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Paramagnon theories below three dimensions

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We study the magnetic behavior of an itinerant, strongly interacting fermion system for varying values of the coupling constant and for arbitrary dimensionalities $1 < d \leq 3$. We show that, at zero temperature, due to quantum effects, the critical exponents are the mean-field ones below two dimensions as well as above. We then focus on the critical behavior of the two-dimensional case, which, within some assumptions that are discussed, exhibits a ferromagnetic rather than antiferromagnetic character. We also compute the effects of critical fluctuations on the low-temperature properties of such systems.

I. INTRODUCTION

As is well known for three-dimensional systems,[1] strong short-range repulsions between itinerant fermions give rise to a magnetic instability when the magnitude of the interaction reaches some critical value at a given temperature. Close to the magnetic transition, the system experiences critical spin fluctuations (the so-called "paramagnons").[2] The simplest description using zero-range interaction and the random-phase approximation (RPA) has proved, in the past, to be quite useful and to give reasonable fits to the experimental data for three-dimensional nearly magnetic systems with magnetic instability, if any, occuring at $T_c = 0$ K. However, for $T_c \neq 0$, the mean-field result does not give correct answers and mode-mode coupling effects[3] must be taken into account. It has been established previously[4] that a direct analogy exists between the system of itinerant interacting fermions at fixed temperature ($T_c = 0$ K), when the dimensionless interaction $\bar{I}$ increases and reaches the critical value 1 and, on the other hand, the system of localized spins on a lattice, with nearest-neighbor interaction J fixed, when the temperature decreases so that the dimensionless temperature $\tau(\sim T/J)$, reaches the value 1, as well. However, in the first case, the continuum of frequencies ω at 0 K, plays a crucial role and brings in important differences as compared to static critical phenomena ususally encountered in the second case. In both cases, however, the study of the Landau-Ginzburg-Wilson (LGW) equation describing the system of interacting critical fluctuations, using group-renormalization techniques, allows to compute critical exponents.[5,6] In that framework, it has been shown that, for $d = 3$,[5,6] and also for $2 < d < 3$,[7] paramagnon theories with zero T_c are "renormalizable," in the sense that the critical exponents are the mean-field ones, and the renormalized value of the coupling constant may then be computed in perturbation. This was due to the quantum effects mentioned above; the presence of ω increases the "effective dimensionality" (which notion was first introduced in Ref. 5), below which mean field breaks down, (i.e., $d_{\text{eff}} = d + 3$, to be compared with 4). On the other hand, itinerant ferromagnets with finite T_c identify with usual *static* critical phenomena problems, and the corresponding critical exponents are thus different from the mean-field ones: indeed, at finite temperature the Matsubara frequencies of the spin fluctuations $\omega_\nu = 2\pi\nu T$ are discrete as compared to the frequency continuum at zero temperature; the most important frequency for criticality is the first one, $\nu = 0$; therefore, everything goes as if the frequency is not present. For very small, although finite T_c, however, a crossover region shows up[6] between mean-field exponents and non-mean-field ones.

At that stage, it appeared interesting to examine what happens for itinerant systems below, and at, two dimensions. Indeed, while the mean-field static susceptibility $\chi(q, \omega = 0) = \chi^0(q, 0)/[1 - I\chi^0(q, 0)]$, diverges for long wavelengths ($q \rightarrow 0$) above two dimensions, it diverges for all q between 0 and $2k_F$ (k_F is the Fermi momentum) at $d = 2$, which thus appears to be a pathological case; finally, χ is meaningless at $d = 1$, since the free-fermion static susceptibility itself $\chi^0(q, 0)$ diverges at $q = 2k_F$.

The main questions we want to answer in the present paper are the following: (a) Is χ^0 finite at $2k_F$ for $1 < d < 2$, or does it diverge as for $d = 1$? (b) If χ^0 is finite for all momenta, what happens when the interaction is turned on in a mean-field approximation? (c) Is the interacting-particle theory "renormalizable" (in the sense defined previously), as is the case for $d > 2$? (d) If so, could the results obtained for $1 < d < 2$ combined with those for $2 < d < 3$ shed light on the pathological case $d = 2$? (e) If not, what can one say about the case $d=2$? (f) Generally speaking, for $1<d\leq 2$, what is the role of the frequency (at $T=0$ K), as compared to its introducing, above two dimensions, an effective dimensionality larger than the real one? (g) Finally, how do the critical fluctuations affect the low-temperature properties of these itinerant systems for low dimensionalities?

In order to answer these questions, we first compute $\chi^0(q,\omega)$ and $\chi(q,\omega)=\chi^0(q,\omega)/[1-I\chi^0(q,\omega)]$ at zero temperature as well as at finite temperature and for all dimensionalities $1\leq d\leq 3$; then we examine the various questions raised above. Our main results are the following: (i) The free-particle dynamic susceptibility $\chi^0(q,\omega)_{T=0}$ remains finite whatever is q, for all d strictly larger than 1; besides, it is sharply peaked, with infinite slopes, at $q=2k_F$ and $\omega=0$, for $1<d<2$ while its maximum occurs at $q=0$ and $\omega=0$, with zero slope, for $2<d\leq 3$. (ii) the value $\chi^0(2k_F,0)/\chi^0(0,0)=(d-1)^{-1}$, for all d, i.e., $1\leq d\leq 3$; this is an exact and remarkably simple result. (iii) The expansion of $\chi^0(q,\omega)$ around $q=2k_F$, $\omega=0$, for $1<d<2$, corresponds to the behavior of a system of particles submitted to long-range forces, whereas the case $d>2$ corresponds to short-range forces. (iv) The maximum of χ^0 being finite for $1<d<2$, one can compute χ in perturbation theory (RPA), for the interacting system; and for increasing values of $\bar{I}=I/E_F$ (E_F is the characteristic energy of the free particles), the system switches, for $d<2$, from a paramagnetic but nearly antiferromagnetic behavior (cf. nearly ferromagnetic for $d>2$), to a real antiferromagnetic state exhibiting spin-density waves. (v) The almost antiferromagnetic case for $1<d<2$ is shown to be "renormalizable" as was the almost ferromagnetic one for $2<d\leq 3$.[5-7] (vi) There is, however, no "effective dimensionality" different from the real one for $d<2$: the continuum of frequencies ω at $T=0$ K is irrelevant and plays no role in contrast to what happens for $d>2$. (vii) The case $d=2$ appears as a borderline. One studies that case separately. We calculate in perturbation the self-energy correction to the two-dimensional fluctuation propagator. If, for simplicity, to render the calculation tractable, one assumes that the vertices at the crossing of two interacting fluctuations are constants or vary smoothly, then, the perturbation series introduces a curvature in the resulting $\tilde{\chi}(q)$ (including the self-energy correction), i.e., a q^2 term arises which was absent in the RPA expression of $\chi(q)$. Thus, for $d=2$, $\tilde{\chi}(q)$ would diverge for $q=0$, $\omega=0$ (ferromagnetic behavior), analogously to what happens above two dimensions. However, the assumption made on the vertices between interacting fluctuations is not obvious, as will be discussed in the text. (viii) Finally, we give expressions for the specific heat, including the effective-mass expression, and the resistivity for such systems, for $1<d\leq 3$.

