limit fails to exist. A scaling function for the spin-spin correlation function, appropriate for nonexceptional paths of approach to the tricritical point in the disordered phase, is calculated through $\epsilon^2$; its large momentum expansion is shown to involve the crossover exponent. These results are generalized to points where $\mathcal{O}$ coexisting phases become critical.

## I. INTRODUCTION

We have seen in the previous paper[1] that model-dependent properties associated with a particular two-dimensional tricritical system could be estimated by approaching two dimensions from below. In this work, we consider universal quantities by approaching two dimensions from above, via the $\epsilon = 3 - d$ expansion. With the transfer matrix we were able to see the global behavior of the phase diagram as one varied parameters in the Hamiltonian, but we were not really able to probe the tricritical region in any detail. Within the $\epsilon$ expansion, one tends to lose sight of the global properties, but, in compensation, one gains almost unlimited access to the scaling regime (in a formal way, at least).

Experience with the $\epsilon_2$ expansion at critical points,[2] where $\epsilon_2 = 4 - d$, obtained by comparison with high-temperature series and experimental results, suggests that one often obtains fairly accurate (say, 20%) numerical estimates from two nontrivial orders of the series evaluated at $\epsilon_2 = 1$. Of course, it is a completely open question as to whether this type of accuracy continues to hold at tricritical points; the reason for this is that $\epsilon = 1$ corresponds to two dimensions, which is known to be special. For example, we shall obtain results for the class of isotropic $n$ component models; when $n = 1$ one expects standard tricritical behavior in two dimensions. Yet, when $n \geqslant 2$, the absence of long-range order in two dimensions must make a two-dimensional tricritical point rather exotic, if it exists at all. Since it would seem unlikely that the series can "know" about this, one is led to be rather suspicious of the expansion when $\epsilon = 1$, and $n \geqslant 2$.

We try to avoid dwelling on calculational details, but we do attempt to convey our methods when they are not completely standard. Our stress is on those matters which one encounters in a tricritical calculation and which are completely absent in a corresponding calculation at critical points. We calculate within the framework of Wilson's (1972) direct Feynman diagram technique[2] and the basic idea is well known: namely, one compares directly the results of perturbation theory with what one anticipates from certain scaling expressions, choosing a coupling constant ($u_6$ here) order by order in $\epsilon$ so that the leading correction to scaling vanishes. However, the additional relevant scaling field at tricritical points (with respect to critical points) does provide additional complications and additional freedom. The additional freedom manifests itself in the choice of a path of approach to the tricritical point; some paths are computationally simpler than others in that they manifestly eliminate a large number of Feynman diagrams from the beginning of a calculation. With a poor choice of path, one may not discover these same diagrams cancelling each other until one is very far into the calculation. The basic freedom here is in deciding what to do with the four point coupling ($u_4$). We discuss two natural choices.

These calculations[3] extend previous results of Stephen and McCauley,[4] who calculated $\eta_t$, $\gamma_t$, and $\phi_t$ through lowest nontrivial order in the $n$-component case; we have been unable to extend the calculation of $\gamma_t$. The exponent calculations are reported in Sec. II of this paper.

In Sec. III, we consider the spin-spin correlation function in the disordered phase. This function is, in general, a function of two arguments. However, we consider it with one of the arguments (containing the crossover exponent) set to zero. This resulting function is then appropriate for comparison with its counterpart at critical points. Our calculation uses the method of Fisher and Aharony,[5] which is explained in detail by them, so we basically report only the result. It is interesting that, in contrast to critical points, one need not be forced to contend with an integral representation of this function[5] because the resulting integral may be evaluated easily in terms of simple functions. This is quite helpful when considering the large-momentum expansion of this function; we find that the expansion contains a new term not present at critical points: even though we though the crossover exponent had been "suppressed," it apparently reappears in the new term.

In Appendix A, we generalize many of these results to order $\Theta$ critical points, where $\Theta$ coexisting phases become critical. Specifically, we provide a general expression for $\eta_\Theta$ which reproduces Wilson's result at critical points ($\Theta=2$) and our new result here for tricritical points ($\Theta=3$) through $\epsilon_\Theta^3$ where $\epsilon_\Theta=2\Theta/(\Theta-1)-d$. The general expression is rather complicated, involving hypergeometric sums. We also provide a general expression for $\gamma_\Theta$ through $\epsilon_\Theta^2$. And finally, we generalize a portion of the spin-spin correlation function calculation, reproducing the result of Fisher and Aharony at critical points, while providing a new integral representation for that case. We point out that the necessary limit for that calculation is trivial in position space, in contrast to the situation in momentum space.

In Appendix B, we present the details of the large-momentum expansion of the spin-spin correlation function calculated in Sec. III. Again, our emphasis is on those details which differ from the critical case.

## II. EXPONENTS

The simple Ising-type Hamiltonian considered in the previous work is inconvenient for $\epsilon$-expansion calculations. However, there are arguments[6] which suggest that it is in the same universality class as (the $n=1$ case of) the more convenient Landau-Ginzburg-Wilson form

$$H=(2\pi)^{-d}\int d^dq\, 2^{-1}\left(m^2+q^2+\frac{q^4}{2\Lambda^2}\right)\sum_{i=1}^{n}\tilde{\sigma}_q^i\tilde{\sigma}_{-q}^i+\int d^dx\left[\left(u_2-\frac{m^2}{2}\right)\sum_1^n(\sigma_x^i)^2+u_4\left(\sum_1^n(\sigma_x^i)^2\right)^2+u_6\left(\sum_1^n(\sigma_x^i)^2\right)^3\right]. \tag{1}$$

The constants (in space) $u_2$, $u_4$, and $u_6$ are the parameters of the effective spin weight function; each spin field component $\sigma_x^i$ ranges over the real line, and $\tilde{\sigma}_q^i$ is its Fourier transform. The term $m^2$, which has been added and subtracted for a more convenient perturbation theory, is defined to be the exact inverse susceptibility $\Gamma_2$ ($p=0$); this is usually denoted by $r$. The partition function is the functional integral $Z=\int d\sigma\, e^{-H}$ and exists only as the formal generator of correlation functions (because the thermodynamic limit has been taken); the correlation functions exist as long as $u_6>0$. Wilson's direct Feynman diagram technique[2] is a calculational tool best motivated, and understood, with some version of an underlying renormalization group formalism. Let us briefly summarize what we need to know about such a formalism, within the context of tricritical behavior, in order to be able to proceed with the calculation.

Consider a particular renormalization group constructed, say, by Wilson's momentum shell integration procedure,[7] acting on the three parameters of the Hamiltonian ($u_2$, $u_4$, and $u_6$) and all other generated by a single iteration of the transformation. In general, there are an infinite number of these others. One discovers[8] that there is a twice unstable fixed point of the transformation at the origin of parameter space for $4>d>3$, one eigendirection leading toward high temperatures and the other toward a critical fixed point. Because of this, one identifies this fixed point as a tricritical fixed point. Furthermore, for sufficiently small $\epsilon=3-d$, this twice unstable fixed point can be followed away from the origin, where the eigenvalues of the linearized recursion relations are (Ising case)[6] $\lambda_1=2+O(\epsilon^2)$, $\lambda_2=1+O(\epsilon)$, $\lambda_3=-2\epsilon+(\epsilon^2)$, and various more negative eigenvalues. $\lambda_3$ is the only "dangerous" irrelevant eigenvalue of order $\epsilon$. It is presumed that, outside of the $\epsilon$ expansion, there exists a fixed point which corresponds to this one as $\epsilon\to0$, and remains twice unstable as $\epsilon\to1$.

Associated with some domain of the tricritical fixed point are an (infinite) set of nonlinear scaling fields[9] ($g_j$) (being nonlinear functions of the parameters). These fields transform multiplicatively by a factor $b^{\lambda_j}$, where $b$ is the length rescaling. The scaling fields are supposed to be analytic functions[9] of the parameters ($u_2,u_4,u_6$, others) within the domain. Thus, the equations $g_1(u_2,u_4,u_6,\text{ others}=0)=g_2(u_2,u_4,u_6,\text{ others}=0)=0$ are expected to admit a solution which is a smooth tricritical line $(u_{2t}(u_6),u_{4t}(u_6))$ in the original three-parameter space. Moreover, one can expect to locate a special tricritical point on this line, given by $(u_{2t}(u_6{}^*),u_{4t}(u_6{}^*))$, which is determined by the one additional constraint $g_3(u_2,u_4,u_6,\text{ others}=0)=0$. Although throughout we denote this special value of $u_6$ by $u_6{}^*$, it is expected not[10] to coincide with the fixed point value of $u_6$ beginning at some high enough order in $\epsilon$ (say, $\epsilon^3$).

