Critical behavior of random resistor networks near the percolation threshold

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We use low-density series expansions to calculate critical exponents for the behavior of random resistor networks near the percolation threshold as a function of the spatial dimension d. By using scaling relations, we obtain values of the conductivity exponent μ. For $d = 2$ we find $\mu = 1.43 \pm 0.02$, and for $d = 3$, $\mu = 1.95 \pm 0.03$, in excellent agreement with the experimental result of Abeles *et al*. Our results for high dimensionality agree well with the results of ϵ-expansion calculations.

I. INTRODUCTION

In this paper, we discuss some new ideas concerning the properties of random resistor networks near the percolation threshold.[1] We continue here the work of an earlier paper.[2] The model we treat is that of an electrical network on a d-dimensional hypercubic lattice of N sites with real conductances σ_{ij} connecting nearest-neighbor pairs of lattice sites i and j. Each σ_{ij} is an independent random variable assuming the values 0 and 1 with respective probabilities $1 - p$ and p. (Thus we will be considering the bond problem.) The macroscopic conductivity Σ is then defined to be the configurational average of $\sigma N^{(2-d)/d}$, where $\sigma = I/V$ and I is the dc current which flows when the potential difference V is applied between two opposite $(d - 1)$-dimensional faces of the hypercube. We may define clusters as being groups of sites which are connected with respect to the unit (i.e., $\sigma_{ij} = 1$) conductances.

The statistics of cluster size and the associated pair connectedness correlation length $\xi(p)$ were shown by Kasteleyn and Fortuin[3] to be related to the thermodynamics of the s-state Potts model[4] in the limit $s \to 1$, if the identification $p = 1 - \exp(-J/kT)$ is made, where J is the nearest-neighbor interaction energy in the Potts model. The s-state Potts Hamiltonian can be written

$$\mathcal{H} = -Js \sum_{\langle ij \rangle} (\delta_{n_i, n_j} - 1) - H \sum_i (\delta_{1, n_i} - 1), \qquad (1)$$

where each of the Potts variables n_i can take on any positive integer value less than or equal to s. Note that if $s = 2$, Eq. (1) becomes the Ising Hamiltonian. The free energy per site is given by the standard prescription

$$F = -(kT/N)\ln[\,\mathrm{Tr}_{\{n_i\}} \exp(-\mathcal{H}/kT)\,]. \qquad (2)$$

If we make the identification for p given above,

then it turns out that the fraction of sites in the infinite cluster $P(p)$ is given by

$$P(p) = \frac{1}{s - 1} \left. \frac{\partial F}{\partial H} \right|_{s=1, H=0}, \qquad (3)$$

and the mean-square cluster size $S(p)$ by

$$S(p) = \frac{1}{s - 1} \left. \frac{\partial^2 F}{\partial H^2} \right|_{s=1, H=0}. \qquad (4)$$

From Eqs. (3) and (4), we see that $P(p)$ and $S(p)$ are analogous to the magnetization and the magnetic susceptibility, respectively, of a ferromagnet.

This relation indicates that the usual scaling-exponent description for critical points can be applied to the percolation threshold p_c, and that the various scaling relations and universality predictions can be expected to hold as well. Since the order parameter $P(p)$ is a probability $P(p) \geq 0$. Thus for $d \geq d_c = 6$, the exponents for cluster statistics near p_c are those of the constrained mean-field theory (MFT) for a positive order parameter with a positive cubic term in the free energy[5]: $\alpha = -1$, $\beta = 1$, $\gamma = 1$, and $\nu = \frac{1}{2}$.

In view of scaling arguments which relate the diluted resistor network and percolation problems, de Gennes[6] has suggested that $d_c = 6$ for the resistor network also. Here we present numerical evidence which confirms that this suggestion is correct. We also discuss a new scaling relation.

II. SERIES FOR THE DILUTED RESISTOR NETWORK

It is possible to determine d_c for the ferromagnetic[7,8] and spin-glass[9] Ising models by analyzing the high-temperature series expansions for the order parameter susceptibilities, as a function of the dimensionality d. This approach has been applied to the site percolation problem by Gaunt *et al*.[10] These studies show that an

analysis of the series based on the assumption of a simple power law scaling form will produce an estimate for the exponent which is too large near d_c, due to the confluent corrections.