Part of these results were reported in a brief communication elsewhere[9]; the present paper is the published version of the 3rd cycle thesis of one of us.[10]

II. FREE-PARTICLE DYNAMIC SUSCEPTIBILITY FOR $1\leq d\leq 3$

We calculate here the dynamic susceptibility $\chi^0(q,\omega)$ for a parabolic band of free fermions, as follows:

$$\chi^0(q,\omega)=\frac{i}{(2\pi)^{d+1}}\int G^0(\epsilon_{\vec{k}},\epsilon)\times G^0(\epsilon_{\vec{k}+\vec{q}},\epsilon+\omega)\,d^dk\,d\epsilon\ , \quad (1)$$

where $\epsilon_{\vec{k}}$ is the kinetic energy of the fermions, $\epsilon_{\vec{k}}=k^2/2$ (in atomic units). $G^0(\epsilon_{\vec{k}},\epsilon)$ is the free-particle Green's function,

$$G^0(\epsilon_{\vec{k}},\epsilon)=[\epsilon-\epsilon_{\vec{k}}+i\eta\,\mathrm{sgn}(|k|-k_F)]^{-1}\ . \quad (2)$$

The space integral $\int d^dk$ is given by the usual[11] brute force successive integrations,

$$\int d^dk=\int\cdots\int k^{d-1}dk\,(\sin\theta_1)^{d-2}d\theta_1\times(\sin\theta_2)^{d-3}d\theta_2\cdots\ . \quad (3)$$

We find closed-form formulas for $\mathrm{Re}\chi^0(q,\omega)$ and $\mathrm{Im}\chi^0(q,\omega)$ in terms of hypergeometric functions[12] as given below; we will use

$$\chi^0(0,0)=2^{2-d}\pi^{-d/2}\Gamma^{-1}(d/2)k_F^{d-2}\ , \quad (4)$$

which is the density of states at the Fermi level for two spin directions. Dividing by the number of particles per unit volume,

$$n^p=2^{2-d}k_F^d d^{-1}\pi^{-d/2}\Gamma^{-1}(d/2)\ , \quad (5)$$

one gets

$$\chi^0(0,0)/n_p=d/2E_F\ , \quad (6)$$

with $E_F=k_F^2/2$. In the above formulas, Γ denotes the gamma function.[12] Setting

$$Q_+=q+2\omega/q\ ,$$

$$Q_-=q-2\omega/q\ , \quad (7)$$

we get

$\mathrm{Re}\chi^0(q,\omega)=$

Q_- \ Q_+	$\leqslant 2k_F$	$\geqslant 2k_F$
$\leqslant 2k_F$	$\chi^0(0,0)\frac{1}{q}\left[\frac{Q_+}{2}F\left(\frac{2-d}{2},1,\frac{3}{2},\left(\frac{Q_+}{2k_F}\right)^2\right)+\frac{Q_-}{2}F\left(\frac{2-d}{2},1,\frac{3}{2},\left(\frac{Q_-}{2k_F}\right)^2\right)\right]$	$\chi^0\frac{(0,0)}{2qd}\left[Q_+\left(\frac{2k_F}{Q_+}\right)F\left(\frac{1}{2},1,\frac{d+2}{2},\left(\frac{2k_F}{Q_+}\right)^2\right)+dQ_-F\left(\frac{2-d}{2},1,\frac{3}{2},\left(\frac{Q_-}{2k_F}\right)^2\right)\right]$
$\geqslant 2k_F$	$-\frac{\chi^0(0,0)}{2qd}\left[Q_-\left(\frac{2k_F}{Q_-}\right)^2F\left(\frac{1}{2},1,\frac{d+2}{2},\left(\frac{2k_F}{Q_-}\right)^2\right)+dQ_+F\left(\frac{2-d}{2},1,\frac{3}{2},\left(\frac{Q_+}{2k_F}\right)^2\right)\right]$	$\frac{\chi^0(0,0)}{2qd}\left[Q_+\left(\frac{2k_F}{Q_+}\right)^2F\left(\frac{1}{2},1,\frac{d+2}{2},\left(\frac{2k_F}{Q_+}\right)^2\right)+Q_-\left(\frac{2k_F}{Q_-}\right)^2F\left(\frac{1}{2},1,\frac{d+2}{2},\left(\frac{2k_F}{Q_-}\right)^2\right)\right]$

(8)

$$\mathrm{Im}\chi^0(q,\omega)=\frac{\sqrt{\pi}}{d-1}\frac{\Gamma(\frac{1}{2}d)}{\Gamma\frac{1}{2}(d-1)}\chi^0(0,0)\frac{k_F}{2q}\left\{\left[1-\left(\frac{Q_-}{2k_F}\right)^2\right]^{(d-1)/2}\left[sgn\left(1-\frac{Q_-}{2k_F}\right)+sgn\left(\frac{Q_-}{2k_F}+1\right)\right]\right.$$
$$\left.-\left[1-\left(\frac{Q_+}{2k_F}\right)^2\right]^{(d-1)/2}\left[sgn\left(1+\frac{Q_+}{2k_F}\right)+sgn\left(-\frac{Q_+}{2k_F}+1\right)\right]\right\}. \tag{9}$$

$F(a,b,c,z)$ is the degenerate hypergeometric series.[12] It can be checked that $\chi^0(q,\omega)$, for fixed q, is maximum for $\omega=0$. Since we will be interested in time-persistent fluctuations, we will consider that case. Hence, we obtain

$$\chi^0(q,0)=\begin{cases}\chi^0(0,0)F\left(\frac{d-2}{2},1,\frac{3}{2},\left(\frac{q}{2k_F}\right)^2\right), & q\leqslant 2k_F\\ \frac{\chi^0(0,0)}{d-1}, & q=2k_F\\ \frac{\chi^0(0,0)}{q}\left(\frac{2k_F}{q}\right)^2F\left(\frac{1}{2},1,\frac{d+2}{2},\left(\frac{2k_F}{q}\right)^2\right), & q\geqslant 2k_F\end{cases} \tag{10}$$

Figure 1 illustrates the variation of $\chi^0(q,0)$ versus q for various values of d.

Then, expanding $\chi^0(q,\omega)$ around its maximum $\omega=0$ and the apppropriate value of q, one gets

for $2\leqslant d\leqslant 3$,
$$\chi^0(q,\omega)\simeq\chi^0(0,0)\left\{\left[1-\left(\frac{d-2}{3}\right)\left(\frac{q}{2k_F}\right)^2\right]+\sqrt{\pi}\frac{\Gamma(d/2)}{\Gamma\left(\frac{d-1}{2}\right)}\frac{\omega}{k_Fq}\right\},$$
$$\frac{q}{2k_F}<<1,\quad \frac{\omega}{k_Fq}<<1\ , \tag{11}$$

for $1<d\leqslant 2$,
$$\chi^0(q,\omega)\simeq\frac{\chi^0(0,0)}{d-1}\left\{\left[1-A^{\pm}(2-d)\left|1-\left(\frac{q}{2k_F}\right)^2\right|^{(d-1)/2}\right]+\frac{\sqrt{\pi}\Gamma(d/2)}{2\Gamma\left(\frac{d-1}{2}\right)}\frac{\omega(d-1)}{k_F^2|1-(q/2k_F)^2|^{(3-d)/2}}\right\},$$
$$\left|1-\left(\frac{q}{2k_F}\right)^2\right|<<1,\quad \frac{\omega}{k_F^2[\,|1-(q/2k_F)^2|\,]}<<1\ . \tag{12}$$

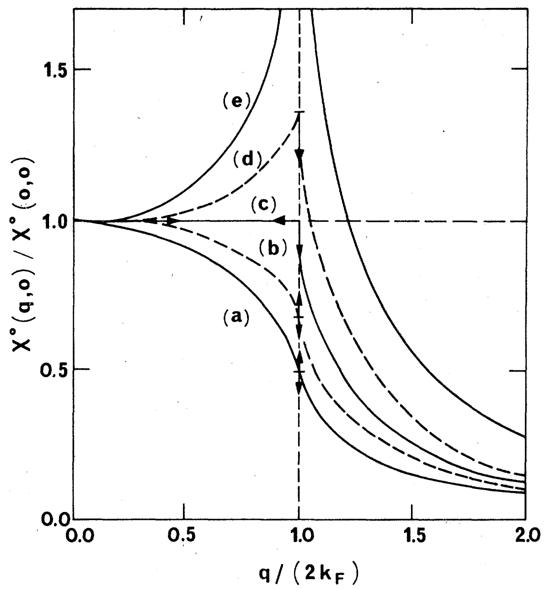

FIG. 1. $\chi^0(q,0)$ normalized vs q for different values of d: (a) $d=3$; (b) $D=2.5$; (c) $d=2$; (d) $d=1.75$; (e) $d=1$.