By considering how lengths, spins, and parameters transform, one obtains a scaling law[11,18] for the one-particle irreducible $2l$ point vertex functions in the original three-parameter space

$$\Gamma_{2l}(p_i=0;u_2,u_4,u_6) = |g_1|^{\nu_t[3-l+\epsilon(l-1)-l\eta_t]} \times \Phi_{2l}(g_2/|g_1|^{\phi_t}, g_3/|g_1|^{\lambda_3/\lambda_1}, \ldots) , \quad (2)$$

where $\phi_t = \lambda_2/\lambda_1$ and all $g_j$ are functions $g_j(u_2,u_4,u_6$, others $=0)$. The functions $\Phi_{2l}$ depend on the sign of $g_1$. The principal temperature instability, represented by $g_1$, is removed by setting $g_1=0$, which locates the critical surface. Thus, fixed $g_1>0$ keeps us in the disordered phase; analyticity of the vertex functions in this phase (as functions of $u_2$ and $u_4$) implies that $\Phi_{2l}$ must be analytic near the origin of every argument that can go to zero in this phase. In particular, we expect analyticity in the first argument. In Fig. 1, we show the $g_1$ axis and $g_2$ axis at the tricritical point in a plane of fixed $u_6$; we define these axes to be the normals to the $g_1=$ constant and $g_2=$ constant contours at the tricritical point in this plane. The axes may be taken to be linear combinations[3] of $(u_2-u_{2t})$ and $(u_4-u_{4t})$.

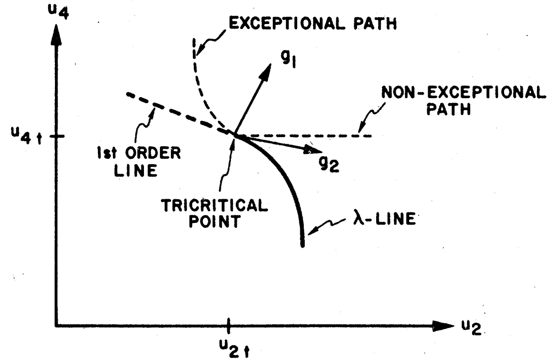



FIG. 1. Schematic of the phase boundary in a $u_2-u_4$ plane at fixed $u_6$. The numerical value of $u_{4t}$ is probably negative. Scaling axes $g_1$ and $g_2$ and examples of exceptional and nonexceptional paths are shown.

Now imagine that, within perturbation theory, we are able to locate $u_{4t}(u_6)$ and we fix $u_4$ to this value. Then, we consider Eq. (2) as $g_1 \to 0^+$. Let us assume, for the moment, that $\Phi_{2l}$ is expandable in a power series near the origin of its first *two* arguments. Then, using $(2-\eta_t)\nu_t = \gamma_t$, and the fact that $g_2$ will be going to zero linearly with $g_1$, we have, for $l=2$,

$$\Gamma_2(p=0)|_{u_4=u_{4t}} = m^2 \underset{g_1\to 0^+}{\sim} g_1^{\gamma_t}[a_0 + a_1 g_3 g_1^{|\lambda_3|/\lambda_1} + a_2(g_3 g_1^{|\lambda_3|/\lambda_1})^2 + \cdots [a_3 g_1^{1-\phi_t}] , \quad (3)$$

where the $a_i$ are constants. Solving Eq. (3) for $g_1$ as a function of $m^2$ and substituting back into Eq. (2) one obtains

$$\Gamma_{2l}(p_i=0)|_{u_4=u_{4t}} \underset{m\to 0}{\sim} m^{2[3-l+\epsilon(l-1)-l\eta_t]/(2-\eta_t)}[b_0 + b_1 g_3(m^{2|\lambda_3|/\lambda_1\gamma_t} + b_2 g_3 m^{4|\lambda_3|/\lambda_1\gamma_t} + \cdots) + b_3 m^{2(1-\phi_t)/\gamma_t} + \cdots] , \quad (4)$$

with new constants $b_i$. Equation (4) is of primary importance to the direct Feynman graph method, where one computes these vertex functions directly as a function of $u_4$, $u_6$, and $m^2$ ($u_2$ is eliminated order by order in favor of $m^2$). We see that the power-law behavior of Eq. (4) requires a constraint (in this case, fixing $u_4$) that ensures that the tricritical point is being approached as $m\to 0$. In perturbation theory near three dimensions one also finds this to be correct; the vertex functions will be found, in general, to be made from graphs which have leading power-law singularities *not* given by Eq. (4) until one imposes a constraint. If the constraint is achieved by fixing $u_4$ to a special constant, as in Ref. 4, we have thus identified that value as $u_{4t}$. There are other possibilities, however, which we discuss below. As an example of the necessity of a constraint, consider that Eq. (4) implies that, within the context of the $\epsilon$ expansion where $\eta = O(\epsilon^2)$, $\Gamma_6$ has leading singularities given by powers of $\ln m/\Lambda$. Nevertheless, one finds in perturbation theory singularities worse than this; in leading order there are $\Lambda/m$ singularities in diagrams contributing to $\Gamma_6$. However, as we have anticipated, fixing $u_4$ to $u_{4t}(u_6)$ removes these singularities to all orders, leaving only powers of $\ln m/\Lambda$. We have, of course, only verified this explicitly through two orders of perturbation theory.

Next, there is the matter of the corrections to the leading power in the vertex functions, also displayed in Eq. (4). Because the leading irrelevant eigenvalue is of order $\epsilon$, one sees that this induces a jumble of $\ln m/\Lambda$ corrections to the leading power, within the formal $\epsilon$ expansion. As discovered by Wilson,[2] we can insist, order by order, that these corrections vanish; as one sees, this is equivalent to insisting that $g_3=0$. Moreover, one sees that setting $g_3=0$ causes these corrections to vanish in every vertex function simultaneously. This additional constraint is implemented by fixing $u_6$ to a special value $u_6{}^*$ self-consistently so that the additional corrections do not occur; one need only use one vertex function to determine this value; $\Gamma_6$ is the most convenient, of course. Both of the special values $u_{4t}$ and $u_6{}^*$ are also functions of how the large momentum cutoff is implemented.

Equation (4) was obtained under the assumption that $u_4 = u_{4t}$. In general, one may want to approach the tricritical point along some other path. One usually distinguishes two types of approach[3] from the disordered phase, which we shall call here nonexceptional and exceptional paths. Nonexceptional paths, such as the one $u_4 = u_{4t}$, are always characterized by $g_2/g_1^{\phi_t} \to 0$. Exceptional paths are characterized by $g_2/g_1^{\phi_t} \to A$, where $A \neq 0$. As long as $-\infty \leq A \leq x_0$, where $x_0$ is of order unity, one can expect $\Phi_{2l}$ ($A$, others $=0$) to exist.[11] Thus, one can also derive the leading power-law behavior for the vertex functions, as a function of $m$ for any type of path of approach, within the above restrictions on $A$. One finds Eq. (4) to describe the correct leading power for *any* path; if one changes paths, the only things that change in Eq. (4) are (i) the amplitude $b_0$ depends on $A$, and (ii) the surviving correction, when $g_3 = 0$, depends on the type of path. However, to summarize, the leading power is path independent when written in terms of $m$, as long as $m > 0$. This, of course, is manifest in the fact that the leading power only depends on $\eta_t$, whose definition does not require mention of a path, whereas the correction involves an exponent ($\gamma_t$) that requires one to specify a type of path. For the moment we shall restrict ourselves to the nonexceptional path $u_4 = u_{4t}$.

Ignoring irrelevant eigenvalues there are only three independent tricritical exponents, which may be taken to be $\eta_t$, $\gamma_t$, and $\phi_t$. All the others may be determined via the tricritical scaling laws.[3] Once one fixes $u_4$, the leading power-law behavior in Eq. (4) may be used to determine $u_6{}^*$ and then $\eta_t$ with Wilson's method in the standard way[2]; as one sees, $\phi_t$ may also be determined by looking at the leading correction to the leading power in any vertex function. In fact, although we have checked that this method works through leading order, it is not convenient for determining $\phi_t$ through next-leading order. One wants to only look at leading powers; for this purpose we use an alternate scaling law, which we now derive from Eq. (2).