We have carried out the same program for the diluted resistor network by analyzing low-concentration expansions for analogous susceptibilities. In order to do this, we must identify appropriate order parameters. For the undiluted ($p = 1$) network, Kasteleyn and Fortuin[3] have shown that the correlation function for the $s \to 0$ state Potts model yields the resistance R_{ij} between lattice points i and j. This correlation function can also be obtained by averaging over the Gaussian density matrix,

$$\rho = \exp\left(-\frac{1}{2}\sum_{m,n}\sigma_{mn}(x_m - x_n)^2\right).$$

These observations suggest that R_{ij} plays the role of a correlation function for resistor networks.[2] Therefore we define the resistive (r), percolative (p), and conductive (c) susceptibilities as

$$\chi_\alpha = \frac{1}{N}\sum_{i,j}\chi_{\alpha ij},$$

where

$$\chi_{rij} = [C_{ij}R_{ij}]_{av} - \frac{1}{N^2}\sum_{k,l}[C_{kl}R_{kl}]_{av}, \qquad (5a)$$

$$\chi_{pij} = [C_{ij}R_{ij}^0]_{av} - \frac{1}{N^2}\sum_{k,l}[C_{kl}R_{kl}^0]_{av}$$

$$= [C_{ij}]_{av} - P^2(p), \qquad (5b)$$

and

$$\chi_{cij} = [C_{ij}R_{ij}^{-1}]_{av} - \frac{1}{N^2}\sum_{k,l}[C_{kl}R_{kl}^{-1}]_{av}$$

$$= [R_{ij}^{-1}]_{av} - \frac{1}{N^2}\sum_{k,l}[R_{kl}^{-1}]_{av}. \qquad (5c)$$

The brackets $[\ \ ]_{av}$ denote a configurational average, and the cluster function C_{ij} is defined by

$$C_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are in the same cluster,} \\ 0 & \text{otherwise.} \end{cases} \qquad (6)$$

For Eq. (5a) we take $C_{ij}R_{ij} = 0$ if $C_{ij} = 0$. Note that $\chi = S(p)$, as defined by Eq. (4).

Our definition of χ_p is the same as that of Kasteleyn and Fortuin, and Harris et al.,[11] but it differs from that of Sykes et al.[12] This difference will not affect the exponents, although it will, of course, lead to series expansions which are different from those of Sykes et al.

The low-density series expansions for the various susceptibilities were calculated in a straightforward manner by the method of cumulants, as discussed extensively in Vol. III of the Domb and Green series.[13] The computational procedure is considerably simplified by the fact that, due to the properties of the cumulant expansion, the weight factor for each diagram is proportional to p^n, where n is the number of bonds in the diagram. Diagram weights are listed in Ref. 14. The series expansions for the susceptibilities χ_r, χ_p, and χ_c are given in Tables I, II, and III, respectively.

III. SCALING RELATIONS AND EXPONENTS

We define the conductivity exponent μ by $\Sigma \sim (p - p_c)^\mu$, for $p \to p_c^+$. The quantity L is defined to be the average resistance between two connected points which are separated by a percolation correlation length, ξ:

$$L = \frac{\chi_r(\xi)}{\chi_p(\xi)} = \frac{\chi_{rij}}{\chi_{pij}}, \text{ where } |x_i - x_j| = \xi. \qquad (7)$$

This definition of L makes sense both above and below p_c. For $p > p_c$ de Gennes[6] has called L the "resistance between nodes," because on a Cayley tree, nodes are spaced a distance ξ apart, on