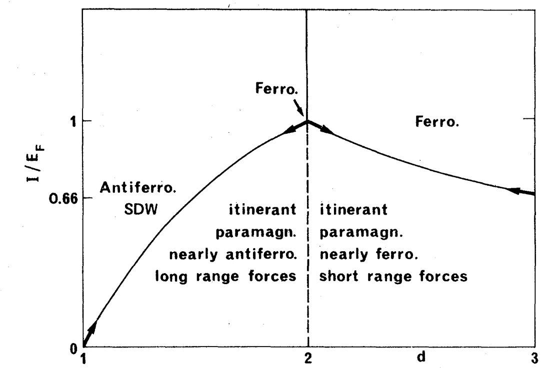

FIG. 2. Phase diagram for the itinerant interacting fermion system at $T=0$ K.

A^+ (or A^-) is a function of d corresponding to $q=2k_F^+$ (or $2k_F^-$). The main features arising from the above formulas figure already in the introduction; let us just add some precisions:

(a) for $\omega=0$, $\chi^0_{\max}$ occurs at $q=0$, for $3\geq d>2$, and at $q=2k_F$, for $2>d\geq 1$, where it is finite except at d exactly equal to 1. (b) $\chi^0(q)$ increases (or decreases) monotonically for $2>d>1$ (or $3\geq d>2$), when q increases from 0 to $2k_F$; $\chi^0(q)$ decreases monotonically towards 0 for all d, for $q>2k_F$. (c) the slope of $\chi^0(q)$ at $q=0$ is zero. (d) the slopes of $\chi^0(q=2k_F)$, on both sides of $2k_F$, are infinite for all d, except for $d=2$ where it is zero for $2k_F^-$ and infinite for $2k_F^+$. (e) for $q\to 0$, we recover, for $1<d<2$ the same expansion we had for $d>2$.[13]

III. DYNAMIC SUSCEPTIBILITY OF THE INTERACTING FERMIONS EVALUATED IN MEAN FIELD FOR $1<d\leq 3$ AND $T=0$ K.

Since $\chi^0(q,\omega)$ remains finite for all d, except $d=1$, one is then allowed to formally write down the RPA series in presence of the spin-spin interaction I

$$\chi(q,\omega)=\chi^0(q,\omega)/[1-I\chi^0(q,\omega)] \tag{13}$$

and since, as was noticed in Sec. II, $\max[\chi^0(q,\omega)]$ occurs at $q=2k_F$, $\omega=0$, for $d<2$, compared to $q=0$, $\omega=0$, for $d>2$, the system will undergo an antiferromagnetic instability in the former case, when I is large enough, and a ferromagnetic one, in the latter case.

In the RPA, the magnetic instability will occur when

$$1-I\chi^0_{\max}=0\ , \tag{14}$$

with

$$\chi^0_{\max}=\frac{d}{2E_F}\left[\theta(d-2)+\frac{1}{d-1}\theta(2-d)\right]\ . \tag{15}$$

Therefore, the instability will take place when I reaches the critical values

$$\begin{aligned}&I_c=2E_F/d\ ,\\ &\text{for } 2<d\leq 3,\ \text{ ferromagnetic instability}\ ,\\ &I_c=2(d-1)/d,\\ &\text{for } 1<d<2,\ \text{ antiferromagnetic instability}\ .\end{aligned} \tag{16}$$

Note that for $d=1$ there is an antiferromagnetic divergence of χ^0 even for $I=0$. The instability near one dimension occurs for very small values of I while between two and three dimensions, it occurs only for I/E_F rather close to 1 (see Fig. 2).

Finally, for $d=2$, $\chi^0(q,0)$ is constant, equal to $\chi^0(0,0)$ for all q such that $0\leq q\leq 2k_F$ (all the q dependence is multiplied by a coefficient proportional to $d-2$, which identically vanishes for $d=2$). $\chi^0_{\max}$ occurs for all these q values; then the Stoner criterion for apparition of magnetism (14) appears meaningless, and the $d=2$ case looks pathological. Note that such a flatness for $\chi^0(q,0)$ vs q would probably disappear in presence of band structure effects in metals; however, the problem still remains for (normal) liquid ^{3}He films, for instance, for which case our model would directly apply.

With the help of (10), we note that the expansion of $\chi^0(q,\omega)$ around its maximum yields

$$1 - I\chi^0(q,\omega) \sim \begin{cases} 1 - I(\chi^0_{\max})_{d>2}\left[1 - \lambda_1 q^2 + ip_1\dfrac{\omega}{q}\right], \\ d > 2 \\ 1 - I(\chi^0_{\max})_{d<2}\left[1 - \lambda_2 q'^{\sigma} + ip_2\dfrac{\omega q'^{\sigma}}{q'}\right], \\ d < 2 \end{cases} \tag{17}$$

where $\lambda_{1,2}, p_{1,2}$ are constants $q' = |2k_F - q|$, and

$$\sigma = (d-1)/2, \quad d < 2 \tag{18}$$

So, the power σ of q' for $d < 2$, is smaller than 2. Therefore, the cases $d < 2$ would correspond to "long-range forces",[14] while those for $d > 2$ correspond to "short-range forces" ($\sim R^{-(d+\sigma)}$), since q appears to the power 2. So here too, $d = 2$ appears as a borderline. All that is illustrated on Fig. 2. The line between the ferromagnetic and the antiferromagnetic parts of the diagram is most likely of first order, while the lines separating the ordered phases from the paramagnetic ones are of second order. The crossing point of these three lines at $d = 2$ could be thought of as an infinite-order Lifshitz point[15] by analogy with problems when q does not appear with a power less than 4; here no power of q appears at all.

Another remark is that the long-range interactions corresponding to (18) below two dimensions, when $d \to 1$, tend to behave like the unscreened Coulomb interection considered by Overhauser,[16] so that the spin-density waves (SDW) of our Fig. 2 are reasonable to expect in this model.

In Sec. IV, we will study the validity of the above mean-field results by constructing the LGW Lagrangian of interacting fluctuations, with each inverse fluctuation propagator given by (16) or (11) in mean field.

IV. LGW LAGRANGIAN FOR THE SYSTEM OF INTERACTING FLUCTUATIONS FOR 1 DIMENSION

Using standard techniques, the LGW Lagrangian describing the system of interacting fluctuations reads

$$H(\psi) = \frac{1}{2}\sum_{q,\omega_n} v_2(q,i\omega_n)|\psi(q,i\omega_n)|^2 + \frac{1}{4\beta N}\sum_{q_i,\omega_i} v_4(q_1,\ldots,q_4,i\omega_1,\ldots,i\omega_4) \times \psi(q_1,i\omega_1)\cdots\psi(q_4,i\omega_4)\cdots \tag{18}$$

$\beta = T^{-1}$ and the v_i's are the vertices interactions. We will consider the expression (18) only up to the quartic term in the perturbation series expansion; this will be justified, *a posteriori*, as it was in Refs. 4–6, since we will show that the paramagnon theories are "renormalizable" for $d > 2$ and $d < 2$. v_2, as usual, is proportional to the inverse mean field fluctuation propagator [Eq. (13)]; as usual too, we will suppose that v_4 varies only smoothly and is reasonably well accounted for by a constant; this is not obvious for the two-dimensional case and we will come back to that point later on.