Near the tricritical point, we have $g_i = (u_2 - u_{2t}) - s_i(u_4 - u_{4t})$, $i = 1, 2$, where $s_1$ is the slope of the phase boundary, and $s_2$ has no special geometric significance. Now, consider an infinitesimal path of constant $m^2 = g_1^{\gamma_t}\Phi_2(x)$, where $x = g_2/g_1^{\phi_t}$ and ignore all other scaling fields, as we are interested in a leading power. Along such a path $dm^2 = 0$, which implies $dg_1/g_1 = dg_2 f(x)/g_2$, where $f(x) = x\Phi_2'/(\Phi x \Phi_2' - \gamma\Phi)$. Thus, $(\partial g_1/\partial u_4)_{m^2}|_{u_4=u_{4t}} \sim x$, and $(\partial g_2/\partial u_4)_{m^2}|_{u_4=u_{4t}} \sim$ const, as the tricritical point is approached along the nonexceptional path $u_4 = u_{4t}$. The scaling expression for $\Gamma_4$ and the above relations may then be used to show that

$$\left(\frac{\partial\Gamma_4}{\partial u_4}\right)_{m^2}\Big|_{u_4=u_{4t}} \sim m^{2(1+\epsilon-2\eta_t)/(2-\eta_t)-2\phi_t/\gamma_t} \quad . \qquad (5)$$

Equation (5) was used to obtain the crossover exponent; one sees that it is very convenient because $\Gamma_4$ is given directly by a power series in $u_4$ and $u_6$. Thus the differentiation is trivial and just reproduces the series with a factor $u_4^{n-1}$ wherever there was a $u_4^n$ originally. For a computation through $\epsilon^2$, we need Stephen and McCauley's result

$$\gamma_t = 1 + 5(n+2)(n+4)\epsilon^2/8(3n+22)^2 \quad ,$$

which we also check independently in Appendix A as a special case of order $\Theta$ critical points.

For our computations, then, we need diagrams contributing to $\Gamma_6(p_i = 0; m)$, $\Gamma_4(p_i = 0; m)$, and $\Gamma_2(p; m = 0)$. The relevant diagrams, combinatoric factors, and integrals are shown in Figs. 2–4 and Table I. Every four-point vertex comes with a factor,[4,12] not included in the tabulations, $u_4 + 3(n+4)u_6 G(x=0; m)$, where $G$ is the free propagator. This leads to[4] $u_{4t} = -3(n+4)u_6 G(0,0) + O(u_6^2)$. Any diagrams omitted from Figs. 2 and 3 are either (i) not singular enough as $m/\Lambda \to 0$ or (ii) in the case of $\Gamma_4$ diagrams, contain no four-point vertices (since we only need $\partial\Gamma_4/\partial u_4$). Certain geometric factors have been removed so that one obtains the vertex functions by summing over graphs, with a factor $u_6(-\tilde{u}_6)^{n-1}$ when there are exclusively ($n$) six-point vertices, and a factor

$$[u_4 + 3(n+4)u_6 G(x=0; m)]^{n-n_6+1}(-\tilde{u}_6)^{n_6-1}$$

when there are $n_6$ six-point vertices and $n$ total vertices. The new coupling $\tilde{u}_6$ is

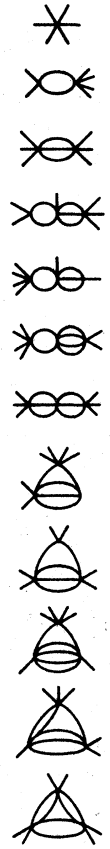
$$\tilde{u}_6 = u_6 \Lambda^{-2\epsilon} 2^{-4} \pi^{-d} [\Gamma(\tfrac{1}{2}d - 1)]^2/(d-2) \quad , \qquad (6)$$

The constants that appear in Table I and Fig. 4 are

$$\begin{aligned} A &= \sqrt{2}(10\ln 2 - 6\ln 3) \quad , \\ C &= -\tfrac{5}{2}\ln 2 + \ln 3 + C_E \quad , \\ J &= \ln 3 + C_E \quad , \\ K &= \tfrac{1}{6}(\tfrac{11}{6} - C_E) \quad , \end{aligned} \qquad (7)$$

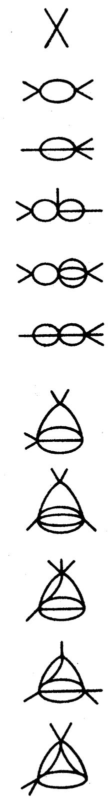
where $C_E$ is Euler's constant. Of these constants, $J$ and $K$ are universal, while $A$ and $C$ depend on the specific form of the large momentum cutoff. A tedious amount of algebra then yields the results

$$\tilde{u}_6{}^*(\epsilon) = 2^{-3}3^{-1}(3n+22)^{-1} \times \epsilon\{1 + \epsilon[Q(n) + 2C]\} + O(\epsilon^3) \quad , \qquad (8)$$



| DIAGRAM | COMBINATORIC FACTOR |
|---|---|
| 1 | $6!$ |
| 2 | $6!\ 2^2\ 3\ (n+14)$ |
| 3 | $6!\ 2^4\ 3\ (3n+22)$ |
| 4a | $6!\ 2^7\ 3^2\ (n^2+14n+60)$ |
| 4b | $6!\ 2^6\ 3^2\ (n+4)(n+14)$ |
| 5 | $6!\ 2^4\ 3^3\ (n+4)(n^2+10n+64)$ |
| 6 | $6!\ 2^7\ 3^2\ (3n^2+30n+92)$ |
| 7a | $6!\ 2^6\ 3^2\ (n+4)(n+14)$ |
| 7b | $6!\ 2^6\ 3^2\ (n^2+36n+188)$ |
| 8 | $6!\ 2^5\ 3^4\ (n+4)(3n+22)$ |
| 9 | $6!\ 2^8\ 3^3\ (5n^2+78n+292)$ |
| 10 | $6!\ 2^5\ 3^2\ (n^3+34n^2+620n+2720)$ |

FIG. 2. Diagrams contributing to $\Gamma_6$ with their combinatoric factors.



| DIAGRAM | COMBINATORIC FACTOR |
|---|---|
| 1 | $4!$ |
| 2 | $4!\ 2^2\ (n+8)$ |
| 3 | $4!\ 2^5\ 3\ (n+4)$ |
| 4 | $4!\ 2^7\ 3\ (n+4)(n+8)$ |
| 5 | $4!\ 2^5\ 3^2\ (n+4)(n^2+8n+36)$ |
| 6 | $4!\ 2^8\ 3^2\ (n+4)^2$ |
| 7 | $4!\ 2^7\ 3\ (n+4)(n+8)$ |
| 8 | $4!\ 2^6\ 3^2\ (n+4)(7n+38)$ |
| 9a | $4!\ 2^8\ 3^3\ (n+4)^2$ |
| 9b | $4!\ 2^8\ 3^2\ 5\ (n+4)(n+8)$ |
| 10 | $4!\ 2^5\ 3^2\ (n+4)(n^2+18n+116)$ |

FIG. 3. Diagrams contributing to $\Gamma_4$ with their combinatoric factors.

$$\eta_t(\epsilon)=[(n+2)(n+4)/12(3n+22)^2]\times\epsilon^2\{1+2\epsilon[Q(n)-\tfrac{1}{3}]\}+O(\epsilon^4)\ , \qquad (9)$$

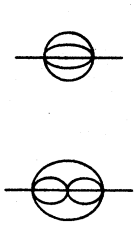
where

$$Q(n)=[\pi^2(n^3+34n^2+620n+2720)+8(53n^2+858n+3304)]\times[16(3n+22)^2]^{-1} \qquad (10)$$

and

$$\phi_t(\epsilon)=\frac{1}{2}+\frac{(6-n)\epsilon}{2(3n+22)}+\frac{(n+4)\epsilon^2}{16(3n+22)^3}\times[\pi^2(n^3+8n^2-496n-2888)-8(19n^2+508n+2428)]+O(\epsilon^3)\ . \qquad (11)$$

In two dimensions ($\epsilon=1$), Eq. (9) yields $\eta_t=0.027$, 0.034, and 0.040 for $n=1,2$, and 3, respectively. The lowest-order term alone accounts



| DIAGRAM | COMBINATORIC FACTOR AND INTEGRAL |
|---|---|
| (1) | $2^5\ 3^2\ (n+2)(n+4)$ <br> $[\frac{1}{6}(\ln p+\frac{1}{3}-C)-K]+\epsilon[\frac{-1}{3}\ln^2 p+(4K-\frac{1}{9})\ln p]$ |
| (2) | $2^9\ 3^3\ (n+2)(n+4)(3n+22)$ <br> $-\frac{1}{6}\ln^2 p+(\frac{1}{3}C+2K)\ln p$ |

FIG. 4. Propagator diagrams with a factor of $p^2$ removed. Constants are defined in the text; in logarithms $p$ should be read as $p/\Lambda$.