TABLE I. Coefficients a_{nm} of the resistive susceptibility expansion defined by

$$\chi_r = 2\sum_{n,m \geq 1} a_{nm}d^n p^m.$$

n	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
1	1	−2	3	13.5	−75	−292.1$\bar{6}$	2642.2$\bar{6}$	20 298	−167 972.398	−1 955 630.113
2	0	4	−12	6.5	130	133.5	−5362.$\bar{3}$	−27 996.2$\bar{6}$	415 426.816	3 633 613.528
3	0	0	12	−48	30	302.$\bar{6}$	2272.4	2 883.$\bar{3}$	−329 864.541	−1 731 450.200
4	0	0	0	32	−160	150	710.$\bar{6}$	3 414.9$\bar{3}$	74 640.989	−101 778.107
5	0	0	0	0	80	−480	640	1 592	3 564.8	145 332.757
6	0	0	0	0	0	192	−1344	2 376	3 157.$\bar{3}$	−1 651.2
7	0	0	0	0	0	0	448	−3 584	7 968	4 693.$\bar{3}$
8	0	0	0	0	0	0	0	1 024	−9 216	24 800
9	0	0	0	0	0	0	0	0	2 304	−23 040
10	0	0	0	0	0	0	0	0	0	5 120

TABLE II. Coefficients b_{nm} of the percolative susceptibility expansion defined by

$$\chi_p = 1 + 2 \sum_{n,m \geq 1} b_{nm} d^n p^m.$$

n	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
1	1	−1	1	4	−17	−58	442	2999	$-22117\frac{1}{3}$	−231788
2	0	2	−4	1	30	31	−899	−4182	$54493\frac{2}{3}$	$431985\frac{2}{3}$
3	0	0	4	−12	4	56	394	498	$-43244\frac{2}{3}$	$-208743\frac{1}{3}$
4	0	0	0	8	−32	20	112	510	$9905\frac{1}{3}$	$-9310\frac{2}{3}$
5	0	0	0	0	16	−80	80	224	484	$16753\frac{1}{3}$
6	0	0	0	0	0	32	−192	272	416	−112
7	0	0	0	0	0	0	64	−448	832	640
8	0	0	0	0	0	0	0	128	−1024	2368
9	0	0	0	0	0	0	0	0	256	−2304
10	0	0	0	0	0	0	0	0	0	512

the average. However, the concept of a node seems somewhat ambiguous in networks which are not trees, and it will not be used here.

We define the exponent ζ by $L \sim (p_c - p)^{-\zeta}$, for $p < p_c$. We expect that L must grow at least as fast as ξ, since clusters become increasingly ramified[15(a)] as p approaches p_c, so that in the critical region the contribution of independent parallel paths can be neglected.[15(b)] Thus, the average resistance between two connected points will increase at least as fast as the distance between them. This means that $\zeta \geq \nu$, where ν is defined, as usual, by $\xi \sim (p_c - p)^{-\nu}$. We also expect that L cannot be greater than the average length of a self-avoiding walk between two points separated by a distance ξ. This gives us an upper bound $\zeta \leq \nu/\nu_s$, where ν_s is the correlation length exponent for self-avoiding walks. Thus, ζ must

satisfy the relation[16]

$$\nu \leq \zeta \leq \nu/\nu_s. \tag{8}$$

We also know that $\nu = \zeta$ for $d = 1$, and that $\zeta = \nu/\nu_s = 1$ for $d \geq 6$, because the self-avoiding walk approximation (which is the appropriate MFT for this problem) gives the correct exponents for $d \geq d_c = 6$.

Following standard procedure, we define ζ' by $L \sim (p - p_c)^{-\zeta'}$, for $p > p_c$. Straley[17] has argued that a scaling picture of the usual sort is valid for this problem; therefore we expect that $\zeta' = \zeta$. Stinchcombe[18] has verified that this relation is true for MFT, by showing that $\zeta' = 1$ for a Cayley tree. De Gennes[6] has shown that $\Sigma \sim L^{-1}\xi^{2-d}$, which gives us the desired scaling relation,

$$\mu = \zeta + (d - 2)\nu. \tag{9}$$

TABLE III. Coefficients c_{nm} of the conductive susceptibility expansion defined by