In order to investigate the validity of mean field, we will study the Ginsburg criterion developed in Ref. 5, i.e., we study the condition under which the following expression converges:

$$\begin{aligned} V(r) &\propto \int\cdots\int \frac{d^dq\,d\omega}{(r_0 + q^2 + \omega/q)^2} \sim r_0^{(d-4)/2+3/2}, \quad d > 2, \\ V(r) &\propto \int\cdots\int \frac{d^dq\,d\omega}{(r_0 + q'^{\sigma} + \omega/(q')^{1-\sigma})^2} \sim r_0^{2(3-d)/(d-1)}, \quad d < 2, \end{aligned} \tag{19}$$

dropping all constant coefficients q' and σ are given in the previous section, and

$$r_0 = 1 - I\chi^0_{\max} \to 0, \quad \text{as well as} \quad q_1 q', \frac{\omega}{q}, \frac{\omega}{q'} \quad . \tag{20}$$

The first formula of (19) was already found and commented in Refs. 5 and 7: $V(r)$ converges for $2 < d \leq 3$, mean field is valid in the sense that the critical exponents will be the Gaussian ones. At this stage, we wish to mention a comment by Noziéres[17] who raised the question of whether, in the ω/q term, which, actually reads $\omega/v_F q$, the bare fermion mass m should appear or the effective mass m^*, containing the first (or more?) paramagnon correction (for instance, $m^* \sim \ln r_0$ in three dimensions[2]). The answer is not obvious since in the Wilson theory, the first term in the denominator of (19) should be the bare r_0 and not an effective one already dressed by fluctuations; in any case and supposing one should indeed, use m^* instead of m (m^* is given in Appendix C, $\sim r_0^{(d-3)/2}$), one would find that $V(r) \sim r_0$, independently of d above two dimensions, so that $V(r)$ still converges for vanishing r_0, but the notion of "effective dimensionality" has disappeared.

As far as the second formula of (19) is concerned, below two dimensions it obviously converges, even in absence of ω. One thus expects classical behavior, for this kind of "long-range forces" case; this result is reasonable since we know from Ref. 14 that a classical regime holds for $2\sigma - d \lesssim 0$, and that condition is indeed satisfied with σ given by (18). We have redrawn Sak's phase diagram in a (d, σ) plane, in our Fig. 3, and we have indicated on the figure where our paramagnon case below two dimensions falls for better clarity. Therefore, the critical exponents will be the classical one, $\eta = 2 - \sigma$, $\gamma = 1$, $\nu = \sigma^{-1}$.

We point out here an amusing remark concerning the case $d \rightarrow 1$: in the discrete Ising model of Migdal,[18] the inverse critical temperature T_c^{-1}, together with the critical exponent ν, diverge like $(d-1)^{-1}$; we can compare that with our $\chi^0_{\max}$ and our ν diverging like $(d-1)^{-1}$ too; however, the comparison stops beyond that, since our critical exponents are of "classical" type as shown above, while Migdal's ones are not.

For $d = 2$, we must remember that for $\omega = 0$, all the momenta dependence has a coefficient proportional to $(d-2)$ and thus identically vanishes when $d = 2$. The frequency and the momentum do not appear separately and no scaling as the ones used to get formulas (19) are possible as noted previously.[5,7,19] We must then study the $d = 2$ case from another point of view.

V. $d = 2$ CASE

Let us first note that, for $d = 2$, we have

$$\nu_2 = r_0 + \omega/k_F q, \quad \text{if } q \lesssim 2k_F \ ,$$

$$\nu_2 = r_0 + (q^2 - 4k_F^2)^{1/2} + \omega/k_F^2 (q^2 - 4k_F^2)^{1/2} \qquad (21)$$

$$\text{if } q \gtrsim 2k_F \ .$$

In the present section, we study in perturbation the self-energy corrections to the bare paramagnon propagator $[\nu_2(q)]^{-1}$, i.e., we calculate the first few diagrams given on Fig. 4 where each wavy line is a bare fluctuation propagator and each dot the bare vertex at the crossing point between two fluctuations; that bare vertex is just one close fermion loop and its value is given by a sum over frequency and momentum of the product of four fermion propagators as shown on Fig. 5 and is identified with ν_4.

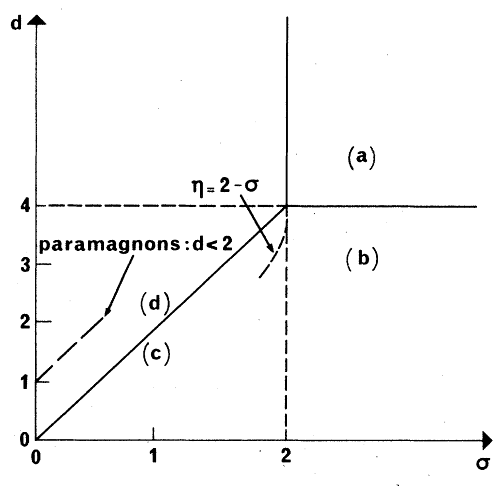

FIG. 3. Summary of the nature of the critical exponents given by Sak (Ref. 14), (where our paramagnon results below two dimensions were added) for short- and long-range forces $1/R^{d+\sigma}$: (a) short-range forces, Gaussian exponents; (b) short-range forces, Wilson-type exponents; (c) long-range forces, Wilson-type exponents; (d) long-range forces, Gaussian exponents, to which case belong the paramagnons below two dimensions.

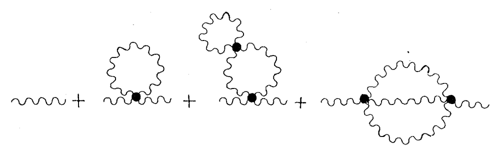

FIG. 4. The first few diagrams in the self-energy correction to the one-paramagnon propagator.

Here comes a crucial hypothesis for the evaluation of this self-energy: usually, for static critical phenomena, in ferromagnetic type of systems, $\nu_4(q_1, q_2, q_3, q_4) \sim \nu_4(0,0,0,0) \sim$ const.; similarly, in three-dimension paramagnon problems with ferromagnetic tendancy one can show easily[20] that $\nu_4(q_1\omega_1, q_2\omega_2, q_3\omega_3, q_4\omega_4) \sim \nu_4(0,0,0,0,0,0,0,0) = $ const. But since we do not know whether, in the two-dimensional case, it is the ferromagnetic or the antiferromagnetic limit which is the correct one, we also do not know whether we can approximate ν_4 by $\nu_4(0,0,0,0,0,0,0,0)$ which is equal to 0, for $d = 2$, or its value, still at zero frequency but for momenta close to $2k_F$, which may diverge.[21] So we should, in principle, in order to evaluate the various diagrams of Fig. 4, keep the momenta and frequency dependence of the ν_4's as it appears in the general expression (18). But then, the calculation becomes extremely difficult; it could be handled only if one could provide a simplified model form for the momenta dependence of ν_4 (see, for instance the form studied for another problem in Ref. 22), but since we failed to compute the general expression for ν_4, we have been unable to approximate it, also. Therefore, we suppose that the singularities of the ν_4's, when integrated over, will not sensibly affect the result one would obtain if one supposes they act as constants. But this has not been proved and appears extremely difficult to clarify.

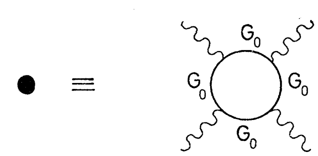

FIG. 5. Vertex between two interacting paramagnons; the wiggly lines are paramagnons, the closed loop is formed by four fermion lines.

