Relevance of ϕ^4 operators in the Edwards-Anderson model

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Field-theoretical formulations of the spin-glass problem possess a symmetry which permits an invariant interaction of third order in the fluctuating fields. In the renormalization-group program one is naturally led to look for infrared-stable fixed points which yield ϵ expansions in $d = 6 - \epsilon$ dimensions. Naive dimensional analysis suggests that quartic interactions will become relevant in four dimensions, hence, precluding the use of ϵ-expansion techniques to describe physics in three dimensions. We show that indeed the anomalous dimensions of quartic operators are such that for Ising and X-Y systems and ϵ expansion around six dimensions should not be extrapolated down to three dimensions. However, for a Heisenberg system the quartic interactions remain irrelevant.

Following the original paper of Edwards and Anderson,[1] Harris, Lubensky, and Chen[2] have recently proposed a field-theoretic model for the paramagnetic to spin-glass transition. Correlation functions are calculated for the following reduced Hamiltonian in the limit $n=0$:

$$H=\int d^dx\left(\frac{1}{4}\sum_{\substack{\alpha,\beta\\ i,j}} Q_{ij}^{\alpha\beta}(x)(-\nabla^2+m^2)Q_{ij}^{\alpha\beta}(x) + \frac{u_0}{3!}\Lambda^{\epsilon/2}\sum_{\substack{\alpha,\beta,\gamma\\ i,j,k}} Q_{ij}^{\alpha\beta}(x)Q_{jk}^{\beta\gamma}(x)Q_{ki}^{\gamma\alpha}(x) + O(Q)^4\right),$$

where

$$Q_{ij}^{\alpha\alpha}=0,\quad \alpha=1,2,\ldots,n;\quad i,j=1,2,\ldots,m,$$
$$Q_{ij}^{\alpha\beta}=Q_{ji}^{\beta\alpha},\quad \alpha,\beta=1,2,\ldots,n;\quad i,j=1,2,\ldots,m. \qquad (1)$$

The propagator for such a field theory has the following form:

$$\langle Q_{ij}^{\alpha\beta}Q_{kl}^{\gamma\delta}\rangle=[1/(q^2+m^2)]X_{ijkl}^{\alpha\beta\gamma\delta},$$
$$\alpha,\beta,\gamma,\delta=1,2,\ldots,n, \qquad (2)$$
$$i,j,k,l=1,2,\ldots,m,$$

where

$$X_{ijkl}^{\alpha\beta\gamma\delta}=\delta_{ik}^{\alpha\gamma}\delta_{jl}^{\beta\delta}+\delta_{il}^{\alpha\delta}\delta_{jk}^{\beta\gamma}-T^{\alpha\beta\gamma\delta}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),$$

and

$$\delta_{ij}^{\alpha\beta}=\delta^{\alpha\beta}\delta_{ij},\quad T^{\alpha\beta\gamma\delta}=\begin{cases}1 \text{ if } \alpha=\beta=\gamma=\delta,\\ 0 \text{ otherwise}.\end{cases} \qquad (3)$$

A calculation on the lines of Brezin, Le Guillou, and Zinn-Justin[3] (BLZ) using renormalized perturbation theory, the necessary integrals having been performed by Amit,[4] yields a stable nontrivial fixed point involving the trilinear interaction only, with fixed-point coupling

$$K_d(u^*)^2=\frac{\epsilon}{4(2m-1)}+\frac{\epsilon^2}{288(2m-1)^3}\times(425m^2-696m+117)+O(\epsilon^3), \qquad (4)$$

where $n=0$ and $K_d=S_d/2(2\pi)^d=1/2^d\Gamma(\frac{1}{2}d)\pi^{d/2}$.

The exponents α and β of particular interest have the following form:

$$\alpha=-1-\frac{(3m+1)\epsilon}{2(2m-1)}+\frac{m(363m^2+1192m-63)\epsilon^2}{144(2m-1)^3}+O(\epsilon^3),$$

$$\beta=1+\frac{(m+1)\epsilon}{4(2m-1)}-\frac{m(373m^2+580m-9)\epsilon^2}{288(2m-1)^3}+O(\epsilon^3).$$

It should be remembered that due to the nature of the $n=0$ technique, β governs the susceptibility χ, that is

$$\chi\sim 1+\lim_{n\to 0}\sum_{\alpha,\beta,i}\langle Q_{ii}^{\alpha\beta}\rangle,$$

i.e.,

$$\chi\sim\begin{cases}1+|t|^\beta, & t<0,\\ 1, & t>0,\end{cases}$$

where t is the reduced temperature.