TABLE I. Leading sigularities of the diagrams in Figs. 2 and 3. In logarithms $m$ should be read as $m/\Lambda$. Various constants are defined in the text.

| Diagram | Integral |
|---|---|
| 1 | $1$ |
| 2 | $1/2m - \epsilon \ln m/m$ |
| 3 | $(-\ln m + C - J) + \epsilon(\ln^2 m + 2J \ln m)$ |
| 4 | $-\ln m/2m$ |
| 5 | $A/m + 2\ln m$ |
| 6 | $\ln^2 m + 2(J - C)\ln m$ |
| 7 | $-\ln m/2m$ |
| 8 | $A/m + \frac{5}{2}\ln m$ |
| 9 | $\frac{1}{2}\ln^2 m + (J - C - 1)\ln m$ |
| 10 | $-\frac{1}{4}\pi^2 \ln m$ |

for less than a tenth of these values. Because of the three-loop graph (diagram 10), both $\eta_t$ and $\phi_t \sim +n$ for large $n$. In fact, Eq. (8) reveals that even at lowest order in $\epsilon$, the spherical-model limit is ill behaved: $u_6{}^*$ is proportional to $n^{-1}$ for large $n$, whereas the free-energy density has a reasonable large-$n$ limit only if[13] $u_6 \sim n^{-2}$. The series for $\phi_t$ produces a *negative* estimate for the Ising case, which is presumably not correct since it would imply, formally, that no crossover occurs. The primary source of the large negative contribution in second order is again the highly irreducible diagram 10. A not unreasonable speculation is that the second-order term has "overcorrected" the wrong sign of the first-order term, and that the exact value of $\phi_t$ is indeed less than $\frac{1}{2}$, although *positive.*

After one finishes the algebra that leads to the results of Eqs. (9) and (11), one cannot help noting certain remarkable cancellations among the various diagrams in Figs. 2 and 3. Specifically, one discovers that in Fig. 2, diagrams 4a, 4b, 7a, and 7b, which potentially contribute to $\eta_t$, in fact do not, but rather cancel a similar contribution at order $\epsilon^3$ from diagram 2 in the same figure. An analogous cancellation occurs with diagrams 4 and 7 in Fig. 3. These diagrams are all distinguished by having four-point vertices; contributing diagrams are distinguished by having exclusively six-point vertices (taking into account the differentiation of $\Gamma_4$). As we anticipated earlier, there is an optimal choice of path of approach which manifestly eliminates these noncontributing diagrams right away. As we also mentioned, the leading power in Eq. (4) is path independent; similarly, Eq. (5) is also path independent, as may be seen by the following argument. Along the path $u_4 = u_{4t}$, one may replace $m^2$ by $g_1^{\gamma_t}$ and rewrite Eq. (5) in terms of $g_1$. However, one may show that along an exceptional path of approach one obtains the correct power by replacing $g_1$ by $g_2^{1/\phi_t}$; but since along an exceptional path $m^2 \sim g_2^{\gamma_t/\phi_t}$, the power of $m$ is invariant. Thus, the subscript $u_4 = u_{4t}$ is superfluous in both Eqs. (4) and (5).

The optimal path is the exceptional path $\Gamma_4(p_i = 0; m) = 0$. This equation may be solved for the curvilinear path of approach $u_4 = \bar{u}_4(m)$, order by order in $u_6$. Through second order this path is

$$\bar{u}_4(m) = -3(n+4)u_6 G(0;m) + 2^2 3^2 (n+4)(n+14) u_6^2 \times \int d^d x\, G^4(x,m) + O(u_6^3) \ .$$

Note that this is different from choosing $u_4 = u_{4t}$, a constant independent of $m$. So, in the perturbation series for $\Gamma_6$, everywhere a factor of $u_4$ appears, the value $\bar{u}_4(m)$ is substituted; similarly, *after* differentiating the series for $\Gamma_4$ one substitutes $\bar{u}_4(m)$. Now one sees why the third-order diagrams with four-point vertices do not contribute: they are immediately eliminated by the first term of $\bar{u}_4(m)$. Also, one sees that the only role of diagram 2 in Figs. 2 and 3 is to provide a subtraction for diagram 8 once $\bar{u}_4(m)$ is substituted. Once the perturbation series is entirely in terms of $u_6$, one proceeds in the standard way to reproduce the results of Eqs. (8)–(11) with much less work.

Although we have not calculated $\gamma_t$ through next order, one may anticipate similar cancellations to those discussed above along the path $u_4 = u_{4t}$. This phenomenon seems to occur in general; at critical points where $\Theta$ coexisting phases become critical, only portions of diagrams made up exclusively of $2\Theta$ point vertices seem to contribute to the exponents; the only role of the other diagrams is to provide subtractions for the former. A further example of this is discussed in the Appendix.

## III. SPIN-SPIN CORRELATION FUNCTION

If we are deep within the tricritical scaling region we can ignore corrections due to irrelevant scaling fields. Then, one has the scaling form for the spin-spin correlation function in the disordered phase,[3]

$$\tilde{G}(p;u_2,u_4) \simeq g_1^{-\gamma_t} \tilde{D}(p g_1^{-\nu_t}, g_2 g_1^{-\phi_t}) \ , \qquad (12)$$

where $p$ is the momentum variable, so that at the tricritical point $\tilde{G} \simeq p^{\eta_t - 2}$. If we set $u_4 = u_{4t}$, and then approach the tricritical point by letting $m^2 \to 0$, we have seen that the second argument vanishes in the limit. Let us adopt the notation of Fisher and Aharony[5] and call this resulting function $\hat{D}(x^2) \equiv D(p g_1^{-\nu_t} t, 0)$, with the usual normalization $\hat{D}(0) = -d\hat{D}(0)/dx^2 = 1$ and $x = p g_1^{-\nu_t}$.

Then, a simple computation in position space, following the method of Fisher and Aharony, yields

$$\hat{D}(x^2)=(1+x^2)^{-1}+\epsilon^2[x^2/(1+x^2)]^2Q(x^2)+O(\epsilon^3)\ ,$$

where $Q$ is determined by the two self-energy graphs with three and five lines. Namely,

$$Q(x^2)=\frac{(n+2)(n+4)}{2(3n+22)^2}x^{-4}\int_0^\infty dy\, y^{-3}[e^{-5y}+(n+4)y^2e^{-3y}]\left[\frac{\sin xy}{xy}-1+\frac{x^2y^2}{6}\right]\ . \tag{13}$$

At critical points one obtains a similar integral representation (see Appendix A); here, however, the integral is easy to evaluate in terms of simple functions and yields the final result

$$Q(x^2)=\frac{(n+2)(n+4)}{2(3n+22)^2}\left\{\left[\left(\frac{1}{12x^2}-\frac{25}{4x^4}\right)\ln\left(1+\frac{x^2}{25}\right)+\left(\frac{5}{2x^3}-\frac{125}{6x^5}\right)\arctan\left(\frac{x}{5}\right)-\frac{11}{36x^2}+\frac{25}{6x^4}\right]\right.$$
$$\left.-(n+4)\left[\frac{1}{2x^4}\ln\left(1+\frac{x^2}{9}\right)+\frac{3}{x^5}\arctan\left(\frac{x}{3}\right)+\frac{1}{54x^2}+\frac{1}{x^4}\right]\right\} \tag{14}$$

Thus, small- and large-momentum expansions[5,14] are immediately obtained. Similarly to the situation at critical points $Q(0)$ is very small and indicates only very small corrections to the Ornstein Zernike line shape. Of course, we know that the scale here is set by $\eta_t$, and since there is an order of magnitude change in next order, one can expect similar numerical changes in $\hat{D}$ from contributions at order $\epsilon^3$.

