Two-dimensional Ising model near T_c: Approximation for small magnetic field

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The study of the equation of state of the two-dimensional Ising model is initiated by use of the recently calculated n-point spin-correlation functions. In the scaling region where $T \to T_c$ and $h = B|1 - T_c/T|^{-15/8}\,(H/kT_c)$ is of order 1 (B is a lattice-dependent constant) the free energy below T_c is of the form $f(T,h) - f(T,0) = \text{const}(1 - T_c/T)^2 \sum_{n=1}^{\infty}(1/n!)C_n^- h^n$ where the lattice-dependent constant is chosen so that $C_1^- = 1$. The constant C_2^- has been studied previously. In this paper we study C_3^- and C_4^- and find as a first approximation $C_3^-/3! \sim -11/72\pi^2$ and $C_4^-/4! \sim 5/189\pi + 59/162\pi^3$. These are in close agreement with the low-temperature-series values of -0.01538 and 0.0195.

I. INTRODUCTION

In 1944 Onsager[1] published his remarkable calculation of the free energy of the two-dimensional Ising model in the absence of an external magnetic field H. The extension of Onsager's calculation to the case $H \neq 0$ has remained as one of the most challenging problems in statistical mechanics.

There are at least two distinct ways in which one can attempt to extend Onsager's calculation to $H \neq 0$. Either one can find an entirely new method of solution which will produce the $H \neq 0$ free energy in one shot, or one can expand the free energy in a power series in H and compute the coefficients in terms of spin-correlation functions evaluated at $H = 0$. Since no one has made any progress with the first approach, it is the purpose of this paper to initiate the use of the recent calculation, carried out in collaboration with Tracy,[2] of n-spin correlation functions of the two-dimensional Ising model at $H = 0$ to study the equation of state by the second of these two procedures.

The energy for the two-dimensional Ising model in the presence of an external field H is

$$\mathcal{E} = -E_1 \sum_{j,k} \sigma_{j,k}\sigma_{j,k+1} - E_2 \sum_{j,k} \sigma_{j,k}\sigma_{j+1,k} - H \sum_{j,k} \sigma_{j,k}, \tag{1.1}$$

where $\sigma_{j,k} = \pm 1$, j (k) specifies the row (column) of the lattice and E_1 (E_2) is the horizontal (vertical) interaction energy. If we consider a finite lattice of $\mathfrak{M}$ rows and $\mathfrak{N}$ columns, the free energy per spin is given by

$$f(T,H) = -kT \lim_{\substack{\mathfrak{M}\to\infty \\ \to\infty}} (\mathfrak{M}\mathfrak{N})^{-1} \ln Z_{\mathfrak{M}\mathfrak{N}}, \tag{1.2}$$

where the partition function $Z_{\mathfrak{M},\mathfrak{N}}$ is

$$Z_{\mathfrak{M},\mathfrak{N}} = \sum_{\{\sigma\}} e^{-\beta\mathcal{E}} \tag{1.3}$$

and the sum is over all $\sigma_{j,k} = \pm 1$. Let the n-spin correlation function at $H = 0^+$ in the infinite lattice be

$$S_n(R_1, R_2, \ldots, R_n) = \lim_{H\to 0^+} \lim_{\substack{\mathfrak{M}\to\infty \\ \mathfrak{N}\to\infty}} \langle \sigma_{R_1}\sigma_{R_2}\cdots\sigma_{R_n}\rangle. \tag{1.4}$$

For the translationally invariant lattice (1.1), S_n is translationally invariant and, in particular,

$$S_1(R_1) = \mathfrak{M} = [1 - (\sinh 2\beta E_1 \sinh 2\beta E_2)^{-2}]^{1/8} \tag{1.5}$$

is the spontaneous magnetization.[3]

The correlation function S_n has the property that if the n variables $R_1, \ldots, R_n$ are divided into two sets $R_1, \ldots R_l$ and $R_{l+1}, \ldots, R_n$ and if the distance between points in each set is fixed while the separation between the two sets becomes infinite, then

$$S_n(R_1, \ldots, R_n) \sim S_l(R_1, \ldots, R_l) S_{n-l}(R_{l+1}, \ldots, R_n). \tag{1.6}$$

We define the connected part of the n-point correlation S_n^c as $S_n(R_1, \ldots, R_n)$ with all the limiting behavior (1.6) subtracted out so that $S_n^c(R_1, \ldots, R_n)$ goes to zero when the separation between any points or sets of points becomes large. Explicitly

$$S_2^c(R_1, R_2) = S_2(R_1, R_2) - \mathfrak{M}^2, \tag{1.7a}$$

$$S_3^c(R_1, R_2, R_3) = S_3(R_1, R_2, R_3) - \mathfrak{M}[S_2(R_1, R_2) + S_2(R_1, R_3) + S_2(R_2, R_3)] + 2\mathfrak{M}^3, \tag{1.7b}$$

and

$$S_4^c(1,2,3,4)=S_4(1,2,3,4)-\mathfrak{M}[S_3(1,2,3)+S_3(1,2,4)+S_3(1,3,4)+S_3(2,3,4)]$$
$$-[S_2(1,2)S_2(3,4)+S_2(1,3)S_2(2,4)+S_2(1,4)S_2(2,3)]$$
$$+2\mathfrak{M}^2[S_2(1,2)+S_2(1,3)+S_2(1,4)+S_2(2,3)+S_2(2,4)+S_2(3,4)]-6\mathfrak{M}^4\,, \tag{1.7c}$$

where in the last line the variable R is suppressed. It is well known that the free energy is given in terms of $S_n^c(R_1, R_2, \ldots, R_n)$ as

$$f(T,H)=f(T,0)-kT\sum_{n=1}^{\infty}\frac{f^{(n)}(T)}{n!}\left(\frac{H}{kT}\right)^n, \tag{1.8}$$

where

$$f^{(n)}(T)=\sum_{R_1,\ldots,R_{n-1}} S_n^c(0,R_1,\ldots,R_{n-1})\,. \tag{1.9}$$

It has recently been shown[2] that for $T<T_c$

$$S_n(R_1,\ldots,R_n)=\mathfrak{M}^n e^{F_n(R_1,\ldots,R_n)}, \tag{1.10}$$

where

$$F_n(R_1,\ldots,R_n)=\sum_{k=2}^{\infty}F_n^{(k)}(R_1,\ldots,R_n), \tag{1.11}$$

with

$$F_n^{(k)}(R_1,\ldots,R_n)=-\frac{1}{2k}[2z_2(1-z_1^2)]^k\int_{-\pi}^{\pi}\frac{d\phi_1}{2\pi}\cdots\frac{d\phi_{2k}}{2\pi}$$
$$\times\prod_{l=1}^{k}\frac{1}{\Delta(\phi_{2l-1},\phi_{2l})}\,\frac{\sin\frac{1}{2}(\phi_{2l-1}+\phi_{2l+1})}{\sin\frac{1}{2}(\phi_{2l}-\phi_{2l+2}+i\epsilon)}\;\mathrm{Tr}[A(1,2)\ldots A(2k-1,2k)]\,. \tag{1.12a}$$

Here $A(2l-1,2l)$ is an $n\times n$ matrix with elements,

$$A(2l-1,2l)\,|_{jj}=0\,, \tag{1.12b}$$

$$A(2l-l,2l)\,|_{jk}=\mathrm{sgn}(M_{jk})\times\exp(-iM_{jk}\phi_{2l-1}-iN_{jk}\phi_{2l}) \tag{1.12c}$$

$\phi_{2k+1}\equiv\phi_1,\quad \phi_{2k+2}\equiv\phi_2\,,$

$R_\alpha=(M_\alpha,N_\alpha)$ (in a row, column notation),

$$M_{\alpha\beta}=M_\alpha-M_\beta,\quad N_{\alpha\beta}=N_\alpha-N_\beta\,, \tag{1.12d}$$

$$\Delta(\phi_1,\phi_2)=(1+z_1^2)(1+z_2^2)-2z_2(1-z_1^2)\cos\phi_1-2z_1(1-z_2^2)\cos\phi_2\,, \tag{1.12e}$$

$z_i=\tanh\beta E_i\quad(i=1,2)\,,$

$\mathrm{sgn}\,x=+1$ if $x>0$, -1 if $x<0$, either ±1 if $x=0$,

and the limit $\epsilon\to0^+$ is understood.