$$\chi_c = 2 \sum_{n,m \geq 1} c_{nm} d^n p^m. \text{[a]}$$

n	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
1	1	−0.5	$0.\overline{3}$	0.25	$-3.\overline{095238}$	$1.\overline{215}$	$47.\overline{935064}$	−1.7736616	−1689.5450467	−1614.9105798
2	0	1	$-1.\overline{3}$	1	$5.390\overline{4761}$	$-12.50\overline{3}$	$-94.1\overline{307359}$	212.1830338	4069.5797606	−5017.3664644
3	0	0	$1.\overline{3}$	−3	$1.10\overline{47619}$	$16.1\overline{93}$	30.4580086	−374.2571215	−2952.7798552	16272.9663969
4	0	0	0	2	−6.4	$3.2\overline{60}$	$23.4\overline{23376}6$	133.8571663	365.7296312	−12434.9013057
5	0	0	0	0	3.2	$-13.\overline{3}$	$10.7\overline{428571}$	37.3949580	142.2575232	2551.1905462
6	0	0	0	0	0	$5.\overline{3}$	$-27.\overline{428571}$	32.1604010	61.8325211	102.2857397
7	0	0	0	0	0	0	$9.\overline{142857}$	−56	88.3699103	91.6504818
8	0	0	0	0	0	0	0	16	$-113.\overline{7}$	$228.385\overline{1}$
9	0	0	0	0	0	0	0	0	$28.\overline{4}$	−230.4
10	0	0	0	0	0	0	0	0	0	51.2

[a] Bars over numbers indicate repeating decimal fractions.

Equation (9) has also been derived by other methods.[2,19] From the definitions (5) and (7), we see that $\chi_r = L\chi_p$ and $\chi_c = L^{-1}\chi_p$, so that we have

$$\gamma_r = \gamma_p + \zeta , \tag{10a}$$

and

$$\gamma_c = \gamma_p - \zeta , \tag{10b}$$

where the exponents γ_α are defined by $\chi_\alpha \sim (p_c - p)^{-\gamma_\alpha}$.

We have used our susceptibility series expansions to obtain γ_r, γ_p, and γ_c directly, and thus check the validity of the scaling relations. The exponents were calculated by both Padé and ratio methods, and, in general, there was good agreement between the two techniques. Best values for these exponents are shown in Table IV. We also display values of the exponent ζ, computed by using the scaling relations Eqs. (10). The numbers in Table IV were computed by assuming the simple power law scaling form $\chi \sim (p - p_c)^{-\gamma}$; thus, the deviations of the exponents from their MFT values for $d \geq 6$ are believed to be spurious.

Our values for γ_p are in excellent agreement with those of Gaunt et al.[10] For $d \geq 3$, our confidence limits are significantly better than theirs. This is possible because the large value of γ_r allows a more precise determination[2,14] of p_c, thus substantially reducing the primary source of uncertainty. It is also probably significant that γ_p is indeed given by the average of γ_r and γ_c, as required by Eqs. (10a) and (10b).

Priest and Lubensky,[20] and Amit[21] have used the connection between these lattice statistics problems and the Potts model to compute ϵ expansions to order ϵ^2 for the various exponents, where $\epsilon = 6 - d$. They find $\gamma_p = 1.19$ for $d = 5$. On this

basis, we feel that there is excellent agreement between the values of γ_p derived from the series expansion and the ϵ expansion, contrary to the opinion expressed by Gaunt et al.,[10] and Kirkpatrick.[22] Recently, Dasgupta et al.[23] have also computed $\zeta = 1 + O(\epsilon^3)$, and Wallace and Young[24] have extended this result to show that $\zeta = 1$ to all orders of perturbation theory near $d = 6$. Our calculations agree with this result, too. They are also consistent with the conjecture that $\zeta = 1$ for $d \geq 4$.