Within the above assumption, the second and third terms of Fig. 4 will just multiply the bare fluctuation by a constant; only the last term will matter since it is the only one which may possibly bring in an extra momentum and frequency dependence. For simplicity, we study that diagram for zero frequency. The calculations are given in Appendix E; we are thus able to show that the corrected $\tilde{\nu}_2(q)$, taking into account the four first self-energy corrections to $\nu_2(q)$, may be written

$$\tilde{\nu}_2(q) = r_0' + \alpha q^2, \quad \text{for} \quad q \leq 2k_F \quad , \tag{22}$$

where α is a positive constant. Under those conditions, the corresponding $\chi^0(q)$ will exhibit a maximum at $q=0$, i.e., a ferromagnetic type of behavior. The Ginsburg criterion reads then as in the cases $d>2$, and thus the theory is renormalizable as well. But we insist that we have used an approximation to arrive at this result, approximation which remains to be justified: indeed, if it would happen that the singularities in the momenta dependence of ν_4 could render the overall sums of the second term in (18) divergent, it would then be forbidden to write down the series (18) where all terms would then diverge and the entire mystery would remain for the $d=2$ problem.

VI. LOW-TEMPERATURE PROPERTIES OF PARAMAGNON SYSTEMS FOR $1 \leq d \leq 3$

a. Critical properties. As we noted previously, at finite temperature the paramagnon frequencies are discrete, the most important one is the frequency zero, and thus, we recover a static critical phenomena type of problem, for $2 < d \leq 3$, for which mean field does not give the correct critical exponents. However, for $1 < d < 2$, as we showed, the frequency does not play any role, so the "classical" behavior holds at finite temperature as it did at zero degree.

b. Specific heat. We generalize here, for arbitrary dimensionalities a calculation which has been derived long ago, at $d=3$.[2] The specific heat is given by

$$C = -T\frac{\partial^2 \Delta F}{\partial T^2} \quad , \tag{23}$$

where ΔF is the excess free energy of the fermion gas, due to paramagnons. Calculations are given in Appendix C. As usual,[2] ΔF is given by

$$\Delta F \propto T \sum_{q,\omega} [\ln(1 - I\chi^0) + I\chi^0] \quad , \tag{24}$$

or

$$\Delta F \propto \int_0^I dI\, \Big(T \sum_{q,\omega} [-\chi(q,\omega) + \chi^0(q,\omega)]\Big) \quad , \tag{25}$$

but the sums over the momenta concern vectors k in d dimensions. A low-temperature expansion, $T/r_0 < 1$ [with r_0, given by (20), $\rightarrow 0$], yields the results of Table I. One must note that the critical fluctuations for $d=2$ are stronger than for $d \neq 2$, so that the crossover lines between a Gaussian type of regime and a Wilson type (see, for instance, Ref. 6) occur at much lower temperature in the former case. As far as m^* is concerned between two and three dimensions, its limit when $d \rightarrow 3$ switches to $\ln r_0$ as it should; note that while m^* appears to be independent of d below two dimensions, the following term in the specific heat strongly depends on d.

TABLE I. Fermion effective-mass correction and the following term in the temperature expansion of the specific heat for various dimensionalities. For $d=2$, r_0' is the value of r_0 renormalized by the paramagnon self-energy correction of Fig. 4.

d	$\left(\frac{m^*}{m}-1\right) = \lim_{T\to 0}\left(\frac{C}{T}\right)$	$C - \left(\frac{m^*}{m}-1\right)T$
3	$-\frac{9}{2}\ln\left(\frac{r_0}{3}\right)$	$\left(\frac{T}{r_0}\right)^3 \ln\left(\frac{T}{r_0}\right)$
between 2 and 3	$\frac{9}{2}\left(\frac{r_0}{3}\right)^{(d-3)/2}$	$\left(\frac{T}{r_0}\right)^d$
2	$\frac{9}{2}\left(\frac{r_0'}{3}\right)^{1/2}$	$\left(\frac{T}{r_0'}\right)^2$
between 1 and 2	$-\ln r_0$	$\left(\frac{T}{r_0}\right)^{2/(3-d)}$

c. Resistivity. We assume, for simplicity, that we consider the scattering of conduction electrons by paramagnons formed by the electrons of the same band, and we evaluate the resulting expression for the resistivity, at low temperature, $T/r_0 < 1$. We calculate

$$\rho \propto \int d\omega \int dx\, x^d (1-x^2)^{(d-3)/2} \times \frac{\omega}{T} n(\omega)[1+n(\omega)]\, \mathrm{Im}\chi(x,\omega) \quad , \tag{26}$$

where $n(\omega) = (e^{\omega/T}-1)^{-1}$ and $x = q/2k_F$. Calculations presented in Appendix D yield the results of Table II.

d. Discussion of the results. We would like to make a few remarks on the results that we have obtained below two dimensions in comparison to what is known elsewhere, first for the $d=1$ case and second for the nearly antiferromagnetic three-dimensional case.

For $d \rightarrow 1$, we have verified that, at finite temperature, $\chi^0 \sim \ln T$,[10] is well known.[23] On the other hand, we have also verified that the resistivity $\rho \sim e^{-\omega_0/T}$ (ω_0 being a frequency cutoff), at low temperature and switches to $\rho \sim T/\omega_0$ at higher temperature, in agreement to what is known.[24]

TABLE II. Low-temperature contribution to the resistivity, for various dimensionalities; the enhancement of the T^2 varies, depending whether one retains (column I), or not (column II), the first momentum contribution in the expansion of the real part of the susceptibility around its maximum.

	ρ	
d	I	II
3	$(T/r_0)^2$	$T^2/r_0^{1/2}$
between 3 and 2	$(T/r_0)^2$	$T^2/r_0^{(4-d)/2}$
2	$(T/r_0)^2$	T^2/r_0
between 2 and 1	$(T/r_0)^2$	$T^2/r_0^{2/(d-1)}$

For the $1 < d < 2$, nearly antiferromagnetic cases, we would like to compare our results with the known results for the $d=3$, nearly antiferromagnetic case, although it seems difficult to elaborate on the result of the comparison. It has been shown that, in general,[25] the linear T term in the specific heat of the three-dimensional AF case is not enhanced, but for particular shapes of the Fermi surface,[26] the coefficient of the T term diverges like $\ln r_0$ (finite cylinder for the Fermi surface) or like r_0^{-1} (infinite cylinder). On the other hand, the resistivity[27] for nearly excitonic three-dimensional systems was calculated to behave like T^2, but with a coefficient diverging like r_0^{-1}. Obviously, the nearly AF character of these cases and ours allows the comparison but the difference of dimensionality, most likely, explains the different degrees of divergences in the coefficients.

We also note[10] that when one crosses the antiferromagnetic line for $1 < d < 2$, in Fig. 2, the paramagnon pseudomodes becomes antiferromagnetic magnons.

Finally, if it can be proven in the future, that the ferromagnetic character of the $d=2$ case is confirmed, independently of the assumption used in the present paper, that would allow to render more quantitative the explanation proposed elsewhere[28] for the measured ferromagnetic (Curie-Weiss) susceptibility of liquid ^{3}He near a surface (while bulk liquid ^{3}He is paramagnetic).