Moreover, it was shown that if $T\to T_c^-$ $(z_1z_2+z_1+z_2-1)$ and all $|R_\alpha-R_\beta|^2=M_{\alpha\beta}^2+N_{\alpha\beta}^2\to\infty$ such that

$$m_{\alpha\beta}=M_{\alpha\beta}[z_2(1-z_1^2)]^{-1/2}\,|z_1z_2+z_1+z_2-1| \tag{1.13a}$$

and

$$n_{\alpha\beta}=N_{\alpha\beta}[z_1(1-z_2^2)]^{-1/2}\,|z_1z_2+z_1+z_2-1| \tag{1.13b}$$

are fixed for all α,β then [calling $r_\alpha=(m_\alpha,\ n_\alpha)$]

$$s_n(r_1,\ldots,r_n)=\lim\mathfrak{M}^{-n}S_n(R_1,\ldots,R_n)=e^{f_n} \tag{1.14}$$

exists with

$$f_n(r_1,\ldots,r_n)=\sum_{k=1}^{\infty}f_n^{(k)}(r_1,\ldots,r_n), \tag{1.15}$$

where

$$f_n^{(k)}(r_1,\ldots,r_n)=-\frac{1}{2k}(2\pi^2)^{-k}\int_{-\infty}^{\infty}dx_1\ldots dx_k\,dy_1\ldots dy_k\prod_{l=1}^{k}\left((1+x_l^2+y_l^2)^{-1}\frac{y_l+y_{l+1}}{x_l-x_{l+1}+i\epsilon}\right)\mathrm{Tr}[a(1)a(2)\ldots a(k)]\,, \tag{1.16}$$

and $a(l)$ is an $n\times n$ matrix with elements

$$a(l)\big|_{jj}=0,\quad a(l)\big|_{jk}=\operatorname{sgn}(m_{jk})\exp(-im_{jk}y_l-in_{jk}x_l)\,. \tag{1.17}$$

In this paper, we will initiate the study of $f(T,H)$ near T_c for $T<T_c$ by using the expression (1.14) in (1.8) and (1.9). We can then write (1.9) for $T\sim T_c$ as

$$f_-^{(n)}(T)\sim \mathfrak{M}^n\,\frac{[z_1z_2(1-z_1^2)(1-z_2^2)]^{(n-1)/2}}{|z_1z_2+z_1+z_2-1|^{2(n-1)}}\int_{-\infty}^{\infty}\prod_{l=1}^{n-1}dm_l\prod_{l=1}^{n-1}dn_l s_n^c(0,r_1,\ldots,r_{n-1})\,. \tag{1.18}$$

Furthermore, as $T\to T_c$

$$\mathfrak{M}^n\left(\frac{[z_1z_2(1-z_1^2)(1-z_2^2)]^{1/2}}{(z_1z_2+z_1+z_2-1)^2}\right)^{n-1}\sim\left(\frac{T_c}{T}-1\right)^{2-(15/8)n}A^{1/8}B^{n-1}\,, \tag{1.19}$$

where

$$A=4(z_{1c}+z_{2c})[\beta_cE_1(1-z_{2c})^{-1}+\beta_cE_2(1-z_{1c})^{-1}]\,, \tag{1.20a}$$

$$B=\frac{[4(z_{1c}+z_{2c})]^{1/8}}{8z_{1c}z_{2c}[\beta_cE_1(1-z_{2c})^{-1}+\beta_cE_2(1-z_{1c})^{-1}]^{15/8}}\,, \tag{1.20b}$$

$$\beta_c=1/kT_c\,,$$

and

$$z_{1c}z_{2c}+z_{1c}+z_{2c}-1=0\,. \tag{1.21}$$

Thus, if we write as $T\to T_c^-$

$$f_-^{(n)}(T)\sim C_n^-(T_c/T-1)^{2-(15/8)n}A^{1/8}B^{n-1} \tag{1.22}$$

the numbers C_n^- will be independent of both temperature and the lattice-dependent constants E_1 and E_2 and are given as $C_1^-=1$ and

$$C_n^-=\int_{-\infty}^{\infty}\prod_{l=1}^{n-1}dm_l\,dn_l\,s_n^c(0,r_1,\ldots,r_{n-1}) \tag{1.23}$$

for $n\geq 2$. Then, defining a scaled magnetic field as

$$h=(H/kT_c)(T_c/T-1)^{-15/8}B\,, \tag{1.24}$$

we may rewrite (1.8) in the region where $T\sim T_c$ and h is fixed as

$$f(T,h)-f(T,0)\sim-8kT_cz_{1c}z_{2c}[\beta_cE_1(1-z_{2c})^{-1}+\beta_cE_2(1-z_{1c})^{-1}]^2\times\left(\frac{T_c}{T}-1\right)^2\sum_{n=1}^{\infty}\frac{C_n^-}{n!}h^n\,. \tag{1.25}$$

In this paper, we restrict our consideration to C_2^-, C_3^-, and C_4^-. Then, using (1.7) and (1.14) in (1.23) we have

$$C_2^-=\int_{-\infty}^{\infty}d^2r\,(e^{f_2(0,r)}-1)\,, \tag{1.26a}$$

$$C_3^-=\int_{-\infty}^{\infty}d^2r_1\,d^2r_2\,(e^{f_3(0,r_1,r_2)}-e^{f_2(0,r_1)}-e^{f_2(0,r_2)}-e^{f_2(r_1,r_2)}+2)\,, \tag{1.26b}$$

and

$$C_4^-=\int_{-\infty}^{\infty}d^2r_1d^2r_2d^2r_3[e^{f_4(0123)}-(e^{f_3(012)}+e^{f_3(013)}+e^{f_3(023)}+e^{f_3(123)})-(e^{f_2(01)+f_2(23)}+e^{f_2(02)+f_2(13)}+e^{f_2(03)+f_2(12)})+2(e^{f_2(01)}+e^{f_2(02)}+e^{f_2(03)}+e^{f_2(12)}+e^{f_2(13)}+e^{f_2(23)})-6]\,. \tag{1.26c}$$

To proceed further we use the representation of f_n as an infinite series and expand the exponentials. In this paper, we study C_2^-, C_3^-, and C_4^- in the approximation of keeping the first connected term $f_n^{c(n)}$ and the related connected terms $f_n^{c(k)}$ $k>n$ which are required to produce a globally rotationally invariant function. In f_2 the term $f_2^{(2)}$ by itself is globally rotationally invariant. However, in a previous publication[4] we saw that $f_3^{(3)}$ was not globally rotationally invariant but that the combination $f_3^{(3)}+f_3^{c(4)}$ does have this property. In Sec. III we will study this question for f_4 and find that $f_4^{c(4)}+f_4^{c(5)}+f_4^{c(6)}$ is the smallest globally rotationally invariant set. Therefore, our leading approximation is

$$f_2(0,r)\sim f_2^{(2)}(0,r), \qquad (1.27a)$$

$$f_3(0,r_1,r_2)\sim f_2(0,r_1)+f_2(0,r_2)+f_2(r_1,r_2)+[f_3^{(3)}(0,r_1,r_2)+f_3^{c(4)}(0,r_1,r_2)], \qquad (1.27b)$$

and

$$f_4(0123)\sim f_2(01)+f_2(02)+f_2(03)+f_2(12)+f_2(13)+f_2(23)+f_3^c(012)+f_3^c(013)+f_3^c(023)+f_3^c(123)+[f_4^{c(4)}(0123)+f_4^{c(5)}(0123)+f_4^{c(6)}(0123)]. \qquad (1.27c)$$

Thus, using (1.27) in (1.26) we find

$$C_2^-\sim I_2^{-(2)}=\int_{-\infty}^{\infty} d^2r f_2^{(2)}(0,r) \qquad (1.28)$$

and

$$C_3^-\sim\int_{-\infty}^{\infty} d^2r_1\, d^2r_2\{f_2(0,r_1)f_2(r_1,r_2)+f_2(0,r_1)f_2(0,r_2)+f_2(0,r_3)f_2(r_2r_3)+[f_3^{(3)}(0,r_1,r_2)+f_3^{c(4)}(0,r_1,r_2)]\}, \qquad (1.29)$$

which, using (1.27a) becomes

$$C_3^-\sim 3(I_2^{-(2)})^2+I_3^{-(4)}, \qquad (1.30)$$

where

$$I_3^{-(4)}=\int_{-\infty}^{\infty} d^2r_1\, d^2r_2[f_3^{(3)}(0,r_1,r_2)+f_3^{c(4)}(0,r_1,r_2)]. \qquad (1.31)$$

Finally,

$$C_4^-\sim 12\int_{-\infty}^{\infty} d^2r_1 f_2^{(2)}(0,r_1)\int_{-\infty}^{\infty} d^2r_2\, d^2r_3[f_3^{(3)}(0,r_2,r_3)+f_3^{c(4)}(0,r_2,r_3)]+16\int_{-\infty}^{\infty} d^2r_1\, d^2r_2\, d^2r_3 f_2^{(2)}(0,r_1)\, f_2^{(2)}(r_1,r_2)f_2^{(2)}(r_2,r_3)+\int_{-\infty}^{\infty} d^2r_1\, d^2r_2\, d^2r_3(f_4^{c(4)}+f_4^{c(5)}+f_4^{c(6)}), \qquad (1.32)$$

which may be written as

$$C_4^-\sim 16(I_2^{-(2)})^3+12I_3^{-(4)}I_2^{-(2)}+I_4^{-(6)}, \qquad (1.33)$$

where

$$I_4^{-(6)}=\int_{-\infty}^{\infty} d^2r_1\, d^2r_2\, d^2r_3\,(f_4^{c(4)}+f_4^{c(5)}+f_4^{c(6)}). \qquad (1.34)$$

In obtaining (1.32) from (1.26) and (1.27) the further approximation is made that in products $\Pi_i f_{n_i}^{(k_i)}$ only terms where $\Sigma_i k_i\leq 6$ are retained.

The integral for $I_2^{-(2)}$ has been evaluated previously[5] in the study of the magnetic susceptibility. A simple evaluation is given here in the Appendix. We find

$$I_2^{-(2)}=1/6\pi\sim 0.053\,051\,647\,70. \qquad (1.35)$$

The integral $I_3^{-(4)}$ is evaluated in Sec. II where we show that

$$I_3^{-(4)}=-\pi^{-2} \qquad (1.36)$$

so that from (1.30)

$$C_3^-/3!\sim -11/72\pi^2\sim -0.015\,479\,625\,2. \qquad (1.37)$$

Finally $I_4^{-(6)}$ is evaluated in Sec. IV where we show that

$$(1/4!)I_4^{-(6)}=\tfrac{4}{9}\pi^{-3}+\tfrac{5}{189}\pi^{-1}, \qquad (1.38)$$

so that from (1.33)

$$C_4^-/4!\sim\tfrac{5}{189}\pi^{-1}+\tfrac{59}{162}\pi^{-3}\sim 0.020\,166\,825\,66. \qquad (1.39)$$

The approximation in these calculations is systematic, in the sense that by retaining further terms in the expansion of f_n a more accurate expansion may be obtained. However, because there is no *a priori* small parameter, it may be questioned how numerically good approximations (1.37) and (1.39) are. One partial answer to this is that for C_2^-, the integral (1.26a) has been carried out numerically[5] with the result that

$$C_2^-=0.053\,102\,5\,893\,\cdots. \qquad (1.40)$$

The close (0.1%) agreement between the C_2^- and the approximation $I_2^{-(2)}$ encourages us to believe that (1.37) and (1.39) are also close approximations to C_3^- and C_4^-.