To gauge the relevance of this simple model to physics in three dimensions we investigate the anomalous dimensions of the quartic interactions. The analogous problem for percolation has been studied by Amit, Wallace, and Zia[5] (AWZ). To calculate the anomalous dimensions we must simultaneously renormalize all operators of the same or lower naive dimension. Operators which are total derivatives may be neglected and using dimensional regularization we need only consider the eight linearly independent operators

$$
\begin{aligned}
A_1(x) &= (1/4!)Q_{ij}^{\alpha\beta}Q_{jk}^{\beta\gamma}Q_{kl}^{\gamma\delta}Q_{li}^{\delta\alpha}\,,\\
A_2(x) &= (1/4!)Q_{ij}^{\alpha\beta}Q_{ij}^{\alpha\beta}Q_{kl}^{\gamma\delta}Q_{kl}^{\gamma\delta}\,,\\
A_3(x) &= (1/4!)Q_{ij}^{\alpha\beta}Q_{ij}^{\alpha\beta}Q_{kl}^{\alpha\gamma}Q_{kl}^{\alpha\gamma}\,,\\
A_4(x) &= (1/4!)Q_{ij}^{\alpha\beta}Q_{kj}^{\alpha\beta}Q_{kl}^{\alpha\gamma}Q_{il}^{\alpha\gamma}\,,\\
A_5(x) &= (1/4!)Q_{ij}^{\alpha\beta}Q_{ij}^{\alpha\beta}Q_{kl}^{\alpha\beta}Q_{kl}^{\alpha\beta}\,,\\
A_6(x) &= (1/4!)Q_{ij}^{\alpha\beta}Q_{kj}^{\alpha\beta}Q_{kl}^{\alpha\beta}Q_{il}^{\alpha\beta}\,,\\
A_7(x) &= (1/3!)Q_{ij}^{\alpha\beta}Q_{jk}^{\beta\gamma}\Box Q_{ki}^{\gamma\alpha}\,,\\
A_8(x) &= (1/2!)Q_{ij}^{\alpha\beta}\Box^2 Q_{ij}^{\alpha\beta}\,,
\end{aligned}
\tag{5}
$$

where summation over the internal indices is to be understood, and

$$\Box = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\mu}\,.$$

Now the N-point vertex function with an insertion of the operator A_a, $a=1,2,\ldots,8$ at zero momentum, denoted by $\Gamma_a^N(q,u_0,\Lambda)$ is not in general multiplicatively renormalizable. Following AWZ we define dimensionless multiplicatively renormalizable vertex functions $\hat{\Gamma}_a^{\{b\}}$ by

$$\sum_{k=1}^{6} S_{i_1\cdots i_8}^{\alpha_1\cdots\alpha_8}\big|^{(k)}\,\hat{\Gamma}_a^{\{k\}} = \Gamma_a^{(4)}(\{q_i\},u_0,\Lambda)\,, \tag{6a}$$

$$R_{i_1\cdots i_6}^{\alpha_1\cdots\alpha_6}\,\hat{\Gamma}_a^{\{7\}} = \left(\frac{\partial}{\partial q_1^2}+\frac{\partial}{\partial q_2^2}+\frac{\partial}{\partial q_3^2}\right) \times \Gamma_a^{(3)}(\{q_i\},u_0,\Lambda)\,, \tag{6b}$$

$$X_{i_1\cdots i_4}^{\alpha_1\cdots\alpha_4}\,\hat{\Gamma}_a^{\{8\}} = \left(\frac{\partial}{\partial q_1^2}\frac{\partial}{\partial q_2^2}\right)\Gamma_a^{(2)}(\{q_i\},u_0,\Lambda)\,, \tag{6c}$$