The large-momentum expansion of Eq. (14) is more interesting. At critical points,[14] the first corrections to the leading power include an energy singularity $\propto g_1^{1-\alpha_t}$ and a regular term $\propto g_1$; these powers do not coincide for $d=3$ (unlike the corresponding powers at critical points for $d=4$). If Eq. (14) is expanded for large $x$ far enough to observe the regular term, one finds that another term besides the ones present at critical points must appear in the expansion in order for the calculation expression to agree with it. The form of this term can be obtained by the field theoretic method. Because the mthod has been described in the literature by its originators[15] and applied to an exhaustive discussion of the critical-point correlation function by the Saclay group,[14] we present in Appendix B only those details of the derivation which differ significantly from the critical case. The result of this calculation for the path $u_4=u_{4t}$ in the disordered phase is

$$D(x^2)\underset{x\to\infty}{\sim}x^{\eta_t-2}[A(\epsilon)+B(\epsilon)x^{(\alpha_t-1)/\nu_t}+C(\epsilon)x^{(\phi_t/\nu_t)-d}+D(\epsilon)x^{-1/\nu_t}+O(x^{-3+\tau})]\ , \tag{15}$$

where $\tau$ is the order $\epsilon$. The coefficient $C(\epsilon)$ of the term not present at critical points begins at order $\epsilon$. When this term is included, the expansion of Eq. (15) is consistent with the expansion of Eq. (14).

## IV. SUMMARY

We have calculated the tricritical exponents $\eta_t$ and $\phi_t$ through two nontrivial orders in $\epsilon$, and a portion of the spin-spin correlation function through leading nontrivial order. Deviations from mean-field theory, in the case of $\eta_t$ and the correlation function, are numerically small and comparable to corresponding deviations found at critical points in three dimensions. With respect to $\phi_t$, the situation is rather ambiguous. The weak dependence of exponents on the symmetry index $n$, characteristic of critical points in three dimensions or higher, is lost at sufficiently large $n$, although the variation for $1\leq n\leq 3$ is still quite small.

## ACKNOWLEDGMENTS

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## APPENDIX A: ORDER-$\Theta$ CRITICAL POINTS (ISING CASE)

For those universal quantities, such as $\eta_\Theta$ and $\gamma_\Theta$ which exist at all critical points where $\Theta$ coexisting phases become critical, it is possible to calculate all of them at once[6] as a function of $\Theta$. This is possible because of the similar structure of the infrared divergences in the appropriate vertex functions near each upper borderline dimension $d_\Theta=2\Theta/(\Theta-1)$. Here we will outline the calculation for $\gamma_\Theta$ through $\epsilon_\Theta^2$, $\eta_\Theta$ through $\epsilon_\Theta^3$, and consider briefly the spin-spin correlation function. In all cases, $\epsilon_\Theta = d_\Theta - d$ and we will often drop the subscript on $\epsilon_\Theta$.

In Wilson's method,[2] $\gamma_\Theta$ is determined by the $m^2 \times$ (powers of $\ln m$) singularities in self-energy (mass counterterm) diagrams generated by the perturbation $\sum_{j=2}^{\Theta} \int d^3x\, u_{2j}\sigma_x^{2j}$. If we approach the order-$\Theta$ critical point along a nonexceptional path, we find (see below) that these singularities do not begin until order $\epsilon^2$ if $\Theta \geq 3$. Thus, writing $\gamma = 1 + \gamma_2\epsilon^2 + \Theta(\epsilon^3)$ and $\Gamma_2(p=0;m) = m^2 = m^2 + [(m_\Theta^2 - m^2) + I(m)]$, where $I(m)$ may be taken to be a sum of graphs through second order in the perturbation, one has $\gamma_2 = 2^{-1}[I(m)]|_{\epsilon^2 m^2 \ln m \text{ part}}$, where the notation means abstract the coefficient of the singularity.

We make use of the generalized vertex,[6] for $2 \leq j \leq \Theta$,

$$\hat{u}_{2j}(m;\epsilon) = \sum_{k=0}^{\Theta-j} \frac{(2j+2k)!}{(2j)!k!2^k} u_{2j+2k}[G(x=0;m)]^k \ . \tag{A1}$$

Then, in terms of these vertices and regular vertices, the sum of graphs is

$$\begin{aligned} I(m) = {} & \sum_{k=1}^{\Theta-1} \frac{(2k+2)!}{k!2^k} u_{2k+2}[G(0,m)]^k \\ & - \sum_{j=2}^{\Theta-1} (2j+2)!\hat{u}_{2j}(m,\epsilon)\hat{u}_{2j+2}(m,\epsilon)F_{2j}(m,\epsilon) \\ & - \sum_{j=2}^{\Theta} (2j)(2j)!\hat{u}_{2j}^2(m,\epsilon)F_{2j-1}(m,\epsilon) \ , \end{aligned} \tag{A2}$$

where the Feynman integral is $F_j = \int d^dx\, [G(x;m)]^j$. Note that the first sum in Eq. (2) contains only regular vertices.

By using the scaling law[6] for $\Gamma_{2\Theta}$, one determines the order-$\Theta$ point coordinates $u_{2j}{}^*$ and the special value $u_{2\Theta}{}^*$. These values are, through leading order,

$$u_{2\Theta-2j}{}^* = \frac{(-1)^j(2\Theta)![G(0,0)]^j}{2^j j!(2\Theta-2j)!} u_{2\Theta}{}^* \tag{A3}$$

and

$$u_{2\Theta}{}^* = [4\pi^{d/2}/\Gamma(\nu)]^{\Theta-1} 2\epsilon \Big/ \binom{2\Theta}{\Theta}^2 \Theta! \ , \tag{A4}$$

where $\nu = \frac{1}{2}d - 1$, and it is to be understood that $d$ may be replaced by $d_\Theta$ wherever appropriate. Substituting these values in Eq. (A1) results in

$$\hat{u}_{2\Theta-2j}{}^* = \frac{(2\Theta)![G(0,m) - G(0,0)]^j}{2^j j!(2\Theta-2j)!} u_{2\Theta}{}^* \ . \tag{A5}$$

The free propagator $G(x,m)$ is the Fourier transform of $(m^2 + q^2 + q^4/2)^{-1}$. This may be written

$$G(x,m) = 2x^{2-d}(c_2^2 - c_1^2)^{-1}[W(c_1x) - W(c_2x)] \tag{A6}$$

where $c_1(m)$ and $c_2(m)$ tend toward $m$ and $\sqrt{2}$ respectively, as $m \to 0$, and $W(x)$ is the free propagator scaling function (proportional to $x^\nu K_\nu$), given by, for $\Theta \geq 3$,

$$W(x) = \Gamma(\nu)\Gamma(1-\nu)4^{-1}\pi^{-d/2}\left(\sum_{m=0}^{\infty} \frac{(x/2)^{2m}}{m!\Gamma(1-\nu+m)} - \sum_{m=0}^{\infty} \frac{(x/2)^{2m+2\nu}}{m!\Gamma(1+\nu+m)}\right) . \tag{A7}$$

From the above, one finds that, for $\Theta \geq 3$,

$$[G(0,m)]^k|_{m^2\ln m} = \delta_{k,\Theta-1}\left(\frac{\Gamma(\nu)\Gamma(1-\nu)}{4\pi^{d/2}\Gamma(1+\nu)}\right)^{\Theta-1} \times 4^{-1}(\Theta-1)(-1)^\Theta\epsilon \ . \tag{A8}$$

Because of Eq. (8) and the fact that the couplings are of order $\epsilon$ one indeed finds the coorrection to $\gamma = 1$ beginning at order $\epsilon^2$. When $\Theta = 2$, there are additional $\ln x$ terms present in the power series for $W(x)$ and this causes the first correction to begin at order $\epsilon$ in that case.[2] In this respect, $d_\Theta = 4$ is quite special. Let us implicitly define three constants $W(0)$, $R_1$, and $R_2$ by rewriting Eq. (7) for small $x$ as $W(x) \simeq W(0)(1 + R_1x^{d-2} + R_2x^2 + \cdots)$. Then, analysis of the Feynman integrals in $d_\Theta$ yields, for $0 \leq j \leq \Theta$,

$$F_{\Theta+j}|_{m^{(d-2)j}\ln m} = -\Omega_\Theta[W(0)]^{\Theta+j} \times \left[\binom{\Theta+j}{j} R_1^j + (2\Theta-1)R_2\delta_{j,\Theta-1}\right] , \tag{A9}$$

where $\Omega_\Theta$ is the area of the unit sphere in $d_\Theta$. Ultimately, one finds that $\gamma_2$ is determined exclusively by the second term in Eq. (9). All the ingredients necessary for the evaluation of the $\epsilon^2m^2\ln m$ piece of $I(m)$ are now present in Eqs. (3)–(5), (8), and (9). Thus, putting everything into Eq. (2) yields

$$\begin{aligned} 2^{-1}[I(m)]|_{\epsilon^2m^2\ln m \text{ part}} = {} & (2\Theta)(2\Theta-1)(\Theta-1)^2/(\Theta-2)\binom{2\Theta}{\Theta}^3 \\ & + (-1)^\Theta\left(\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\right)^{\Theta-1}\left[\Theta!(\Theta-1)\Big/\binom{2\Theta}{\Theta}^2\right]\left[S_1 + S_2 - \binom{2\Theta}{\Theta}\Big/(\Theta-1)!2^\Theta\right] , \end{aligned} \tag{A10}$$

where the sums $S_1$ and $S_2$ are, with $[x]$ the greatest integer in $x$,

$$S_1 = \sum_{j=1}^{[\Theta/2]} [(\Theta - 2j)! 2^{2j-1} j! (j-1)!]^{-1} \ ,$$

$$S_2 = \sum_{j=0}^{[(\Theta-1)/2]} [(\Theta - 2j - 1)! 2^{2j} (j!)^2]^{-1} \ . \tag{A11}$$