The numbers C_n^- for $n=1,2,3,4$, and 5 have been studied by means of low-temperature series by Essam and Hunter.[6] From their calculation of $f^{(n)}(T)$ on the square ($E_1=E_2$) lattice for $T\to T_c^-$, we find the lattice-independent C_n^- of Table I. We

TABLE I. Values of $C_n^-/n!$

n	Series expansion of Essam and Hunter	Leading approximation from this paper
1	1 (exact)	1 (exact)
2	$0.026\,70 \pm 0.000\,12$	$\frac{1}{12\pi} \sim 0.026\,526$
3	$-0.015\,38 \pm 0.000\,16$	$-\frac{11}{72\pi^2} \sim -0.015\,480$
4	0.0195 ± 0.0004	$\frac{5}{189\pi} + \frac{59}{162\pi^3} \sim 0.020\,167$
5	-0.048 ± 0.006	

also give these the error ϵ quoted by Essam and Hunter. Note that C_3^- of (1.37) agrees with C_3^- of Table I to within the given error and C_4^- of (1.39) agrees to $1\frac{1}{2}$ the given error. Furthermore, by (1.40), since for $n=2$ our leading approximation is more accurate than the numerical value of Essam and Hunter, it is perhaps not unreasonable to expect the same to be true for $n=3$ and 4.

II. EVALUATION OF $I_3^{-(4)}$

To evaluate $I_3^{-(4)}$ of (1.31), we consider first the region where the vertical coordinates m_1 and m_2 satisfy

$$0<m_1<m_2 . \tag{2.1}$$

In the trace in the integrand of (1.16) there is one distinct term (of weight 6) for $k=3$ and 3 distinct connected terms (of weight 4) for $k=4$. These are graphically represented in Fig. 1. We saw previously[4] that these four diagrams taken together are rotationally invariant across the lines $m_1=m_2$ or $m_1=0$. We further saw that $f_3^{(3)}$ could be very conveniently combined with the term of $f_4^{c(4)}$ represented by Fig. 1(b) by changing the signs of one of the $i\epsilon$ in (1.16). The integrals of diagrams 1(c) and 1(d) over the region (2.1) are equal by symmetry consideration. Thus, if we take into account that the contributions from the five other regions similar to (3.1) will give identical contributions we have

$$I_3^{-(4)} = 6(Z_1 + 2Z_2), \tag{2.2}$$

where after integrating (1.16) over the y variables

$$\begin{aligned} Z_1 = -\tfrac{1}{2}(2\pi)^{-4} \int_0^\infty dm_2 \int_0^{m_2} dm_1 \int_{-\infty}^\infty dn_1\, dn_2 \int_{-\infty}^\infty dx_1\, dx_2\, dx_3\, dx_4 (1+x_1^2)^{-1/2}(1+x_2^2)^{-1/2}(1+x_3^2)^{-1/2}(1+x_4^2)^{-1/2} \\ \times \left(\frac{-(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}}{x_1-x_2}\right)\left(\frac{(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}}{x_2-x_3+i\epsilon}\right) \\ \times \left(\frac{(1+x_3^2)^{1/2}-(1+x_4^2)^{1/2}}{x_3-x_4}\right)\left(\frac{-(1+x_4^2)^{1/2}-(1+x_1^2)^{1/2}}{x_4-x_1-i\epsilon}\right) \\ \times \exp\{-m_1[(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}]-(m_2-m_1) \\ \times [(1+x_3^2)^{1/2}+(1+x_4^2)^{1/2}] - i n_1 (x_1-x_2) - i(n_1-n_2)(x_3-x_4)\} . \end{aligned} \tag{2.3a}$$

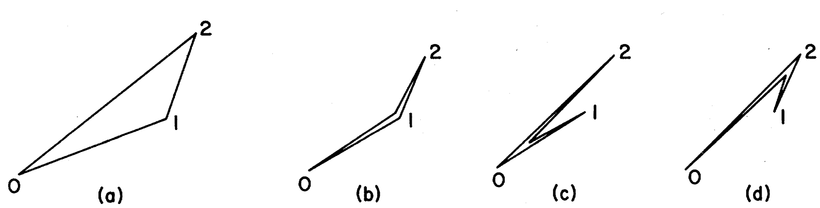

FIG. 1. Four unlabeled connected graphs of three and four lines that contribute to f_3 in the region $0<m_1<m_2$.

and

$$Z_2=-\tfrac{1}{2}(2\pi)^{-4}\int_0^\infty dm_2\int_0^{m_2} dm_1\int_{-\infty}^\infty dn_1\,dn_2\int_{-\infty}^\infty dx_1\,dx_2\,dx_3\,dx_4(1+x_1^2)^{-1/2}(1+x_2^2)^{-1/2}(1+x_3^2)^{-1/2}(1+x_4^2)^{-1/2}$$
$$\times\left(\frac{(1+x_1^2)^{1/2}-(1+x_2^2)^{1/2}}{x_1-x_2}\right)\left(\frac{-(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}}{x_2-x_3}\right)$$
$$\times\left(\frac{(1+x_3^2)^{1/2}-(1+x_4^2)^{1/2}}{x_3-x_4}\right)\left(\frac{-(1+x_4^2)^{1/2}+(1+x_1^2)^{1/2}}{x_4-x_1}\right)$$
$$\times\exp\{-m_1[(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}]-m_2[(1+x_3^2)^{1/2}+(1+x_4^2)^{1/2}]$$
$$-in_1(x_1-x_2)-in_2(x_3-x_4)\}. \tag{2.3b}$$

Consider first Z_1. The n_i integrals are easily done to give δ functions. Then the x_2 and x_4 integrals are carried out and we find

$$Z_1=\tfrac{1}{2}(2\pi)^{-2}\int_0^\infty dm_1\int_{m_1}^\infty dm_2\int_{-\infty}^\infty dx_1\,dx_3\,(1+x_1^2)^{-3/2}(1+x_3^2)^{-3/2}x_1x_3\left(\frac{(1+x_1^2)^{1/2}+(1+x_3^2)^{1/2}}{x_1-x_3+i\epsilon}\right)^2$$
$$\times\exp[-m_1 2(1+x_1^2)^{1/2}-(m_2-m_1)2(1+x_3^2)^{1/2}]. \tag{2.4}$$

Then using the variable $m_{21}=m_2-m_1$ instead of m_2 the integrals over m_1 and m_{21} are done and we obtain

$$Z_1=\tfrac{1}{8}(2\pi)^{-2}\int_{-\infty}^\infty dx_1\,dx_3(1+x_1^2)^{-2}(1+x_3^2)^{-2}\,x_1x_3\left(\frac{(1+x_1^2)^{1/2}+(1+x_3^2)^{1/2}}{x_1-x_3+i\epsilon}\right)^2. \tag{2.5}$$

We treat Z_2 in a similar manner. First carry out the n_i integrations and then do the x_2 and x_4 integrals to obtain

$$Z_2=-\tfrac{1}{2}(2\pi)^{-2}\int_0^\infty dm_2\int_0^{m_2} dm_1\int_{-\infty}^\infty dx_1\int_{-\infty}^\infty dx_3\quad(1+x_1^2)^{-3/2}(1+x_3^2)^{-3/2}x_1x_3\left(\frac{(1+x_1^2)^{1/2}-(1+x_3^2)^{1/2}}{x_1-x_3}\right)^2$$
$$\times\exp[-2m_1(1+x_1^2)^{1/2}-2m_2(1+x_3^2)^{1/2}]. \tag{2.6}$$

The integrand is symmetric in x_1 and x_3 so the m_1 integrand may be extended from $0\to m_2$ to $0\to\infty$ if we divide by 2. Therefore,

$$2Z_2=-\tfrac{1}{8}(2\pi)^{-2}\int_{-\infty}^\infty dx_1\,dx_3(1+x_1^2)^{-2}(1+x_3^2)^{-2}\;x_1x_3\left(\frac{(1+x_1^2)^{1/2}-(1+x^2)^{1/2}}{x_1-x_3}\right)^2. \tag{2.7}$$

We may now add (2.5) and (2.7) to obtain

$$Z_1+2Z_2=\tfrac{1}{2}(2\pi)^{-2}\int_{-\infty}^\infty dx_1\,dx_3\,(1+x_1^2)^{-3/2}\;(1+x_3^2)^{-1/2}x_1x_3(x_1-x_3+i\epsilon)^{-2}. \tag{2.8}$$