where

$$
\begin{aligned}
S_{i_1\cdots i_8}^{\alpha_1\cdots\alpha_8}\big|^{(1)} &= X_{i_1i_2ij}^{\alpha_1\alpha_2\alpha\beta}X_{i_3i_4jk}^{\alpha_3\alpha_4\beta\gamma}X_{i_5i_6kl}^{\alpha_5\alpha_6\gamma\delta}X_{i_7i_8li}^{\alpha_7\alpha_8\delta\alpha}\big|_{\text{sym.}}\,,\\
S_{i_1\cdots i_8}^{\alpha_1\cdots\alpha_8}\big|^{(2)} &= X_{i_1i_2ij}^{\alpha_1\alpha_2\alpha\beta}X_{i_3i_4ij}^{\alpha_3\alpha_4\alpha\beta}X_{i_5i_6kl}^{\alpha_5\alpha_6\gamma\delta}X_{i_7i_8kl}^{\alpha_7\alpha_8\gamma\delta}\big|_{\text{sym.}}\,,\\
S_{i_1\cdots i_8}^{\alpha_1\cdots\alpha_8}\big|^{(3)} &= X_{i_1i_2ij}^{\alpha_1\alpha_2\alpha\beta}X_{i_3i_4ij}^{\alpha_3\alpha_4\alpha\beta}X_{i_5i_6kl}^{\alpha_5\alpha_6\alpha\gamma}X_{i_7i_8kl}^{\alpha_7\alpha_8\alpha\gamma}\big|_{\text{sym.}}\,,\\
S_{i_1\cdots i_8}^{\alpha_1\cdots\alpha_8}\big|^{(4)} &= X_{i_1i_2ij}^{\alpha_1\alpha_2\alpha\beta}X_{i_3i_4kj}^{\alpha_3\alpha_4\alpha\beta}X_{i_5i_6kl}^{\alpha_5\alpha_6\alpha\gamma}X_{i_7i_8il}^{\alpha_7\alpha_8\alpha\gamma}\big|_{\text{sym.}}\,,\\
S_{i_1\cdots i_8}^{\alpha_1\cdots\alpha_8}\big|^{(5)} &= X_{i_1i_2ij}^{\alpha_1\alpha_2\alpha\beta}X_{i_3i_4ij}^{\alpha_3\alpha_4\alpha\beta}X_{i_5i_6kl}^{\alpha_5\alpha_6\alpha\beta}X_{i_7i_8kl}^{\alpha_7\alpha_8\alpha\beta}\big|_{\text{sym.}}\,,\\
S_{i_1\cdots i_8}^{\alpha_1\cdots\alpha_8}\big|^{(6)} &= X_{i_1i_2ij}^{\alpha_1\alpha_2\alpha\beta}X_{i_3i_4kj}^{\alpha_3\alpha_4\alpha\beta}X_{i_5i_6kl}^{\alpha_5\alpha_6\alpha\beta}X_{i_7i_8il}^{\alpha_7\alpha_8\alpha\beta}\big|_{\text{sym.}}\,,\\
R_{i_1\cdots i_6}^{\alpha_1\cdots\alpha_6} &= X_{i_1i_2ij}^{\alpha_1\alpha_2\alpha\beta}X_{i_3i_4jk}^{\alpha_3\alpha_4\beta\gamma}X_{i_5i_6ki}^{\alpha_5\alpha_6\gamma\alpha}\big|_{\text{sym.}}\,,
\end{aligned}
\tag{7}
$$

where sym. means the tensors are to be symmetrized in the external indices.

Renormalized vertex functions $\hat{\Gamma}_{Ra}^{\{b\}}$ are now defined by

$$\hat{\Gamma}_{Ra}^{\{b\}}(\{q_i\},u,\mu) = Z_{ac}Z^{l_b/2}\Gamma_c^{\{b\}}(\{q_i\},u_0,\Lambda)\,,$$

where l_b is the number of external legs on the vertex function $\hat{\Gamma}_c^{\{b\}}$. Z is the conventional wave-function renormalization constant, given in terms of the renormalized coupling constant u (a factor K_d has been absorbed into u^2):

$$Z = 1 + (4m/3\epsilon)u^2 + O(u^4)\,.$$

The matrix Z_{ab} is determined by imposing the renormalization conditions

$$\hat{\Gamma}_{Ra}^{\{b\}}\big|_{\text{sym.pt.}} = \delta_a^b$$

where our convention for the symmetry point is external momenta

$$q_i^2 = \mu^2\,.$$

With these normalization conditions the renormalization-group (RG) equation satisfied by $\hat{\Gamma}_a^{\{b\}}$ can be obtained in the usual way:

$$\left[\left(\mu\frac{\partial}{\partial\mu}+\beta(u)\frac{\partial}{\partial u}-\tfrac{1}{2}l_d\,\eta(u)-\frac{\epsilon}{2}(4-l_d)\right)\times\delta_{ab}-\gamma_{ab}(u)\right]\hat{\Gamma}_{Rb}^{\{d\}}=0\,, \tag{8}$$

where

$$\gamma_{ab}(u) = -Z_{ac}\,\mu\frac{d}{d\mu}\bigg|_{u_0,\Lambda}Z_{cb}^{-1} + \frac{\epsilon}{2}\sum_c Z_{ac}(l_c-4)Z_{cb}^{-1}\,,$$