Now we combine our series expansion results with the scaling relation, Eq. (9). In Table IV we list what we believe are best values of the correlation length exponent ν, based on all information available to us.[25] We also know that $\nu_s = 0.75$ for $d = 2$,[26] $\nu_s = 0.588$ for $d = 3$,[27] and $\nu_s = \frac{1}{2}$ for $d \geq 4$.[28] Thus, our values for ζ satisfy the inequalities, Eq. (4). Using Eq. (9) we obtain values for the conductivity exponent; these are listed in Table IV. We find that $\mu = 1.43 \pm 0.02$ for $d = 2$, and $\mu = 1.95 \pm 0.03$ for $d = 3$. Our result for $d = 3$ is in excellent agreement with the experiment of Abeles et al.,[29] who measured $\mu = 1.9 \pm 0.2$ in amorphous $W - Al_2O_3$ cermet films. By building models of resistor networks, Watson and Leath[30] obtained $\mu = 1.38 \pm 0.12$ for $d = 2$, and Adler et al.[31] found $\mu = 2$ for $d = 3$. However, extensive computer simulations by Kirkpatrick[32] and by Straley[33] give $\mu = 1.1 \pm 0.1$ and $\mu = 1.10 \pm 0.05$ for $d = 2$, and $\mu = 1.6 \pm 0.1$ and $\mu = 1.70 \pm 0.05$ for $d = 3$, respectively, in disagreement with our results. Their $d = 2$ results are difficult to reconcile with the scaling theory, since inserting $\mu = 1.1$ into Eq. (9) gives $\zeta = 1.1$ which does not satisfy $\zeta \geq \nu$.

We believe that the discrepancy between our results and those of Kirkpatrick and Straley is associated with the fact that our values are obtained via a low-density series expansion, whereas theirs are obtained by using data for $p > p_c$. We remind the reader that Sykes et al.[34] found that high-density series expansions for the percolation problem are poorly behaved. This would lead one to expect that corrections to the asymptotic scaling behavior should be unusually large for $p > p_c$, thus making the extraction of critical exponents from the data of Kirkpatrick and Straley extremely difficult. In view of the connection[3,23] with the Potts model, we believe that it is unlikely that the high-density exponents actually differ from their low-density counterparts.

TABLE IV. Exponents for percolation conductivity.

d	2	3	4	5	6	MFT
γ_r [a]	3.8	2.78	2.45	2.19	2.09 [d]	2
γ_p [a]	2.42	1.66	1.40	1.17	1.08 [d]	1
γ_c [a]	0.99	0.54	0.35	0.15	0.07 [d]	0
ζ [a]	1.43	1.12	1.05	1.02	1.01 [d]	1
ν [b]	1.34	0.83	0.66	0.57	0.50	$\frac{1}{2}$
μ [c]	1.43	1.95	2.37	2.73	3.01 [d]	3

[a] This work; the error bars are about ± 2 in the last decimal place quoted.

[b] These numbers represent the authors subjectively weighted average of the available data (see Ref. 25).

[c] Calculated via the scaling relation, $\mu = \zeta + (d - 2)\nu$.

[d] These values do not take confluent corrections into account. In fact, we believe that the MFT values are correct for $d = 6$.

IV. SUMMARY

In this paper we analyzed the properties of randomly diluted hypercubic resistor networks

near the percolation threshold p_c as a function of the probability p that each resistor is present. The following results were obtained.

(i) A resistive susceptibility χ_r and a conductive susceptibility, χ_c, were defined. Low-density series expansions for these functions and for the percolative susceptibility χ_p were computed to order p^{10}.

(ii) Ratio and Padé techniques were applied to the series expansions, and values of the susceptibility exponents γ_r, γ_c, and γ_p were obtained as functions of d. For d near six, there is good agreement between these results and the results of ϵ expansion calculations.

(iii) The characteristic resistance L was defined as $L = \chi_r(\xi)/\chi_p(\xi)$, where ξ is the percolation correlation length. It follows from this definition that the exponent ζ, associated with the divergence of L at p_c is given by $\zeta = \gamma_r - \gamma_p = \gamma_p - \gamma_c$. The values

of ζ obtained via this relation from the series expansions obey the inequalities $\nu \leqslant \zeta \leqslant \nu/\nu_s$, where ν and ν_s are the correlation length exponents for percolation and for self-avoiding walks, respectively. For d near 6, there is little or no dependence of ζ on d.

(iv) By using the scaling relation $\mu = \zeta + (d-2)\nu$ values of the conductivity exponent μ were obtained.

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