If we now write

$$(x_1-x_3+i\epsilon)^{-2}=\tfrac{1}{2}\left(\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_1}\right)(x_1-x_3+i\epsilon)^{-1} \tag{2.9}$$

and integrate by parts we find

$$Z_1+2Z_2=\tfrac{1}{4}(2\pi)^{-2}\int_{-\infty}^\infty dx_1\,dx_3\,(1+x_1^2)^{-5/2}(1+x_3^2)^{-5/2}\;[3x_1^2x_3^2-(1+x_1^2)(1+x_3^2)]$$
$$=\tfrac{1}{2}(2\pi)^{-2}\left[\left(\int_{-\infty}^\infty dx(1+x^2)^{-3/2}\right)^2-3\int_{-\infty}^\infty dx_1(1+x_1^2)^{-3/2}\int_{-\infty}^\infty dx_3(1+x_3^2)^{-5/2}+\frac{3}{2}\left(\int_{-\infty}^\infty dx(1+x^2)^{-5/2}\right)^2\right]. \tag{2.10}$$

These integrals are readily evaluated as beta functions[7] and we obtain

$$Z_1+2Z_2=\tfrac{1}{8}\pi^{-1}\left\{\Gamma(\tfrac{3}{2})^{-2}-3[\Gamma(\tfrac{3}{2})(\tfrac{5}{2})]^{-1}+\tfrac{3}{2}\Gamma(\tfrac{5}{2})^{-2}\right\}$$
$$=-\tfrac{1}{6}\pi^{-2}. \qquad (2.11)$$

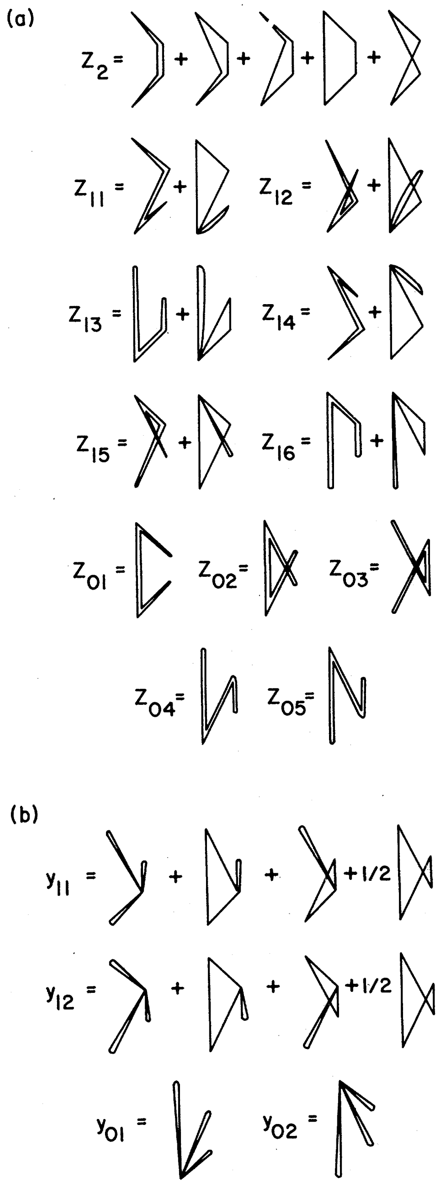

FIG. 2. Thirty-one unlabeled connected graphs of four, five, and six lines which contribute to f_4 in the region $m_1<m_2<m_3<m_4$. There are two generic types of graphs for $k=6$ which we call z and y. The first index in the subscript indicates half of the number of effective $i\epsilon$ terms in the six-line graphs. The graphs are grouped into sets which are combined using various signs of $i\epsilon$ in the six line graph.

Substituting (2.11) in (2.2) we obtain (1.36).

III. COMBINATION OF FOUR-POINT GRAPHS

There are 31 different connected unlabeled graphs of 4, 5, and 6 lines that contribute to f_4 when the vertical coordinates are ordered

$$m_1<m_2<m_3<m_4. \qquad (3.1)$$

These are shown in Fig. 2. To interpret this figure, several remarks are needed: (*i*) There are two generic six-line diagrams which we call z and y as indicated on the figure. (*ii*) By the symbol on the left of each set of diagrams we mean the sum of all *labeled* diagrams represented by the unlabeled graphs shown. (*iii*) The first subscript indicates half the number of poles where the $i\epsilon$ prescription is needed in the six-line graphs. (*iv*) Along with the six-line graphs which have poles, we group the related four- and five-line graphs which are related by changing the sign of some of the $i\epsilon$. (*v*) The relation between labeled and unlabeled graphs is illustrated in Fig. 3.

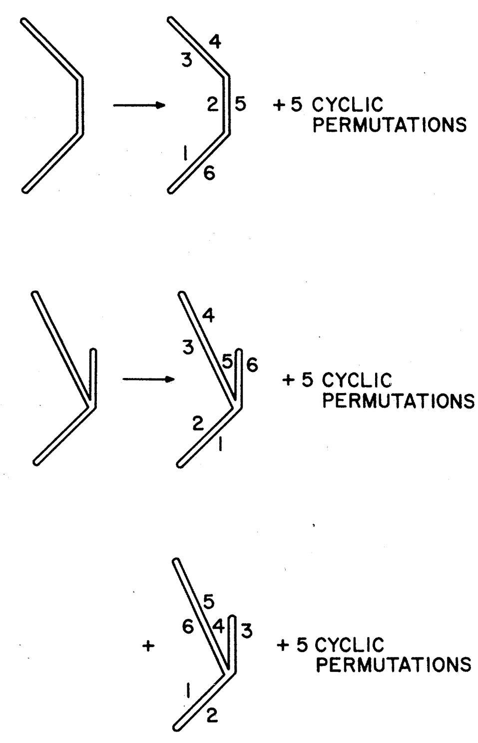

FIG. 3. Relation between labeled and unlabeled six-line graphs.

The combination of terms referred to in point 4 has been previously discussed[8] for the Z diagrams. From that discussion we have the following

$$z_2=\tfrac{1}{2}(z_{2+}+z_{2-}), \tag{3.2}$$

where

$$z_{2\epsilon_1}=i^6\tfrac{1}{2}(2\pi)^{-6}\int_{-\infty}^{\infty}\prod_{l=1}^{6}dx_l\,(1+x_l^2)^{-1/2}$$
$$\times\left(\frac{(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}}{x_1-x_2+i\epsilon}\right)\left(\frac{(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}}{x_2-x_3+i\epsilon_1}\right)\left(\frac{(1+x_3^2)^{1/2}-(1+x_4^2)^{1/2}}{x_3-x_4}\right)$$
$$\times\left(\frac{-(1+x_4^2)^{1/2}-(1+x_5^2)^{1/2}}{x_4-x_5-i\epsilon_1}\right)\left(\frac{-(1+x_5^2)^{1/2}-(1+x_6^2)^{1/2}}{x_5-x_6-i\epsilon}\right)\left(\frac{-(1+x_6^2)^{1/2}+(1+x_1^2)^{1/2}}{x_6-x_1}\right)$$
$$\times\exp\{-m_{21}[(1+x_1^2)^{1/2}+(1+x_6^2)^{1/2}]-m_{32}[(1+x_2^2)^{1/2}+(1+x_5^2)^{1/2}]$$
$$-m_{43}[(1+x_3^2)^{1/2}+(1+x_4^2)^{1/2}]-in_{12}(x_1-x_6)-in_{23}(x_2-x_5)-in_{34}(x_3-x_4)\}, \tag{3.3}$$

$$z_{11}=i^6\tfrac{1}{2}(2\pi)^{-6}\int_{-\infty}^{\infty}\prod_{l=1}^{6}dx_l\,(1+x_l^2)^{-1/2}\left(\frac{-(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}}{x_1-x_2}\right)\left(\frac{(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}}{x_2-x_3+i\epsilon}\right)\left(\frac{(1+x_3^2)^{1/2}-(1+x_4^2)^{1/2}}{x_3-x_4}\right)$$
$$\times\left(\frac{-(1+x_4^2)^{1/2}-(1+x_5^2)^{1/2}}{x_4-x_5-i\epsilon}\right)\left(\frac{-(1+x_5^2)^{1/2}+(1+x_6^2)^{1/2}}{x_5-x_6}\right)\left(\frac{(1+x_6^2)^{1/2}-(1+x_1^2)^{1/2}}{x_6-x_1}\right)$$
$$\times\exp\{-m_{31}[(1+x_2^2)^{1/2}+(1+x_5^2)^{1/2}]-m_{43}[(1+x_3^2)^{1/2}+(1+x_4^2)^{1/2}]-m_{21}[(1+x_1^2)^{1/2}+(1+x_6^2)^{1/2}]$$
$$-in_{13}(x_2-x_5)-in_{34}(x_3-x_4)-in_{12}(x_6-x_1)\}, \tag{3.4a}$$

$$z_{12}=z_{11}|r_2\longleftrightarrow r_3 \tag{3.4b}$$

$$z_{13}=z_{11}|r_2\to r_4,\quad r_3\to r_2,\quad r_4\to r_3 \tag{3.4c}$$

z_{14}, z_{15}, and z_{16} are obtained from z_{11}, z_{12}, and z_{13}, respectively, by up-down reversal, and lastly

$$z_{01}=i^6\tfrac{1}{2}(2\pi)^{-6}\int_{-\infty}^{\infty}\prod_{l=1}^{6}dx_l\,(1+x_l^2)^{-1/2}\left(\frac{-(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}}{x_1-x_2}\right)\left(\frac{(1+x_2^2)^{1/2}-(1+x_3^2)^{1/2}}{x_2-x_3}\right)\left(\frac{-(1+x_3^2)^{1/2}+(1+x_4^2)^{1/2}}{x_3-x_4}\right)$$
$$\times\left(\frac{(1+x_4^2)^{1/2}-(1+x_5^2)^{1/2}}{x_4-x_5}\right)\left(\frac{-(1+x_5^2)^{1/2}+(1+x_6^2)^{1/2}}{x_5-x_6}\right)\left(\frac{(1+x_6^2)^{1/2}-(1+x_1^2)^{1/2}}{x_6-x_1}\right)$$
$$\times\exp\{-m_{41}[(1+x_2^2)^{1/2}+(1+x_5^2)^{1/2}]-m_{43}[(1+x_3^2)^{1/2}+(1+x_4^2)^{1/2}]$$
$$-m_{21}[(1+x_1^2)^{1/2}+(1+x_6^2)^{1/2}]-in_{14}(x_2-x_5)-in_{43}(x_3-x_4)-in_{12}(x_6-x_1)\}, \tag{3.5a}$$

$$z_{02}=z_{01}|r_2\longleftrightarrow r_3, \tag{3.5b}$$

$$z_{03}=z_{01}|r_1\to r_3,\ r_2\to r_1,\ r_3\to r_4,\ r_4\to r_2, \tag{3.5c}$$

$$z_{04}=z_{01}|r_2\to r_4,\ r_3\to r_2,\ r_4\to r_3, \tag{3.5d}$$

$$z_{05}=z_{01}|r_1\to r_2, r_2\to r_3, r_3\to r_1. \tag{3.5e}$$

It is useful to note that the forms (3.2) and (3.4) are independent of the sign of ϵ. From time to time we will make this manifest by explicitly averaging over both signs.