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APPENDIX A: EVALUATION OF Re$\chi^0(q,\omega)$ AND Im$\chi^0(q,\omega)$ AT ZERO TEMPERATURE

$$\mathrm{Re}\chi^0(q,\omega) = -\int \frac{d^dk}{(2\pi)^d}(f_{\vec{k}} - f_{\vec{k}+\vec{q}})\frac{1}{\omega - (\epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}})} , \tag{A1}$$

where $f_{\vec{k}}$ is the Fermi distribution of occupied states,

$$f_{\vec{k}} = 0 \ \text{ if } \ \epsilon_{\vec{k}} > \epsilon_F ,$$

$$f_{\vec{k}} = 1 \ \text{ if } \ \epsilon_{\vec{k}} < \epsilon_F ,$$

and

$$\epsilon_{\vec{k}} = \tfrac{1}{2}k^2 ,$$

$$\mathrm{Im}\chi^0(q,\omega) = \pi \int \frac{d^dk}{(2\pi)^d}(f_{\vec{k}} - f_{\vec{k}+\vec{q}})\,\delta(\omega - (\epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}})) . \tag{A2}$$

(a) Re$\chi^0(q,\omega)$ can be written in the form

$$-\int_0^{k_F} \frac{k^{d-1}dk}{(2\pi)^d} K_{d-1} \int (\sin\theta)^{d-2} \times \left[\frac{1}{kq\cos\theta - \frac{1}{2}q^2 - \omega} + \frac{1}{kq\cos\theta + \frac{1}{2}q^2 - \omega}\right] d\theta , \tag{A3}$$

where K_{d-1} is the area of the $(d-1)$-dimensional sphere with $Q_+ = 2\omega/q + q$ and $Q_- = 2\omega/q - q$. The problem reduces to evaluating

$$(1/2q)[\chi^0(Q_+,0) + \chi^0(Q_-,0)] . \tag{A4}$$

Now

$$\chi^0(q,0) = -\frac{1}{(2\pi)^d}\int_0^{k_F} k^{d-1} I(k,q) , \tag{A5}$$

$$I(k,q) = \int (\sin\theta)^d \left[\frac{q^2}{2(kq)^2 - \frac{1}{2}q^4}\left(\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}\right) + \frac{2(kq)^2}{2(kq)^2 - \frac{1}{2}q^4}\left(\frac{1}{kq\cos\theta - \frac{1}{2}q^2} - \frac{1}{kq\cos\theta + \frac{1}{2}q^2}\right)\right] d\theta . \tag{A6}$$

With the help of

$$\int_0^\pi \frac{\sin^d\theta}{1\pm\cos\theta}\,d\theta=\sqrt{\pi}\frac{\Gamma(\frac{1}{2}(d-1))}{\Gamma(\frac{1}{2}d)} \tag{A7}$$

[$\Gamma(x)$ is the Euler function],

$$\int_0^\pi \frac{(\sin\theta)^d}{\cos\theta+q/2k}\,d\theta=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(d+1))}{\Gamma(\frac{1}{2}(d+2))}\left(\frac{2k}{q}\right)^d\left[\left(\frac{q}{2k}\right)^2-1\right]^{(d-1)/2}F\left[\frac{d+1}{2},\frac{d}{2},\frac{d+2}{2},\left(\frac{2k}{q}\right)^2\right] \text{ if } \frac{q}{2k}\geq 1$$

$$=\pi\left[1-\left(\frac{q}{2k}\right)^2\right]^{(d-1)/2}\cot\left(\frac{d+1}{2}\right)-2^{d-1}B\left(\frac{d-1}{2},\frac{d+1}{2}\right)F\left[1-d;1;\frac{3-d}{2};\frac{1}{2}\left(1+\frac{q}{2k}\right)\right] \text{ if } \frac{q}{2k}<1\ , \tag{A8}$$

where $F(a,b,c,;z)$ is the hypergeometric series.[12]

$$\chi^0(q,0)=-\frac{1}{(2\pi)^d}\left(\int_0^{q/2}k^{d-1}I(k,q\geq 2k)\,dk+\int_{q/2}^{k_F}k^{d-1}I(k,q\leq 2k)\,dk\right) \text{ for } q\leq 2k_F\ , \tag{A9}$$

so that

$$\chi^0(q,0)=\chi^0(0,0)F\left[\frac{2-d}{2};1;\frac{3}{2};\left(\frac{q}{2k_F}\right)^2\right] \text{ for } q\leq 2k_F\ , \tag{A10}$$

$$\chi^0(0,0)=\frac{2^{2-d}\pi^{-d/2}}{\Gamma(d/2)}k_F^{d-2}\ . \tag{A11}$$

Furthermore,

$$F\left(\frac{2-d}{2}\ ,1;\frac{3}{2},1\right)=\frac{1}{d-1} \text{ for } q\geq 2k_F\ , \tag{A12}$$

$$\chi^0(q,0)=-\frac{1}{(2\pi)^d}\int_0^{k_F}k^{d-1}I(k,q\geq 2k)\,dk\ , \tag{A13}$$

$$\chi^0(q,0)=\frac{1}{d}\left(\frac{2k_F}{q}\right)^2\chi^0(0,0)F\left[\frac{1}{2};1;\frac{d+2}{2},\left(\frac{2k_F}{q}\right)^2\right]\ . \tag{A14}$$

Finally, according to how Q_+ and Q_- compare with $2k_F$, one gets the results given in the text, formulas (8).

(b) $\mathrm{Im}\chi^0(q,\omega)$ can be written in the form

$$\frac{\chi^0(0,0)}{2}\frac{\Gamma(d/2)}{\Gamma((d-1)/2)}\frac{1}{\sqrt{\pi}}k_F^{2-d}\int_{-\infty}^{+\infty}dt\int_0^{k_F}k^{d-1}\,dk\int_{-1}^{+1}(1-u^2)^{(d-3)/2}(e^{i(Q_-q/2-kqu)t}-e^{i(Q_+q/2-kqu)t})\,du\ . \tag{A15}$$

Recalling that

$$I_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\sum_{p=0}^{\infty}\sum_{k=0}^{2p}(-1)^p\frac{\alpha_p}{(p)!}\frac{(2p)!}{(k)!}\frac{1}{z^{2p-k+1}}[(-1)^{2p-k}e^z-e^{-z}]\ , \tag{A16}$$

with

$$\alpha_p=(\nu-\tfrac{1}{2})(\nu-\tfrac{3}{2})\cdots(\nu+\tfrac{1}{2}-p)\ , \tag{A17}$$

$$\int_0^1 v^{d/2}I_\nu(zv)\,dv=\frac{1}{z}I_{\nu+1}(z)\ , \tag{A18}$$

$$\sum_{p=0}^{\infty}\frac{\alpha_p}{(p)!}\left[-\left(\frac{x}{2k_F}\right)^2\right]^p=\left[1-\left(\frac{x}{2k_F}\right)^2\right]^{\nu-1/2}\ , \tag{A19}$$

one gets

$$\mathrm{Im}\chi^0(q,\omega)=\frac{\sqrt{\pi}}{d-1}\frac{\Gamma(d/2)}{\Gamma((d-1)/2)}\left(\frac{k_F}{2q}\right)\chi(0,0)\left\{\left[1-\left(\frac{Q_-}{2k_F}\right)^2\right]^{(d-1)/2}\left[\mathrm{sgn}\left(\frac{Q_-}{2k_F}+1\right)-\mathrm{sgn}\left(\frac{Q_-}{2k_F}-1\right)\right]\right.$$
$$\left.-\left[1-\left(\frac{Q_+}{2k_F}\right)^2\right]^{(d-1)/2}\left[\mathrm{sgn}\left(\frac{Q_+}{2k_F}+1\right)-\mathrm{sgn}\left(\frac{Q_+}{2k_F}-1\right)\right]\right\} . \quad \text{(A20)}$$

APPENDIX B: TWO-DIMENSIONAL SUSCEPTIBILITY AT FINITE TEMPERATURE

We wish to evaluate

$$\chi^0(q,0,T)=\frac{1}{2\pi^2}\int_0^\infty k\,dk\,\frac{1}{e^{(k^2-k_F^2)/2T}+1}\times\int_0^{2\pi}d\phi\,\frac{1}{q^2-4k^2\cos^2\phi} \quad \text{(B1)}$$

for $T/E_F \ll 1$ and $q^2/T \ll 1$,

$$\chi^0(q,0,T)=\frac{1}{2\pi^2}\int_0^{q/2}\frac{2\pi k\,dk}{q(q^2-4k^2)^{1/2}}\frac{1}{e^{(k^2-k_F^2)/2T}+1} , \quad \text{(B2)}$$

$$\chi^0(q,0,T)\simeq\frac{1}{8\pi}\int_0^1\frac{du}{\sqrt{u}}\left[1-e^{-(k_F^2-q^2/4)/2T}-\frac{q^2}{8T}u\right]$$
$$\simeq\frac{1}{4\pi}\left[1-e^{-k_F^2/2T}-\frac{1}{12}\frac{q^2}{T}e^{-k_F^2/2T}\right] . \quad \text{(B3)}$$