Now, a remarkable combinatoric cancellation occurs because, apparently the entire second line of Eq. (10) vanishes, i.e.,

$$S_1 + S_2 = \begin{pmatrix} 2\Theta \\ \Theta \end{pmatrix} / [(\Theta - 1)! 2^{\Theta}]$$

when $\Theta \geqslant 3$. We do not have a proof of this amusing identity, but it is easy to check for every $\Theta$ that one might try. Hence, one obtains the final very simple result

$$\gamma_\Theta = 1 + \left[ (2\Theta)(2\Theta - 1)(\Theta - 1)^2 / (\Theta - 2) \begin{pmatrix} 2\Theta \\ \Theta \end{pmatrix}^3 \right] \times \epsilon_\Theta^2 + O(\epsilon_\Theta^3) \ . \tag{A12}$$

And, of course, Wilson's result applies when $\Theta = 2$. When $\Theta = 3$, one finds $\gamma_t = 1 + 3\epsilon^2/200$, in agreement with Stephen and McCauley's result[4] at $n = 1$. The above cancellation has the effect that $\gamma$ is determined solely by a portion of the $m^2 \ln m$ singularity in the single second-order self-energy graph with $2\Theta - 1$ internal lines; this is the only graph, through this order, made up entirely of $2\Theta$-point vertices. One can expect this cancellation to occur not only in the Ising case, but in the $n$-component case also.

Next, we consider the calculation of $\eta$; this calculation, being in third order, is more complicated than the one just outlined, and not entirely airtight, although we have confidence in the final result. The weakness of the calculation is that the evaluation of diagram 5 in the figure required an assumption about the singularity structure that could be explicitly checked when $\Theta = 2$ and 3, but we were not able to prove it in general. This is probably not very serious because, in the context of Wilson's method, this diagram seems to contribute only "old" information toward the evaluation of $\eta$. If our assumption is wrong, it will mean that there are additional terms in the final result when $\Theta \geqslant 4$. Thus, the result of this section must be taken to be a conjecture when $\Theta \geqslant 4$. We calculate along the exceptional path determined by setting all of the intermediate vertices to zero, leaving only $\Gamma_2$ and $\Gamma_{2\Theta}$; contributing diagrams are shown in Table II and Fig. 5. All vertices are $2\Theta$-point vertices. As usual, one obtains $\Gamma_{2\Theta}$ by the sum over diagrams

$$u_{2\Theta} \sum (\text{combinatoric}) \times (\text{integral}) \times (-\tilde{u}_{2\Theta})^{n-1}$$

when there are $n$ vertices, with a new coupling

$$\tilde{u}_{2\Theta} = u_{2\Theta} \Lambda^{(1-\Theta)\epsilon_\Theta} 2^{2-2\Theta} \pi^{-d(\Theta-1)/2} \times [\Gamma(\tfrac{1}{2}d - 1)]^{\Theta-1} / (d-2) \ . \tag{A13}$$

The generalization of the constants, that appeared in the triritical case is

$$C = -\int_0^\infty dx \ln x \frac{d}{dx} \left( \frac{x^{(\Theta+1)/(\Theta-1)}}{(x^2+1)^{\Theta/(\Theta-1)}} \right) \ ,$$

$$J = \int_0^\infty dx \ln x \frac{d}{dx} [\overline{W}(x)]^{\Theta} \ , \tag{A14}$$

$$K = \int_0^\infty dx \ln x \frac{d}{dx} \times \int_0^\pi d\bar{\theta} (\sin\theta)^{2/(\Theta-1)} \left( \frac{e^{ix\cos\theta} - 1}{x^2} \right) \ ,$$

TABLE II. Combinatoric factors and leading singularities in the diagrams in Fig. 5.

| Diagram | | Integral |
|---|---|---|
| 1 | $\frac{(2\Theta)!^3}{2(\Theta!)^3}$ | $(-\ln m + C - J) + \epsilon(\Theta - 1)(\frac{1}{2}\ln^2 m + J \ln m)$ |
| 2 | $\frac{(2\Theta)!^4}{2(\Theta!)^4}$ | $\ln^2 m + 2(J - C)\ln m$ |
| 3 | $\frac{(2\Theta)!^2 s}{l_1!^2 l_2!^2 l_3!^2}$ | $\delta_{l_1,\Theta}[\frac{1}{2}\ln^2 m + (J - C)\ln m] - A \ln m$ |
| 4 | $(2\Theta)(2\Theta)!$ | $\left[\frac{1}{2\Theta}\left(\ln p + \frac{1}{d_\Theta} - C\right) - \frac{2K}{\Theta - 1}\right] + \epsilon\left[-\frac{1}{d_\Theta}\ln^2 p + \left(4K - \frac{2}{d_\Theta^2}\right)\ln p\right]$ |
| 5 | $\frac{\Theta(2\Theta)!^3}{\Theta!^3}$ | $-\frac{1}{2\Theta}\ln^2 p + \left(\frac{C}{\Theta} + \frac{4K}{\Theta - 1}\right)\ln p$ (conjecture when $\Theta \geqslant 4$) |

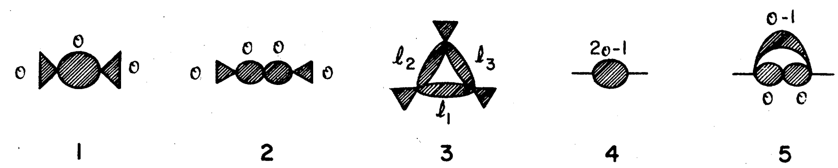


FIG. 5. Diagrams necessary for the computation of $\eta_\Theta$ through $\epsilon_\Theta^3$ with constants defined in the Appendix. All vertices are $2\Theta$ point vertices and numbers next to the cross-hatched blobs indicate the number of lines running between the vertices or the number of external lines.

where $\overline{W}(x)=W(x)/W(0)$, and the normalized angular variable is given by $d\overline{\theta}=d\theta/\omega$, where $\omega=\int_0^\pi d\theta\,(\sin\theta)^{2/(\Theta-1)}$. We have taken the liberty of changing propagators to $G(x,m)=(x^2+\Lambda^{-2})^{(2-d)/2}\times W(mx)$. Only the relevant singular terms of the diagrams, which are not cancelled by the path condition, are shown. This is in contrast to Table I, which contains *all* the leading singular terms, so one should be careful when comparing. The numbers next to the blobs indicate the number of lines running between the vertices. For the "triangle" shaped graphs there are two additional numbers in the figure that must be specified. For these graphs all of the following expressions and the expressions in the figure only hold when $l_1+l_2+l_3=2\Theta$ and $l_1\geq l_2\geq l_3\geq 1$ which labels all these graphs once. In the combinatoric factors for these graphs there is the symmetry factor $s$, defined by

$$s=\begin{cases}1/3!, & l_1=l_2=l_3\\ 1/2!, & l_1=l_2\neq l_3 \text{ or } l_1=l_2=l_3\\ 1, & \text{otherwise}\end{cases}$$

Then, in the Feynman integral, there is the coefficient of the relevant part of the $\ln m$ divergence, $A(l_1,l_2,l_3)$ and we find

$$A(l_1,l_2,l_3)=\frac{\Theta-1}{2(\Theta-l_1)}\left[{}_3F_2\left(\frac{l_3}{\Theta-1},\frac{l_3-1}{\Theta-1},\frac{\Theta-l_1}{\Theta-1};\frac{\Theta}{\Theta-1},\frac{2\Theta-l_1-1}{\Theta-1};1\right)-\delta_{\Theta,l_1}\right]$$
$$+\frac{\Theta-1}{2(\Theta-l_2)}\left[{}_3F_2\left(\frac{l_3}{\Theta-1},\frac{l_3-1}{\Theta-1},\frac{\Theta-l_2}{\Theta-1};\frac{\Theta}{\Theta-1},\frac{2\Theta-l_2-1}{\Theta-1};1\right)\right]\ , \tag{A16}$$

where ${}_3F_2$ is the hypergeometric function with three numerator $(a,b,c)$ and two denominator $(d,e)$ arguments. Equation (A16) is not as bad as it looks, because note that $e=c+1$, so that when each hypergeometric sum is written out in terms of $\Gamma$ functions there are really only three instead of five. It is a matter of taste as to whether (A16) or the explicit sums provide more compact expressions. It is also worth noting that the ${}_3F_2$ functions above contribute only a 1, their first term of their power series, when $l_3=1$.