A similar method exists to combine the diagrams in y_{11} (and y_{12}). The four terms which contribute to y_{11} are [see Fig. 2(b)]

$$y_{11}=y_{11,1}+y_{11,2}+y_{11,3}+\tfrac{1}{2}y_{11,4} \tag{3.6}$$

where

$$y_{11,1}=\tfrac{1}{2}i^6(2\pi)^{-6}\int_{-\infty}^{\infty}\prod_{l=1}^{6}dx_l(1+x_l^2)^{-1/2}\left(\frac{-(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}}{x_1-x_2}\right)\left(\frac{(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}}{x_2-x_3+i\epsilon}\right)\left(\frac{(1+x_3^2)^{1/2}-(1+x_4^2)^{1/2}}{x_3-x_4}\right)$$
$$\times\left(\frac{-(1+x_4^2)^{1/2}+(1+x_5^2)^{1/2}}{x_4-x_5}\right)\left(\frac{(1+x_5^2)^{1/2}-(1+x_6^2)^{1/2}}{x_5-x_6}\right)\left(\frac{-(1+x_6^2)^{1/2}-(1+x_1^2)^{1/2}}{x_6-x_1+i\epsilon}\right)$$
$$\times(\exp\{-m_{21}[(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}]-m_{42}[(1+x_3^2)^{1/2}+(1+x_4^2)^{1/2}]$$
$$-m_{32}[(1+x_5^2)^{1/2}+(1+x_6^2)^{1/2}]-in_{21}(x_1-x_2)-in_{24}(x_3-x_4)-in_{23}(x_5-x_6)\}+r_3\longleftrightarrow r_4),\tag{3.7a}$$

$$y_{11,2}=i^5(2\pi)^{-5}\int_{-\infty}^{\infty}\prod_{l=1}^{5}dx_l(1+x_l^2)^{-1/2}\left(\frac{(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}}{x_1-x_2+i\epsilon}\right)\left(\frac{(1+x_2^2)^{1/2}-(1+x_3^2)^{1/2}}{x_2-x_3}\right)$$
$$\times\left(\frac{-(1+x_3^2)^{1/2}+(1+x_4^2)^{1/2}}{x_3-x_4}\right)\left(\frac{(1+x_4^2)^{1/2}-(1+x_5^2)^{1/2}}{x_4-x_5}\right)\left(\frac{-(1+x_5^2)^{1/2}+(1+x_1^2)^{1/2}}{x_5-x_1}\right)$$
$$\times\exp\{-m_{21}(1+x_1^2)^{1/2}-m_{32}[(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}]-m_{42}(1+x_4^2)^{1/2}-m_{41}(1+x_5^2)^{1/2}$$
$$-in_{12}x_1-in_{23}(x_2-x_3)-in_{24}x_4-in_{41}x_5\},\tag{3.7b}$$

$$y_{11,3}=i^5(2\pi)^{-5}\int_{-\infty}^{\infty}\prod_{l=1}^{5}dx_l(1+x_l^2)^{-1/2}\left(\frac{(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}}{x_1-x_2+i\epsilon}\right)\left(\frac{(1+x_2^2)^{1/2}-(1+x_3^2)^{1/2}}{x_2-x_3}\right)$$
$$\times\left(\frac{-(1+x_3^2)^{1/2}+(1+x_4^2)^{1/2}}{x_3-x_4}\right)\left(\frac{(1+x_4^2)^{1/2}-(1+x_5^2)^{1/2}}{x_4-x_5}\right)\left(\frac{-(1+x_5^2)^{1/2}+(1+x_1^2)^{1/2}}{x_5-x_1}\right)$$
$$\times\exp\{-m_{21}(1+x_1^2)^{1/2}-m_{42}[(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}]-m_{32}(1+x_4^2)^{1/2}-m_{31}(1+x_5^2)^{1/2}$$
$$-in_{12}x_1-in_{24}(x_2-x_3)-in_{23}x_4-in_{31}x_5\},\tag{3.7c}$$

and

$$y_{11,4}=-i^4(2\pi)^{-4}\int_{-\infty}^{\infty}\prod_{l=1}^{4}dx_l(1+x_l^2)^{-1/2}\left(\frac{(1+x_1^2)^{1/2}-(1+x_2^2)^{1/2}}{x_1-x_2}\right)\left(\frac{-(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}}{x_2-x_3}\right)$$
$$\times\left(\frac{(1+x_3^2)^{1/2}-(1+x_4^2)^{1/2}}{x_3-x_4}\right)\left(\frac{-(1+x_4^2)^{1/2}+(1+x_1^2)^{1/2}}{x_4-x_1}\right)$$
$$\times\exp[-m_{31}(1+x_1^2)^{1/2}-m_{32}(1+x_2^2)^{1/2}-m_{42}(1+x_3^2)^{1/2}-m_{41}(1+x_4^2)^{1/2}$$
$$-in_{13}x_1-in_{32}x_2-in_{24}x_3-in_{41}x_4].\tag{3.7d}$$

We combine these diagrams together in terms of the function

$$y_{11\epsilon_1\epsilon_2}=\tfrac{1}{2}i^6(2\pi)^{-6}\int_{-\infty}^{\infty}\prod_{l=1}^{6}dx_l(1+x_l^2)^{-1/2}\left(\frac{-(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}}{x_1-x_2}\right)\left(\frac{(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}}{x_2-x_3+i\epsilon_1}\right)$$
$$\times\left(\frac{(1+x_3^2)^{1/2}-(1+x_4^2)^{1/2}}{x_3-x_4}\right)\left(\frac{-(1+x_4^2)^{1/2}+(1+x_5^2)^{1/2}}{x_4-x_5}\right)$$
$$\times\left(\frac{(1+x_5^2)^{1/2}-(1+x_6^2)^{1/2}}{x_5-x_6}\right)\left(\frac{-(1+x_6^2)^{1/2}-(1+x_1^2)^{1/2}}{x_6-x_1+i\epsilon_2}\right)$$
$$\times(\exp\{-m_{21}[(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}]-m_{42}[(1+x_3^2)^{1/2}+(1+x_4^2)^{1/2}]$$
$$-m_{32}[(1+x_5^2)^{1/2}+(1+x_6^2)^{1/2}]-in_{21}(x_1-x_2)-in_{24}(x_3-x_4)-in_{23}(x_5-x_6)\}+r_3\longleftrightarrow r_4).\tag{3.8}$$

Thus, using

$$\frac{1}{z-i\epsilon}=\frac{1}{z+i\epsilon}+2\pi i\delta(z)\,, \tag{3.9}$$

we have

$$y_{11,++}=y_{11,1}\,, \tag{3.10a}$$

$$y_{11,+-}=y_{11,1}+y_{11,2}+y_{11,3}\,, \tag{3.10b}$$

$$y_{11,-+}=y_{11,1}+y_{11,2}+y_{11,3}\,, \tag{3.10c}$$

$$y_{11,--}=y_{11,1}+2y_{11,2}+2y_{11,3}+4\nu_{11,4}\,. \tag{3.10d}$$

Hence, we obtain the desired combination

$$y_{11}=\tfrac{1}{8}(y_{11,++}+3y_{11,+-}+3y_{11,-+}+y_{11,--})\,. \tag{3.11}$$

We may now examine the question of global rotational invariances as we did in a previous publication[4] for three-point graphs. We may show that the sum of all the z and y graphs is globally rotationally invariant, but that either the z or the y graphs taken separately have discontinuities.

IV. EVALUATION OF $I_4^{-(6)}$

We evaluate $I_4^{-(6)}$ by integrating the z and y graphs separately over $\vec{r}_2$, $\vec{r}_3$, and $\vec{r}_4$. In Sec. III, we have given the explicit expression for the terms of z and y in the region (3.1). The integrals over the other 4!-1 regions will be equal by symmetry. We treat the z and y graphs in separate subsections.