APPENDIX C: CONTRIBUTION OF PARAMAGNONS TO THE SPECIFIC HEAT

1. Effective-mass calculations

(a) $2\le d\le 3$. One has

$$\lim_{T\to 0}(C/T)\sim m^*/m-1 , \quad \text{(C1)}$$

where

$$C=-T\frac{\partial^2\Delta F}{\partial T^2} , \quad \text{(C2)}$$

with

$$\Delta F=-\frac{6V}{(2\pi)^{d+1}}\sqrt{\pi}\frac{\Gamma(d/2)}{\Gamma((d-1)/2)}\int_0^\infty\frac{\omega\,d\omega}{e^{\omega/T}-1}\times\int_0^{\bar I}d\lambda\int_0^{2k_F}\frac{k^{d-2}\,dk}{[1-\lambda+\lambda(k/2k_F)^2]^2} \quad \text{(C3)}$$

and

$$\bar I=I\chi^0(0,0) \quad \text{(C4)}$$

when $1-\bar I\to 0$; with the help of

$$\int_0^\infty\frac{x\,dx}{e^x-1}=\zeta(2) \quad \text{(C5)}$$

[$\zeta(x)$ is the Rieman function[12]],

$$\int_0^{2k_F}\frac{k^{d-2}\,dk}{r_0+\bar I(k/2k_F)^2}=\frac{1}{2}(2k_F\bar I)^{d-1}\times B\left(\frac{d-1}{2},\frac{4-d}{2}\right)r_0^{(d-3)/2} , \quad \text{(C6)}$$

$$r_0=1-\bar I ,$$

$$\Delta F=-3V\zeta(2)(2k_F\bar I)^{d-1}B\left(\frac{d-1}{2},\frac{4-d}{2}\right)\times\frac{\sqrt{\pi}}{(2\pi)^{d+1}}\frac{\Gamma(d/2)}{\Gamma((d-1)/2)}\bar I r_0^{(d-3)/2}T^2 , \quad \text{(C7)}$$

whence

$$m^*/m-1\propto r_0^{(d-3)/2} . \quad \text{(C8)}$$

(b) Straightforward calculations yield

$$\Delta F=-\frac{12V}{(2\pi)^{d+1}}\frac{1}{(d-1)^2}\sqrt{\pi}\,\zeta(2)\times\frac{\Gamma(d/2)\Gamma((2-d)/2)}{\Gamma(\frac{3}{2})\Gamma((1-d)/2)\Gamma((d-1)/2)}\times\frac{1}{(2k_F)^{3-d}}\ln\left(\frac{r_0+\bar I A}{r_0}\right)T^2 , \quad \text{(C9)}$$

where $r_0=1-I\chi^0(2k_F,0)$,

$$A=-\frac{\Gamma(\frac{3}{2})\Gamma((1-d)/2)}{\Gamma((2-d)/2)} , \quad \text{(C10)}$$

whence

$$m^*/m-1\propto -\ln r_0 . \quad \text{(C11)}$$

2. Following term of the expansion of ΔF in powers of T

(a) $2\le d\le 3$. We obtain it from

$$\Delta F=-\frac{6VKd}{(2\pi)^{d+1}}\int_0^{\bar I}\frac{d\lambda}{\lambda(1-\lambda)}\int_0^\infty\frac{d\omega}{e^{\omega/T}-1}\times\int_{\omega/v_F}^\infty k^{d-1}\frac{\alpha\omega k}{k^2+(\alpha\omega)^2}dk , \quad \text{(C12)}$$

$$\alpha = \frac{\sqrt{\pi}\Gamma(d/2)}{\Gamma((d-1)/2)} \frac{\lambda}{1-\lambda} . \tag{C13}$$

After some manipulations

$$\Delta F = -\frac{6KdV}{(2\pi)^{d+1}} \frac{\Gamma(d+1)\zeta(d+1)\Gamma((d+1)/2)\Gamma(d/2)\Gamma((3-d)/2)}{2\Gamma((d+2)/2)(d-1)} \left[\frac{\Gamma((d-1)/2)}{\sqrt{\pi}\Gamma(d/2)}\right]^d T^{d+1} \left[\frac{\bar{I}}{1-\bar{I}}\right]^d , \tag{C14}$$

whence

$$C \sim (T/r_0)^d . \tag{C15}$$

(b) $1 < d < 2$.

$$\Delta F = -\frac{6Vkd}{(2\pi)^{d+1}} (2k_F)^{d-1} \int_0^{\bar{I}} \frac{d\lambda}{\lambda(1-\lambda)} \int_0^\infty \frac{d\omega}{e^{\omega/T}-1} \int_{\omega/v_F}^\infty dk \frac{\beta\omega k k^{-(d+3)/2}}{k^{3-d}+(\beta\omega)^2} , \tag{C16}$$

where

$$\beta = \sqrt{\pi}\frac{\Gamma(d/2)}{\Gamma((d-1)/2)} \frac{1}{(2k_F)^{d-1}} \frac{\lambda}{1-\lambda} , \quad \bar{I} = I\chi^0 \ (2k_F, 0) , \tag{C17}$$

$$\Delta F = -\frac{6Vkd}{(2\pi)^{d+1}} \frac{1}{(2k_F)^{(d-1)^2/(3-d)}} \left[\frac{\sqrt{\pi}\Gamma(d/2)}{\Gamma((d-1)/2)}\right]^{2/(3-d)} \tag{C18}$$

$$\Gamma\left[\frac{7-3d}{2(3-d)}\right]\Gamma\left[\frac{1}{3-d}\right]\Gamma\left[\frac{5-d}{2(3-d)}\right]\Gamma\left[\frac{5-d}{3-d}\right]\zeta\left[\frac{5-d}{3-d}\right]\left[\frac{\bar{I}}{1-\bar{I}}\right]^{2/(3-d)} T^{(5-d)/(3-d)} ,$$

whence

$$C \sim \left[\frac{T}{r_0}\right]^{2/(3-d)} . \tag{C19}$$

APPENDIX D: CONTRIBUTION OF PARAMAGNONS TO THE RESISTIVITY

The expression for the resistivity reads

$$\rho = \rho_0 \frac{m}{2E_F} \int_0^\infty d\omega \int_0^\lambda dx\, x^d (1-x^2)^{(d-3)/2} \frac{\omega}{T} \times n(\omega)[1+n(\omega)]\, \mathrm{Im}\chi(x,\omega) , \tag{D1}$$

where $\lambda \neq 1$ is a constant and $x = q/2k_F$.