To obtain the final answer one has to sum over all triangle graphs. Let us call this sum

$$B=\sum_{\substack{l_1\geq l_2\geq l_3\geq 1\\ l_1+l_2+l_3=2\Theta}}\frac{(2\Theta\,!)^2}{l_1!^2 l_2!^2 l_3!^2}(s)(A)\ . \tag{A17}$$

Then, in terms of $B$, one obtains the simple result (conjecture)

$$n_\Theta=\frac{4(\Theta-1)^2}{\begin{pmatrix}2\Theta\\ \Theta\end{pmatrix}^3}\epsilon_\Theta^2+\frac{4(\Theta-1)^3}{\begin{pmatrix}2\Theta\\ \Theta\end{pmatrix}^3}\left[\frac{8(B-\Theta)}{\begin{pmatrix}2\Theta\\ \Theta\end{pmatrix}^3}-\frac{(\Theta-1)}{2\Theta}\right]$$
$$\times\epsilon_\Theta^3+O(\epsilon_\Theta^4)\ . \tag{A18}$$

Let us illustrate that this is a viable formula by recovering previous results at $\Theta=2,3$. First, consider $\Theta=2$; the sum defining $B$ will admit only one triple (2,1,1); the hypergeometric sums contribute only a 1 because $l_3=1$ and so $B=(\frac{3}{2})4!$ Subsitution of this into (A18) yields agreement with Wilson. Next, let $\Theta=3$; the sum for $B$ admist three triples: two are trivial because $l_3=1$ and one is nontrivial and generates the sum $\sum_{n=0}^{\infty}(2n+1)^{-2}=\pi^2/8$. Thus, one obtains

$$B=6!^2(2^{-9}3^{-1}\pi^2+2^{-4}3^{-2}-2^{-8}3^{-2})$$

and substitution yields agreement with Eq. (9) in the text when $n=1$. In general, one obtains infinite sums that must be evaluated numerically. These small $\Theta$ results indicate that $B$ may increase as fast as $\Theta!$ as $\Theta \to \infty$, resulting in large estimates at sufficiently large $\Theta$ in two dimensions.

The single-argument spin-spin correlation function may also be generalized. Rather than do this completely, however, let us only consider the contribution of diagram 4 in the figure at this function. With this result, we will see how Fisher and Aharony's result[5] at critical points and the result in the text are part of the same general, yet simple expression. Moreover, this will provide a new integral representation for this function at critical points.

In the order-$\Theta$ case we have, as before,

$$\hat{D}(x^2) = (1+x^2)^{-1} + \epsilon_\Theta^2 [x^2/(1+x^2)]^2 (2\Theta)\eta_\Theta Q(x^2;\Theta) \ ,$$

where we have now removed a factor (equal to $4\eta$ at critical points) in order to conform as closely as possible to Fisher and Aharony's notation. The function $Q$ is given by a sum of graphs $Q=\sum_{j=0}^{\Theta-2} Q_{2\Theta-2j-1}(x^2;\Theta)$, where the subscript $2\Theta-2j-1$ is the number of internal lines. Let us only consider the graph with $2\Theta-1$ lines; we find, with $\nu=(\Theta-1)^{-1}$,

$$Q_{2\Theta-1}(x^2;\Theta) = \frac{\nu 2^{2/\nu-\nu}}{[\Gamma(\nu)]^{1+2/\nu}} x^{-4} \times \int_0^\infty dy\, y^{\nu-1}[K_\nu(y)]^{1+2/\nu} \times \left[ \Gamma(1+\nu) 2^\nu x^{-\nu} y^{-\nu} J_\nu(xy) - 1 + \frac{x^2y^2}{4(1+\nu)} \right] . \quad \text{(A19)}$$

The appropriate limit that leads to (A19) is quite trivial in position space, in contrast to the situation in momentum space.[5]

Now specializing to $\Theta=2$, one may expand the Bessel function $J_1(xy)$ about the origin to yield the expansion of $Q_3(u;2)$ about the origin,

$$Q_3(u;2) = \sum_{k=2}^{\infty} \frac{(-1)^k a_{2k} u^{k-2}}{2^{2k-1} k!(k+1)!} \ , \quad \text{(A20)}$$

where $a_{2k} = \int_0^\infty dy\, y^{2k}[K_1(y)]^3$. These integrals may be evaluated numerically to show that $Q_3(u;2)$ agrees with Fisher and Aharony's result for their function $Q(u)$. One can also check the large momentum expansion from the expression

$$Q_3(x^2;2) = 2x^{-5} \int_0^\infty dy \left[ K_1\left(\frac{y}{x}\right)\right]^3 \left[ 2yJ_1(y) - 1 + \frac{y^2}{8} \right] . \quad \text{(A21)}$$

In a similar way, one can verify that for $\Theta=3$, Eq. (A19) agrees with the appropriate part of Eq. (13) in the text.

## APPENDIX B: LARGE-ARGUMENT EXPANSION OF $\hat{D}(x^2)$

The basic method[14,15] is to combine Wilson operator-product expansions with the Callan-Symanzik equation. Zimmermann[16] has discussed Wilson expansions in perturbation theory using his normal product formalism which is general enough to apply immediately to the present case. Through this method (defining normal products by making all subtractions required for the case $d=3$) one can obtain the following expansion:

$$\begin{aligned} G_{2l+2}^{M,0}&(q_1,\ldots,q_{2l},\tfrac{1}{2}p_0+k,\tfrac{1}{2}p_0-k;p_1,\ldots,p_M) \\ &= \Delta_1(k) G_{2l}^{M+1,0}(q_1,\ldots,q_{2l};p_0,\ldots,p_M) \\ &\quad + \Delta_2(k) G_{2l}^{M,1}(q_1,\ldots,q_{2l};p_1,\ldots,p_M;p_0) \\ &\quad + R_{2l}^M(k;q_1,\ldots,p_M) \ , \end{aligned} \quad \text{(B1)}$$

where for large $k$ (by power counting with $d=3$),

$$\Delta_j(k) \sim k^{-2-j+\tau_j}, \quad \tau_j \sim \Theta(\epsilon), \quad j=1,2 \quad \text{(B2)}$$

and, unless $l=0$ and $M<2$,

$$R_{2l}^M(k;\cdots) \sim k^{-5+\tau}, \quad \tau \sim \Theta(\epsilon) \ . \quad \text{(B3)}$$

In Eq. (B1), the connected Green's function of $2l$ spins, $M$ insertions of $\sum_j (\sigma_x^j)^2$, and $N$ insertions of $[\sum_j (\sigma_x^j)^2]^2$ is denoted $G_{2l}^{M,N}$; and the spin indices associated with the momenta $\frac{1}{2}p_0+k, \frac{1}{2}p_0-k$ are set equal and summed (no other case is required in the disordered phase). We do not require the explicit expressions for the $\Delta_j$ which can be obtained through the normal product method. It should be noted that Eq. (B1) is not written for the one particle irreducible vertices $\Gamma_{2l}^{M,N}$. When translating (B1) into a corresponding expansion for the $\Gamma$'s it will be necessary in all but the simplest cases to include extra, explicit terms, comparable to the $\Delta_2$ term, in order that the remainder remain negligible. The necessity of the extra terms is easily understood by contemplating the expansion for $\hat{D}^{-1}$, which is generated from Eq. (15) of the text by the binomial theorem; it also contains an extra term proportional to $[B(\epsilon)]^2$, which is comparable, near $d=3$, to the terms involving $C(\epsilon)$ and $D(\epsilon)$. The extra terms in the expansion of the $\Gamma$'s are responsible for this extra term in the expansion of $\hat{D}^{-1}$.