A. Evaluation of z integrals

To integrate the z graphs we proceed in several stages. First consider the region (3.1).

$$Z_2=\int d^2r_2\,d^2r_3\,d^2r_4\,z_2(r_1,r_2,r_3,r_4)\,, \tag{4.1}$$

where z_2 is given by (3.2). The n_{12}, n_{23}, n_{34} integrals are done to give δ functions. Moreover, the integral over the vertical coordinates in the region (3.1) is just the integral over the region

$$m_{21}>m_{32}>0\,,\quad m_{43}>0\,. \tag{4.2}$$

Since z_2 depends only on m_{21}, m_{32}, m_{43}, the m_i integrals are also easily done and we obtain

$$Z_2=2^{-6}\sum_{\epsilon,\epsilon_1}(2\pi)^{-3}\int_{-\infty}^{\infty}dx_1\,dx_2\,dx_3 \times(1+x_1^2)^{-2}(1+x_2^2)^{-3/2}(1+x_3^2)^{-2}x_1x_3 \times\left(\frac{(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}}{x_1-x_2+i\epsilon}\right)^2 \times\left(\frac{(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}}{x_2-x_3+i\epsilon_1}\right)^2 \tag{4.3}$$

when we have taken advantage of the remark after (3.5) to average over the sign of ϵ.

Secondly, define for the region (3.1)

$$Z_1=\int d^2r_2\,d^2r_3\,d^2r_4(z_{11}+z_{12}+z_{13})\,, \tag{4.4}$$

we can carry out this integral by directly substituting (3.4) for the three terms and keeping the restriction (3.1). However, it is far more convenient to consider z_{11}, z_{12}, or z_{13} as functions of m_{31}, m_{43}, and m_{21} where

$$z_{11}:\; m_{43}>0,\; m_{31}>m_{21}>0\,, \tag{4.5a}$$

$$z_{12}:\; m_{43}>0,\; m_{21}>m_{31}>0,\; m_{21}<m_{31}+m_{43}\,, \tag{4.5b}$$

$$z_{13}:\; m_{43}>0,\; m_{31}>0,\; m_{21}>m_{31}+m_{43}\,. \tag{4.5c}$$

Here (4.5a) is the same as (3.1), (4.5b) is the same as $m_1<m_3<m_2<m_4$ and (4.5c) is the same as $m_1<m_3<m_4<m_2$. The sum of all three regions in (4.5) is the single region

$$m_{43}>0,\quad m_{31}>0,\quad m_{21}>0\,. \tag{4.6}$$

Thus the m_i integrals are easily done and we find

$$Z_1=-2^{-4}(2\pi)^{-3}\int_{-\infty}^{\infty}dx_1\,dx_2\,dx_3(1+x_1^2)^{-2}(1+x_2^2)^{-3/2} \times(1+x_3^2)^{-2}x_1x_3 \times\left(\frac{(1+x_1^2)^{1/2}-(1+x_2^2)^{1/2}}{x_1-x_2}\right)^2 \times\left(\frac{(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}}{x_2-x_3+i\epsilon}\right)^2. \tag{4.7}$$

By up-down symmetry, the contribution from the graphs z_{14}, z_{15}, and z_{16} is also Z_1 which is best expressed in the equivalent form

$$Z_1=-2^{-4}(2\pi)^{-3}\int_{-\infty}^{\infty}dx_1\,dx_2\,dx_3(1+x_1^2)^{-2}(1+x_2^2)^{-3/2} \times(1+x_3^2)^{-2}x_1x_3 \times\left(\frac{(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}}{x_1-x_2+i\epsilon}\right)^2 \times\left(\frac{(1+x_2^2)^{1/2}-(1+x_3^2)^{1/2}}{x_2-x_3}\right)^2. \tag{4.7a}$$

Finally, we consider under the restriction (3.1)

$$Z_0=\int d^2r_2\,d^2r_3\,d^2r_4(z_{01}+z_{02}+z_{03}+z_{04}+z_{05}) \tag{4.8}$$

with z_{0i} given by (3.5). We now consider (4.8) with the integrands all given by (3.5a) which is a function of the variable m_{21}, m_{41}, and m_{43} with the restriction equivalent to (3.1) that

$$m_{21}>0,\quad m_{41}>0,\quad\text{and } m_{43}>0 \tag{4.9}$$

in all regions and

$$z_{01}:\ m_{41}>m_{43}+m_{21}\,, \tag{4.10a}$$

$$z_{02}:\ m_{43}+m_{21}>m_{41},\quad m_{41}>m_{43},\quad m_{41}>m_{21}\,, \tag{4.10b}$$

$$z_{03}:\ m_{43}>m_{41},\quad m_{21}>m_{41}\,, \tag{4.10c}$$

$$z_{04}:\ m_{21}>m_{41}>m_{43}\,, \tag{4.10d}$$

$$z_{05}:\ m_{43}>m_{41}>m_{21}\,. \tag{4.10e}$$

The sum of these five regions is just the single region (4.9). Therefore, the n_i and m_i integrals are easily done and we find

$$Z_0=2^{-4}(2\pi)^{-3}\int_{-\infty}^{\infty}dx_1\,dx_2\,dx_3(1+x_1^2)^{-2}(1+x_2^2)^{-3/2}\times(1+x_3^2)^{-2}x_1x_3\left(\frac{(1+x_1^2)^{1/2}-(1+x_2^2)^{1/2}}{x_1-x_2}\right)^2\times\left(\frac{(1+x_2^2)^{1/2}-(1+x_3^2)^{1/2}}{x_2-x_3}\right)^2. \tag{4.11}$$

Accordingly, if we call Z the contribution to $I_4^{-(6)}$ from the z graphs satisfying (3.1), we use (4.3), (4.7) and (4.11) and obtain

$$\begin{aligned}Z&=Z_2+2Z_1+Z_0\\&=\frac{1}{4}\sum_{\epsilon,\epsilon_1}(2\pi)^{-3}\int_{-\infty}^{\infty}dx_1\,dx_2\,dx_3(1+x_1^2)^{-3/2}\\&\quad\times(1+x_2^2)^{-1/2}(1+x_3^2)^{-3/2}\\&\quad\times x_1x_3(x_1-x_2+i\epsilon)^{-2}(x_2-x_3+i\epsilon_1)^{-2}.\end{aligned} \tag{4.12}$$

We now proceed to evaluate the triple integral. First use the change of variable

$$x_i=\tfrac{1}{2}(\xi_i-\xi_i^{-1})\,. \tag{4.13}$$

To find

$$Z=(2\pi)^{-3}\int_0^{\infty}\frac{d\xi}{\xi}[w(\xi)]^2\,, \tag{4.14}$$

where

$$\begin{aligned}w(\xi)&=\overline{w}(x)\\&=\frac{1}{2}\sum_{\epsilon}\int_{-\infty}^{\infty}dx'x'(1+x'^2)^{-3/2}(x-x'+i\epsilon)^{-2}.\end{aligned} \tag{4.15}$$

If we integrate this by parts the boundary terms vanish and we find

$$\begin{aligned}w(\xi)&=\frac{1}{2}\sum_{\epsilon}\int_{-\infty}^{\infty}dx'\,(1+x'^2)^{-1/2}\frac{d}{dx'}(x-x'+i\epsilon)^{-2}\\&=\frac{d^2}{dx^2}\frac{1}{2}\sum_{\epsilon}\int_{-\infty}^{\infty}dx\,(1+x'^2)^{-1/2}(x-x'+i\epsilon)^{-1}.\end{aligned} \tag{4.16}$$

Using (4.13) we thus find

$$w(\xi)=2(1+\xi^{-2})^{-1}\frac{d}{d\xi}2(1+\xi^{-2})^{-1}\frac{d}{d\xi}I(\xi)\,, \tag{4.17}$$

where

$$I(\xi)=\sum_{\epsilon}\int_0^{\infty}\frac{d\xi'}{\xi'}\frac{1}{\xi-\xi^{-1}-\xi'+\xi'^{-1}+i\epsilon}. \tag{4.18}$$

The integrand may be factorized and decomposed by partial fractions to give

$$I(\xi)=\sum_{\epsilon}\int_0^{\infty}d\xi'\left(\frac{1}{\xi-\xi'+i\epsilon}+\frac{1}{\xi'+\xi^{-1}}\right)(\xi+\xi^{-1})^{-1}. \tag{4.19}$$

Therefore,

$$I(\xi)=(4\ln\xi)/(\xi+\xi^{-1}) \tag{4.20}$$

and hence from (4.17)

$$w(\xi)=16\frac{\xi^3}{(1+\xi^2)^5}[3(1-\xi^4)+2(\xi^4-4\xi^2+1)\ln\xi]. \tag{4.21}$$

Thus, substituting in (4.14) and using the variable

$$t=\xi^2\,, \tag{4.22}$$

we are left with

$$Z=(2\pi)^{-3}2^7\int_0^{\infty}dt\,t^2(1+t)^{-10}\times[3(1-t^2)+(t^2-4t+1)\ln t]^2. \tag{4.23}$$

To proceed further we expand the square and write

$$Z=2^4\pi^{-3}(9A+6B+C)\,, \tag{4.24}$$

where

$$A=\int_0^{\infty}dt\,(1+t)^{-8}t^2(1-t)^2\,, \tag{4.25a}$$

$$B=\int_0^{\infty}dt\,(1+t)^{-9}t^2(1-t)(t^2-4t+1)\ln t\,, \tag{4.25b}$$

and

$$C=\int_0^{\infty}dt\,(1+t)^{-10}t^2(t^2-4t+1)^2\ln^2 t. \tag{4.25c}$$

The first integral is easily expressed in terms of beta functions[7] as

$$\begin{aligned}A&=2[B(3,5)-B(4,4)]\\&=[\Gamma(8)]^{-1}2[\Gamma(3)\Gamma(5)-\Gamma^2(4)]\\&=2^3\times 3\frac{1}{7!}\,.\end{aligned}\tag{4.26}$$