(a) For $2 \leqslant d \leqslant 3$,

$$\mathrm{Im}\chi(x,\omega) \simeq B\frac{\omega}{x}\frac{1}{(r_0+x^2)+B^2\omega^2/x^2}, \quad \text{for } \frac{\omega}{E_F x} \ll 1 ,$$

$$B = 2\sqrt{\pi}\frac{\Gamma(d/2)}{\Gamma((d-1)/2)}\frac{\bar{I}}{(2k_F)^2}$$

$$= 0 \ \text{otherwise} , \tag{D2}$$

so that

$$\rho = \rho_0 \frac{m}{4E_F} T \int_0^\lambda dx\, x^{d+1}(1-x^2)^{(d-3)/2} \times \frac{\partial}{\partial T}\left[\ln z - \frac{1}{2z} - \psi(z)\right] , \tag{D3}$$

with

$$z = (r_0 + x^2)x/2\pi BT . \tag{D4}$$

Thus, for

$$\frac{T}{r_0} \ll 1 ,$$

$$\rho = \rho_0 \frac{Bm}{48E_F}(2\pi B)^2 \frac{\Gamma((2d-1)/2)\Gamma((4-d)/2)}{\Gamma((d-1)/2)} \times \frac{T^2}{r_0^{(4-d)/2}} . \tag{D5}$$

N.B. If we had dropped the x^2 in (r_0+x^2) in $\mathrm{Im}\chi(x,\omega)$ we would have obtained only $(T/r_0)^2$

(b) For $1 < d < 2$,

$$\mathrm{Im}\chi(x,\omega) = \begin{cases} \dfrac{B'\omega}{x(1-x^2)^{(3-d)/2}} & \text{if } \dfrac{\omega}{E_F(1-x)} \ll 1 , \\ 0 & \text{otherwise} \end{cases} \tag{D6}$$

$$B' = \bar{I}\sqrt{\pi}\frac{\Gamma(d/2)}{\Gamma((d-1)/2)}\frac{1}{(2k_F)^2} , \tag{D7}$$

so that

$$\rho = \rho_0 \frac{Bm}{4E_F} T \int_0^\lambda dx\, x^{d+1} \frac{\partial}{\partial T}\left[\ln z - \frac{1}{2z} - \psi(z)\right] , \tag{D8}$$

$$z = x[r_0 + (1-x^2)^{(d-1)/2}](1-x^2)^{(3-d)/2}1/2\pi BT \quad . \tag{D9}$$

Thus, for $T/r_0 \ll 1$,

$$\rho = \rho_0 \frac{Bm}{48E_F}(2\pi B)^2 \frac{1}{2(d-2)} \times \frac{\Gamma(2/(d-1))\Gamma(2(d-2)/(d-1))}{\lambda^{4(d-3)/(d-1)}} \frac{T^2}{r_0^{2/(d-1)}} \quad . \tag{D10}$$

The same N.B. as before applies here.

APPENDIX E: EVALUATION OF A SECOND-ORDER GRAPH FOR $d=2$ AT ZERO TEMPERATURE

We calculate here the fourth diagram of Fig. 4

$$I(\vec{k}) = \int\int d^2q_1\, d^2q_2\, v_2^{-1}(\vec{q}_1)\, v_2^{-1}(\vec{q}_2)\, v_2^{-1}[\vec{k}-(\vec{q}_1+\vec{q}_2)]\, v_4^{-2}(\vec{q}_1,\vec{q}_2,\vec{k}) \quad , \tag{E1}$$

$|\vec{k}| \leq 2k_F$.

We assume that the singularities of v_4 are weak as compared to those of v_2^{-1} so that we set $u_0 = v_4$,

$$v_2(q) = r_0 \text{ if } q \leq 2k_F \quad ,$$

$$v_2(q) = r_0 + (q^2 - 4k_F^2)^{1/2} \text{ if } q \geq 2k_F \quad , \tag{E2}$$

$$I(\vec{k}) = \int d^2x\, e^{-i\vec{k}\cdot\vec{x}}[v_2^{-1}(\vec{x})]^3 \quad , \tag{E3}$$

$$v_2^{-1}(\vec{x}) = \int d^2q\, v_2^{-1}(\vec{q})\, e^{+i(\vec{q}\cdot\vec{x})} \quad , \tag{E4}$$

$$= \frac{2\pi}{r_0}\int_0^{2k_F} J_0(kx)k\, dk + 2\pi\int_{2k_F}^{\infty} \frac{J_0(kx)k\, dk}{r_0 + (q^2 - 4k_F^2)^{1/2}} \quad . \tag{E5}$$

We get an upper limit for $v_2^{-1}(\vec{x})$ when we set $r_0 = 0$ in the second integral so that

$$v_2^{-1}(\vec{x}) = \frac{2\pi}{r_0}(2k_F)\frac{J_1(2k_Fx)}{x} + 2\pi(2k_F)\frac{\cos 2k_Fx}{x} \quad , \tag{E6}$$

$$I(\vec{q}) = \frac{4\pi k_F}{r_0}u_0^2\left[\int_0^\infty \frac{J_1^3(2k_Fx)J_0(qx)\, dx}{x^2} + 3r_0\int_0^\infty \frac{J_1^2(2k_Fx)J_0(qx)\cos 2k_Fx\, dx}{x^2} + 3r_0^2\int_0^\infty \frac{J_1(2k_Fx)J_0(qx)\cos^2(2k_Fx)\, dx}{x^2} + r_0^3\int_0^\infty \frac{\cos^3(2k_Fx)J_0(qx)\, dx}{x^2}\right] \quad , \tag{E7}$$

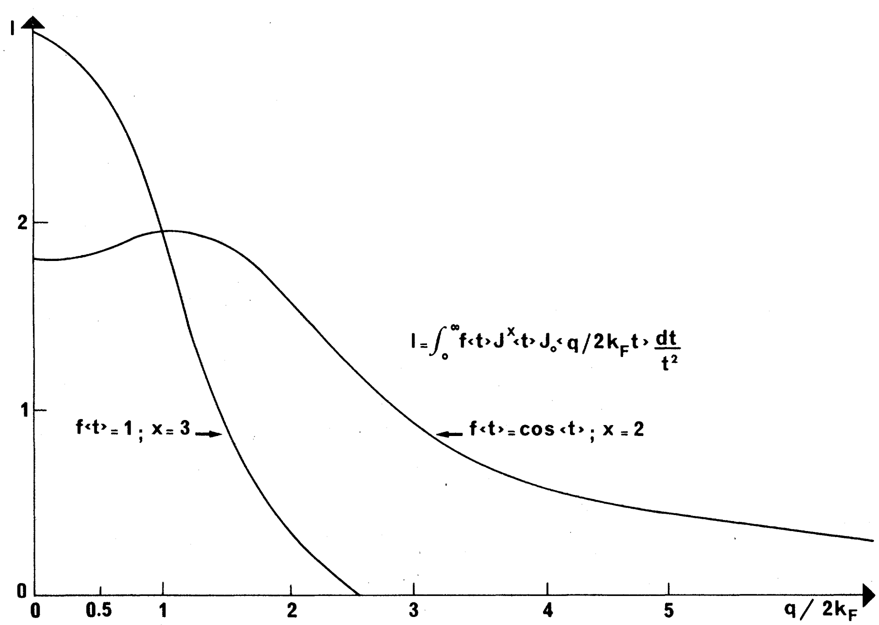

FIG. 6. Result of the numerical calculation presented in Appendix E.

$$I(\vec{q}) = \frac{4\pi k_F}{r_0} u_0^2 [(A) + (B) + (C) + (D)] \quad . \quad \text{(E8)}$$

(A) has been evaluated numerically (see Fig. 6). An expansion around its maximum for $q = 0$ yields

$$(A) = 0.2(2k_F) - \frac{\sqrt{3}}{2} \frac{q^2}{2k_F} \quad . \quad \text{(E9)}$$

(B) has also been evaluated numerically and displays a maximum for $q = 2k_F$ but with a vanishing contribution when $r_0 \rightarrow 0$. (C) and (D) have a zero value. Thus,

$$\nu_2^{-1}(\vec{q}) = r_0' + \alpha q^2 \quad \text{for} \quad q < 2k_F \quad , \quad \text{(E10)}$$

with

$$\alpha = \left(\frac{4\pi k_F}{r_0} \right)^3 \frac{\sqrt{3}}{2} \frac{u_0^2}{2k_F} \quad ,$$

and r_0' is equal to r_0 plus the constant values of the second and third Hartree diagrams of Fig. 4.

The new self-consistent susceptibility has a maximum for $q = 0$ and a mean-field theory built from it will give a ferromagnetic instability.

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