Besides Eq. (B1), the other ingredients of the method are the field theoretic renormalization-group equations[17]

$$[\mathfrak{D} + d + l(\eta - d - 2) - M\nu^{-1}]G_{2l}^{M,0}(sq_1, \ldots, sp_M) = 0 \;, \tag{B4}$$

$$[\mathfrak{D} + d + l(\eta - d - 2) - M\nu^{-1} - \phi\nu^{-1}]\hat{G}_{2l}^{M,1}(sq_1, \ldots, sp_{M+1}) = 0 \;, \tag{B5}$$

and the precursor of the Callan-Symanzik equation,

$$\frac{\partial}{\partial m^2} G_{2l}^{M,N}(q_1, \ldots, q_{2l}; p_1, \ldots, p_M; k_1, \ldots, k_N) = -\frac{G_{2l}^{M+1,N}(q_1, \ldots, q_{2l}; p_1, \ldots, p_M, 0; k_1, \ldots, k_N)}{[m^4 G_2^{1,0}(00;0)]} \;. \tag{B6}$$

The notation $\hat{G}$ in Eq. (B5) stands for

$$\hat{G}_{2l}^{M,1}(q_1, \ldots, q_{2l}; p_1, \ldots, p_M; p_0) = G_{2l}^{M,1}(q_1, \ldots, q_{2l}; p_1, \ldots, p_M; p_0) - KG_{2l}^{M+1,0}(q_1, \ldots, q_{2l}; p_0, \ldots, p_M) \tag{B7}$$

where

$$K \equiv G_2^{0,1}(00;;0)/G_2^{1,0}(00;0) \tag{B8}$$

is chosen so that $\hat{G}_2^{0,1}(00;;0) = 0$. $\mathfrak{D}$ stands for the differential operator,

$$\mathfrak{D} = (\eta - 2)m^2 \frac{\partial}{\partial m^2} - s\frac{\partial}{\partial s} + w_4 \Lambda^{1+\epsilon}\frac{\partial}{\partial u_4} + w_6 \Lambda^{2\epsilon}\frac{\partial}{\partial u_6} \tag{B9}$$

and the quantities $\eta$, $\nu$, $\phi$, $w_4$, and $w_6$ are dimensionless functions of $\Lambda^{-1-\epsilon}u_4$, $\Lambda^{-2\epsilon}u_6$, and $m/\Lambda$ (with Greek letters assigned to agree with the standard notation for tricritical exponents). These functions may be determined order by order in $\epsilon$ by imposing Eqs. (B4) and (B5) on the perturbation series. A discussion of the facts which lead to (B4) and (B5) can be found in Ref. 17. As for why it is $\hat{G}$ and not $G$ that appears in Eq. (B5), we note that

$$\left(\frac{\partial}{\partial u_4}\right)_m G_{2l}^{M,0}(q_1, \ldots, q_{2l}; p_1, \ldots, p_M) = \hat{G}_{2l}^{M,1}(q_1, \ldots, q_{2l}; p_1, \ldots, p_M; 0)$$

[compare Eq. (5) of the text]. The right-hand sides of Eqs. (B4) and (B5) are not strictly zero but can be neglected[17] whenever quantities of the order $\Lambda^{-1}(\ln\Lambda)^{\text{power}}$ are negligible in the renormalized theory, i.e., when $m$ and all momenta are small compared to $\Lambda$, order by order in $\epsilon$. Equation (B6) is easily obtained by differentiating $G_{2l}^{M,N}$ with respect to $u_2$ at fixed $u_4$ and $u_6$, and applying the definition of $m^2$ as the inverse susceptibility; $m^{-2} = G_2^{0,0}(p=0)$.

In order to obtain equations for $\Delta_1$ and $\Delta_2$, we define

$$\hat{\Delta}_1(k) = \Delta_1(k) + K\Delta_2(k) \;, \tag{B10}$$

with $K$ as in Eq. (B8). Then Eq. (B1) can be rewritten

$$G_{2l+2}^{M,0}(k, -k, \cdots) = \hat{\Delta}_1(k) G_{2l}^{M+1,0}(\cdots) + \Delta_2(k)\hat{G}_{2l}^{M,1}(\cdots) + \mathcal{O}(k^{-5}) \;. \tag{B11}$$

Now applying (B4), (B5), and (B6) to (B11) and using $\hat{G}_2^{0,1}(00;;0) = 0$, we obtain

$$(\mathfrak{D} + \eta + \nu^{-1} - 2 - d)\hat{\Delta}_1(sk) = (\mathfrak{D} + \eta + \phi\nu^{-1} - 2 - d)\Delta_2(sk) = \mathcal{O}(k^{-5}) \;, \tag{B12}$$

$$\left[\frac{\partial}{\partial m^2}\hat{\Delta}_1(k)\right] - K'\Delta_2(k) = \frac{\partial}{\partial m^2}\Delta_2(k) = \mathcal{O}(k^{-5}) \;, \tag{B13}$$

where $K' = \partial K/\partial m^2$. We now have enough information to determine the asymptotic expansion of $\hat{\mathfrak{D}}$ through terms of order $k^{-4}$.

The tricritical point is identified with the fixed point of Eqs. (B4) and (B5) located at $u_4 = \hat{u}_4$, $u_6 = \hat{u}_6$ such that

$$w_4(\hat{u}_4, \hat{u}_6, m=0) = w_6(\hat{u}_4, \hat{u}_6, m=0) = 0 \;. \tag{B14}$$

It is easy to show that if Eq. (B14) is satisfied, then $\hat{u}_4 = u_{4t}$ and $\hat{u}_6 = u_6{}^*$ where $u_{4t}$ and $u_6{}^*$ are determined by applying the scaling law to the perturbation series as discussed in Sec. II of the text. For when (B14) holds, we can replace $\mathfrak{D}$ by $\mathfrak{D}_t = (\eta_t - 2)m^2(\partial/\partial m^2) - s(\partial/\partial s)$ and use $\eta_t = \eta(\hat{u}_4, \hat{u}_6, m=0)$, etc. in (B4) and (B5). This turns (B4) and (B5) into the scaling laws

$$[\mathfrak{D}_t + d + l(\eta_t - d - 2) - M\nu_t^{-1}] \times G_2^{M,0}(sq_1, \ldots, sp_M) = 0 \;, \tag{B15}$$

$$[\mathfrak{D}_t + d + l(\eta_t - d - 2) - M\nu_t^{-1} - \phi_t\nu_t^{-1}] \times \hat{G}_{2l}^{M,1}(sq_1, \ldots, sp_{M+1}) = 0 \;. \tag{B16}$$

Now Eqs. (B15) and (B16) suffice to determine $\eta_t$, $\nu_t$, $\phi_t$, and $u_{4t} = \hat{u}_4$, and $u_6{}^* = \hat{u}_6$ order by order in $\epsilon$, as discussed in Sec. II. Thus (B15) and (B16) are the appropriate equations for the path $u_4 = u_{4t}$, $u_6 = u_6{}^*$. Replacing (B4) and (B16) eliminates corrections to the leading powers depending on deviations $u_4 - u_{4t}$, $u_6 - u_6{}^*$ and nonleading powers of $m/\Lambda$, while retaining corrections in different powers of momenta. Now, (B12) and (B13) can be used to give

$$\Delta_2(k) = c_2 k^{\eta_t + (\phi_t/\nu_t) - 2 - d} + \Theta(k^{-5}) \ ,$$

$$\hat{\Delta}_1(k) = c_1 k^{\eta_t + (1/\nu_t) - 2 - d} \times [1 + c_1'(km^{2/(\eta_t - 2)})^{(\phi_t - 1)/\nu_t}] + \Theta(k^{-5}) \ . \tag{B17}$$

Applying (B1) with (B17) to $G_2^{2,0}(k, -k; 00)$ and integrating (B6) twice gives

$$G_2^{0,0}(k, -k) \sim k^{\nu_t - 2}(A + Bx^{(\alpha_t - 1)/\nu_t} + Cx^{\phi_t/\nu_t - d} + Dx^{-1/\nu_t}) + \Theta(k^{-5}) \ ; \tag{B18}$$

where we have set $x = km^{2/(\eta_t - 2)}$. The definition Eq. (12) of the text of $\tilde{D}$ and the identification $m^2 \sim g_1^{\gamma_t}$ turn Eq. (B18) into the expansion of $\hat{D}$ given in Eq. (15).

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