Secondly we write B as

$$\begin{aligned}B&=\frac{\partial}{\partial\epsilon}\int_0^\infty dt\,(1+t)^{-9}t^\epsilon t^2(1-t)(t^2-4t+1)\big|_{\epsilon=0}\\&=-\frac{\partial}{\partial\epsilon}\int_0^\infty dt\,(1+t)^{-9}t^{2+\epsilon}(t^3-5t^2+5t-1)\big|_{\epsilon=0}\\&=-\frac{\partial}{\partial\epsilon}[B(6+\epsilon,3-\epsilon)-5B(5+\epsilon,4-\epsilon)+5B(4+\epsilon,5-\epsilon)-B(3+\epsilon,6-\epsilon)]\big|_{\epsilon=0}\\&=-\frac{1}{8!}\frac{\partial}{\partial\epsilon}[\Gamma(6+\epsilon)\Gamma(3-\epsilon)-5\Gamma(5+\epsilon)\Gamma(4-\epsilon)+5\Gamma(4+\epsilon)\Gamma(5-\epsilon)-\Gamma(3+\epsilon)\Gamma(6-\epsilon)]\big|_{\epsilon=0}\,.\end{aligned}\tag{4.27}$$

Thus, using the recursion relation for $\Gamma(z)$

$$\begin{aligned}B&=\frac{1}{8!}\frac{\partial}{\partial\epsilon}[(5+\epsilon)(4+\epsilon)(3+\epsilon)-5(4+\epsilon)(3+\epsilon)(3-\epsilon)+5(3+\epsilon)(3-\epsilon)(4-\epsilon)-(3-\epsilon)(4-\epsilon)(5-\epsilon)]\,\Gamma(3-\epsilon)\Gamma(3+\epsilon)\big|_{\epsilon=0}\\&=-\frac{1}{8!}\frac{\partial}{\partial\epsilon}4\epsilon\Gamma(3-\epsilon)\Gamma(3+\epsilon)\big|_{\epsilon=0}\\&=-\frac{2}{7!}\,.\end{aligned}\tag{4.28}$$

Thirdly, we write

$$C=\frac{\partial^2}{\partial\epsilon^2}\int_0^\infty dt(1+t)^{-10}t^{2+\epsilon}\,(t^4-8t^3+18t^2-8t+1)\big|_{\epsilon=0}\tag{4.29}$$

which is expressed in terms of Γ functions as

$$C=\frac{1}{9!}\frac{\partial^2}{\partial\epsilon^2}[\Gamma(7+\epsilon)\Gamma(3-\epsilon)-8\Gamma(6+\epsilon)\Gamma(4-\epsilon)+18\Gamma(5+\epsilon)\Gamma(5-\epsilon)-8\Gamma(4+\epsilon)\Gamma(6-\epsilon)+\Gamma(3+\epsilon)\Gamma(7-\epsilon)]\big|_{\epsilon=0}\,.\tag{4.30}$$

Thus, using the recurrence relation for Γ we have

$$\begin{aligned}C&=\frac{1}{9!}\frac{\partial^2}{\partial\epsilon^2}[(6+\epsilon)(5+\epsilon)(4+\epsilon)(3+\epsilon)-8(5+\epsilon)(4+\epsilon)(9-\epsilon^2)+18(16-\epsilon^2)(9-\epsilon^2)\\&\qquad-8(9-\epsilon^2)(4-\epsilon)(5-\epsilon)+(6-\epsilon)(5-\epsilon)(4-\epsilon)(3-\epsilon)]\,\Gamma(3+\epsilon)\Gamma(3-\epsilon)\big|_{\epsilon=0}\\&=\frac{1}{8!}\frac{\partial^2}{\partial\epsilon^2}[4(12-\epsilon^2)\Gamma(3+\epsilon)\Gamma(3-\epsilon)]\big|_{\epsilon=0}\\&=\frac{1}{8!}\frac{\partial^2}{\partial\epsilon^2}\left(4(12-\epsilon^2)(4-5\epsilon^2)\frac{\epsilon\pi}{\sin\pi\epsilon}\right)\Big|_{\epsilon=0}\\&=\frac{1}{7!}8(-8+\pi^2)\,.\end{aligned}\tag{4.31}$$

Thus, collecting terms we obtain from (4.24)

$$Z=64(7!\,\pi^3)^{-1}(2\pi^2+35)\,.\tag{4.32}$$

B. Evaluation of y integrals

The y graphs are shown in Fig. 2. In terms of these we need to calculate, under the restriction (3.1), the integral

$$Y=\int d^2r_2\, d^2r_3\, d^2r_4(y_{11}+y_{12}+y_{01}+y_{02})$$
$$=2(Y_1+Y_0)\,, \tag{4.33}$$

where

$$Y_1=\int d^2r_2\, d^2r_3\, d^2r_4 y_{11}\,, \tag{4.34a}$$

$$Y_0=\int d^2r_2\, d^2r_3\, d^2r_4 y_{01}\,. \tag{4.34b}$$

We first study Y_0. The integrand is

$$y_{01}=i^6\tfrac{1}{2}(2\pi)^{-6}\int_{-\infty}^{\infty}\prod_{l=1}^{6}dx\,(1+x_l^2)^{-1/2}\left(\frac{(1+x_1^2)^{1/2}-(1+x_2^2)^{1/2}}{x_1-x_2}\right)\left(\frac{-(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}}{x_2-x_3}\right)$$
$$\times\left(\frac{(1+x_3^2)^{1/2}-(1+x_4^2)^{1/2}}{x_3-x_4}\right)\left(\frac{-(1+x_4^2)^{1/2}+(1+x_5^2)^{1/2}}{x_4-x_5}\right)$$
$$\times\left(\frac{(1+x_5^2)^{1/2}-(1+x_6^2)^{1/2}}{x_5-x_6}\right)\left(\frac{-(1+x_6^2)^{1/2}+(1+x_1^2)^{1/2}}{x_6-x_1}\right)$$
$$\times(\exp\{-m_{41}[(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}]-m_{31}[(1+x_3^2)^{1/2}+(1+x_4^2)^{1/2}]$$
$$-m_{21}[(1+x_5^2)^{1/2}+(1+x_6^2)^{1/2}]-in_{14}(x_1-x_2)-in_{13}(x_3-x_4)-in_{12}(x_5-x_6)\}+r_2\to r_3)\,. \tag{4.35}$$

To integrate this over the restricted region (3.1) it is useful to rewrite y_{01} as

$$y_{01}=i^6\tfrac{1}{6}(2\pi)^{-6}\int_{-\infty}^{\infty}\prod_{l=1}^{6}dx_l(1+x_l^2)^{-1/2}\left(\frac{(1+x_1^2)^{1/2}-(1+x_2^2)^{1/2}}{x_1-x_2}\right)\left(\frac{-(1+x_2^2)^{1/2}+(1+x_3^2)^{1/2}}{x_2-x_3}\right)$$
$$\times\left(\frac{(1+x_3^2)^{1/2}-(1+x_4^2)^{1/2}}{x_3-x_4}\right)\left(\frac{-(1+x_4^2)^{1/2}+(1+x_5^2)^{1/2}}{x_4-x_5}\right)$$
$$\times\left(\frac{(1+x_5^2)^{1/2}-(1+x_6^2)^{1/2}}{x_5-x_6}\right)\left(\frac{-(1+x_6^2)^{1/2}+(1+x_1^2)^{1/2}}{x_6-x_1}\right)$$
$$\times(\exp\{-m_{41}[(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}]-m_{31}[(1+x_3^2)^{1/2}+(1_1x_4^2)^{1/2}]-m_{21}[(1+x_5^2)^{1/2}+(1+x_6^2)^{1/2}]$$
$$-in_{14}(x_1-x_2)-in_{13}(x_3-x_4)-in_{12}(x_5-x_6)\}+\ldots)\,, \tag{4.36}$$

where the dots stand for the five permutations of 2, 3, and 4. The integral of (4.36) over the region (3.1) is the same as the integral of the first term in (4.36) over the region

$$m_{41}>0,\quad m_{31}>0,\quad m_{21}>0\,. \tag{4.37}$$

Therefore,

$$Y_0=(\tfrac{1}{3})2^{-4}(2\pi)^{-3}\int_{-\infty}^{\infty}dx_1\,dx_3\,dx_5(1+x_1^2)^{-2}(1+x_3^2)^{-2}(1+x_5^2)^{-2}x_1x_3x_5$$
$$\times\left(\frac{(1+x_1^2)-(1+x_3^2)^{1/2}}{x_1-x_3}\right)\left(\frac{(1+x_3^2)^{1/2}-(1+x_5^2)^{1/2}}{x_3-x_5}\right)\left(\frac{(1+x_5^2)^{1/2}-(1+x_1^2)^{1/2}}{x_5-x_1}\right). \tag{4.38}$$

Next, we study Y_1 where y_{11} is given by (3.11) and $y_{11,\epsilon_1\epsilon_2}$ by (3.8). The integral of (3.8) over the region (3.1) is the same as the integral of the first term of (3.8) over the region

$$m_{21}>0,\quad m_{42}>0,\quad m_{32}>0\,. \tag{4.39}$$

Therefore,

$$Y_1 = 2^{-4}(2\pi)^{-3}\int_{-\infty}^{\infty} dx_1\,dx_3\,dx_5(1+x_1^2)^{-2}(1+x_3^2)^{-2}(1+x_5^2)^{-2}x_1x_3x_5[(1+x_1^2)^{1/2}+(1+x_3^2)^{1/2}]$$
$$\times\frac{(1+x_3^2)^{1/2}-(1+x_5^2)^{1/2}}{x_3-x_5}[(1+x_5^2)^{1/2}+(1+x_1^2)^{1/2}]$$
$$\times\frac{1}{8}\left[\left(\frac{1}{x_1-x_3+i\epsilon}\right)\left(\frac{1}{x_5-x_1+i\epsilon}\right)+3\left(\frac{1}{x_1-x_3-i\epsilon}\right)\left(\frac{1}{x_5-x_1+i\epsilon}\right)\right.$$
$$\left.+3\left(\frac{1}{x_1-x_3+i\epsilon}\right)\left(\frac{1}{x_5-x_1-i\epsilon}\right)+\left(\frac{1}{x_1-x_3-i\epsilon}\right)\left(\frac{1}{x_5-x_1-i\epsilon}\right)\right]. \tag{4.40}$$

Now we may add Y_0 and Y_1 and obtain

$$Y=\frac{1}{3}2^{-3}(2\pi)^{-3}\int_{-\infty}^{\infty} dx_1\,dx_3\,dx_5(1+x_1^2)^{-2}(1+x_3^2)^{-2}(1+x_5^2)^{-2}$$
$$\times x_1x_3x_5[4(1+x_1^2)^{1/2}(x_3^2-x_5^2)-2(1+x_3^2)^{1/2}(x_5^2-x_1^2)-2(1+x_5^2)^{1/2}(x_1^2-x_3^2)]$$
$$\times\frac{1}{x_3-x_5+i\epsilon}\times\frac{1}{8}\left[\left(\frac{1}{x_1-x_3+i\epsilon}\right)\left(\frac{1}{x_5-x_1+i\epsilon}\right)+3\left(\frac{1}{x_1-x_3-i\epsilon}\right)\left(\frac{1}{x_5-x_1+i\epsilon}\right)\right.$$
$$\left.+3\left(\frac{1}{x_1-x_3+i\epsilon}\right)\left(\frac{1}{x_5-x_1-i\epsilon}\right)+\left(\frac{1}{x_1-x_3-i\epsilon}\right)\left(\frac{1}{x_5-x_1-i\epsilon}\right)\right]. \tag{4.41}$$

If in the 2nd term in the first bracket we let $3\to1\to5$ and in the 3rd term $5\to1\to3$ we obtain

$$Y=-\frac{1}{3}2^{-4}(2\pi)^{-3}\int_{-\infty}^{\infty} dx_1\,dx_3\,dx_5(1+x_1^2)^{-3/2}(1+x_3^2)^{-2}(1+x_5^2)^{-2}\;x_1x_3x_5(x_3+x_5)$$
$$\times\left[3\left(\frac{1}{x_1-x_3+i\epsilon}\right)\left(\frac{1}{x_5-x_1+i\epsilon}\right)-\left(\frac{1}{x_1-x_3-i\epsilon}\right)\left(\frac{1}{x_5-x_1+i\epsilon}\right)\right.$$
$$\left.-\left(\frac{1}{x_1-x_3-i\epsilon}\right)\left(\frac{1}{x_5-x_1+i\epsilon}\right)-\left(\frac{1}{x_1-x_3-i\epsilon}\right)\left(\frac{1}{x_5-x_1-i\epsilon}\right)\right]. \tag{4.42}$$

We now use

$$1/(z\pm i\epsilon)=P(1/z)\mp\pi i\delta(z) \tag{4.43}$$

to find

$$Y=(\tfrac{1}{3})2^{-2}(2\pi)^{-3}\int_{-\infty}^{\infty} dx_1\,dx_3\,dx_5(1+x_1^2)^{-3/2}(1+x_3^2)^{-2}\;(1+x_5^2)^{-2}x_1x_3x_5(x_3+x_5)$$
$$\times\left(\pi i\frac{P}{x_1-x_3}\delta(x_5-x_1)+\pi i\frac{P}{x_5-x_1}\delta(x_1-x_3)+\pi^2\delta(x_5-x_1)\delta(x_1-x_3)\right). \tag{4.44}$$

After performing the x_1 integral using the δ function the imaginary part vanishes because of antisymmetry under $3\leftrightarrow5$ exchange. Therefore,

$$Y=\frac{1}{3}\frac{1}{16}\pi^{-1}\int_{-\infty}^{\infty} dx_1x_1^4(1+x_1^2)^{-11/2}. \tag{4.45}$$

This last integral is a beta function. Thus, we obtain

$$Y=(3\times5\times7\times9\pi)^{-1}. \tag{4.46}$$

C. Final result

We may now combine (4.32) and (4.46) and obtain the final result

$$\frac{1}{4!}I_4^{-(6)}=\tfrac{4}{9}\pi^{-3}+\tfrac{5}{189}\pi^{-1}. \tag{4.47}$$

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APPENDIX

In this Appendix, we evaluate

$$C_2^{-(2)}=\int_{-\infty}^{\infty} d^2r f_2^{(2)}(0,r)\,. \qquad \text{(A1)}$$

From (1.16) we find

$$f_2^{(2)}(0,r)=\tfrac{1}{2}(2\pi^2)^{-2}\int_{-\infty}^{\infty} dx_1\,dx_2\,dy_1\,dy_2 \times\left(\frac{1}{1+x_1^2+y_1^2}\right)\left(\frac{1}{1+x_2^2+y_2^2}\right) \times\frac{y_1+y_2}{x_1-x_2}\,\frac{y_2+y_1}{x_2-x_1} \times e^{im(y_1-y_2)+in(x_1-x_2)}\,, \qquad \text{(A2)}$$

which is more symmetrically written by letting $x_2\to -x_2$ and $y_2\to -y_2$ as

$$f_2^{(2)}(0,r)=-\tfrac{1}{2}(2\pi^2)^{-2}\int_{-\infty}^{\infty} dx_1\,dx_2\,dy_1\,dy_2 \times\left(\frac{1}{1+x_1^2+y_1^2}\right)\left(\frac{1}{1+x_2^2+y_2^2}\right) \times\left(\frac{y_1-y_2}{x_1+x_2}\right)^2 e^{-im(y_1+y_2)-in(x_1+x_2)}. \qquad \text{(A3)}$$

For the moment consider $m>0$ and do the y integrals by closing on the poles $y_i=-i(1+x_i^2)^{1/2}$. Then

$$f_2^{(2)}(0,r)=\tfrac{1}{2}(2\pi)^{-2}\int_{-\infty}^{\infty} dx_1\,dx_2(1+x_1^2)^{-1/2}(1+x_2^2)^{-1/2} \times\left(\frac{(1+x_1^2)^{1/2}-(1+x_2^2)^{1/2}}{x_1+x_2}\right)^2 \times\exp\{-m[(1+x_1^2)^{1/2}+(1+x_2^2)^{1/2}] - in(x_1+x_2)\}\,, \qquad \text{(A4)}$$

and, if we use

$$x_i=\sinh\xi_i \qquad \text{(A5)}$$

we obtain

$$f_2^{(2)}(0,r)=\tfrac{1}{2}(2\pi)^{-2}\int_{-\infty}^{\infty} d\xi_1\,d\xi_2\left(\frac{e^{\xi_1}-e^{\xi_2}}{e^{\xi_1}+e^{\xi_2}}\right)^2 \times\exp[-m(\cosh\xi_1+\cosh\xi_2) - in(\sinh\xi_1+\sinh\xi_2)]\,. \qquad \text{(A6)}$$

Now shift the contour ξ_i to $\xi_i'=\xi_i+i\theta$, where

$$m\cosh\xi_i+in\sinh\xi=r\cosh\xi_i' \qquad \text{(A7)}$$

with $r=(m^2+n^2)^{1/2}$, to obtain

$$f_2^{(2)}(0,r)=\frac{1}{2}(2\pi)^{-2}\int_{-\infty}^{\infty} d\xi_1'\,d\xi_2'\left(\frac{e^{\xi_1'}-e^{\xi_2'}}{e^{\xi_1'}+e^{\xi_2'}}\right)^2 \times\exp[-r(\cosh\xi_1'+\cosh\xi_2')]\,. \qquad \text{(A8)}$$

This is explicitly rotationally invariant and the restriction $m>0$ may be removed.

To now evaluate $C_2^{-(2)}$ let

$$e^{\xi_i}=(1/r)s_i\,. \qquad \text{(A9)}$$

Then,

$$f_2^{(2)}(0,r)=\tfrac{1}{2}(2\pi)^{-2}\int_0^{\infty}\frac{ds_1}{s_1}\,\frac{ds_2}{s_2}\left(\frac{s_1-s_2}{s_1+s_2}\right)^2 \times\exp[-\tfrac{1}{2}(s_1+s_2)] \times\exp[-\tfrac{1}{2}r^2(s_1^{-1}+s_2^{-1})]\,. \qquad \text{(A10)}$$

We integrate over r to obtain

$$I_2^{-(2)}=2\pi\int_0^{\infty} r\,dr\,f_2^{(2)}(0,r) =\tfrac{1}{2}(2\pi)^{-1}\int_0^{\infty} ds_1\,ds_2\left(\frac{s_1-s_2}{s_1+s_2}\right)^2 \times\frac{e^{-(s_1+s_2)/2}}{s_1+s_2}\,. \qquad \text{(A11)}$$

Then let

$$s_1=\lambda\alpha_1,\quad s_2=\lambda\alpha_2 \qquad \text{(A12a)}$$

with

$$\alpha_1+\alpha_2=1 \qquad \text{(A12b)}$$

and do the λ integral to obtain the final result

$$C_2^{-(2)}=(2\pi)^{-1}\int_0^1 d\alpha_1\,d\alpha_2\,\delta(1-\alpha_1-\alpha_2)(\alpha_1-\alpha_2)^2 =1/6\pi\,. \qquad \text{(A13)}$$

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