$$\beta(u) = \mu\frac{d}{d\mu}\bigg|_{u_0,\Lambda}u\,,$$

$$\eta(u) = \mu\frac{d}{d\mu}\bigg|_{u_0,\Lambda}\ln Z\,.$$

To illustrate the conditions for irrelevance of the quartic operators we follow AWZ and solve the RG equation at the fixed point u^* for the vertex function $\Gamma_a^{(4)}$:

$$\Gamma_\alpha^{(4)}(\{q_i\},u^*,\mu) = (\mu/k)^{2\eta+\lambda_\alpha}g(q_i/k)\,,$$

where α denotes an insertion of a linear combination of the operators A_a, $a=1,2,\ldots,8$ corresponding to the eigenvalue λ_α of the matrix $\gamma_{ab}(u^*)$. On the other hand the solution of the RG equation for the vertex function $\Gamma^{(4)}$ itself (i.e., without ϕ^4 insertions) yields, at the fixed point,

$$\Gamma^{(4)}(\{q_i\},u^*,\mu) = (\mu/k)^{2\eta}k^{-2+\epsilon}f(q_i/k)\,.$$

Hence, irrelevance, i.e., the condition that as $k\to 0$, q_i/k finite, $\Gamma^{(4)}$ dominates $\Gamma_\alpha^{(4)}$ for all eigenvalues λ_α, $\alpha=1,2,\ldots,8$ is simply expressed by

$$2-\epsilon-\lambda_\alpha > 0\,. \tag{9}$$

Our task is thus simply to evaluate the eigenvalues λ_α. A straightforward, although lengthy, calculation yields the following form for γ_{ab} (to first order):

$$\gamma = \begin{bmatrix} -\frac{8}{3}(8m-9)u^2 & 16u^2 & 8u^2 & 8u^2 & -40u^2 & 8(m-4)u^2 & \frac{2}{3}(1-2m)u & 0 \\ 64u^2 & -\frac{40}{3}mu^2 & 0 & 0 & 0 & -64u^2 & \frac{8}{3}u & 0 \\ 0 & 0 & -\frac{16}{3}mu^2 & -16u^2 & 8mu^2 & 16u^2 & 0 & 0 \\ 0 & 0 & -8u^2 & -\frac{8}{3}(2m+3)u^2 & 8u^2 & 8(m+1)u^2 & 0 & 0 \\ 8u^2 & 4mu^2 & 0 & -16u^2 & -\frac{40}{3}mu^2 & 8u^2 & \frac{2}{3}u & 0 \\ 4(m+1)u^2 & 4u^2 & -8u^2 & -8u^2 & 0 & \frac{4}{3}(3-7m)u^2 & \frac{1}{3}(m+1)u & 0 \\ 64(m-1)u^3 & -48u^3 & -64u^3 & -64u^3 & 160u^3 & -32(m-4)u^3 & \frac{2}{3}u^2-\epsilon/2 & -\frac{8}{9}mu \\ -96(m-1)u^4 & 72u^4 & 96u^4 & 96u^4 & 240u^4 & 48(m-4)u^4 & 2(1-3m)u^3 & \frac{4}{3}mu^2-\epsilon \end{bmatrix}, \quad (10)$$

where the limit $n \to 0$ has already been taken and m is the dimension of the magnetic order parameter of the pure system.

As with AWZ the first two eigenvalues of (10) may be determined exactly using two identities derived from the equation of motion at the fixed point; they are $\lambda_1 = -\epsilon$, $\lambda_2 = -\frac{1}{3}\epsilon[(7m-3)/(2m-1)]$ and both irrelevant. To obtain the other eigenvalues we computed for the special cases $m = 1, 2, 3, 4, 5, 6$. The results are (to two decimal places)

$$m=1:\ \lambda/\epsilon = 0.67, 0.67, 0.15 \pm i\,5.08, -3.33, -8.96,$$

$$m=2:\ \lambda/\epsilon = 0.02, -0.53, -0.84 \pm i2.54, -1.85, -4.61,$$

$$m=3:\ \lambda/\epsilon = -0.17, -0.76, -1.05 \pm i\,1.86, -1.56, -3.68, \quad (11)$$

where all λ's are negative for larger values of m.

Of course these eigenvalues are correct only to order ϵ; higher orders in ϵ will surely be important. The strongest statement one may make concerns the signs of the λ's. They determine if the corresponding operators have become relevant as d is lowered towards four, where the naive dimension of the ϕ^4 operators is zero ($\epsilon = 2$), according to Eq. (9). The appearance of operators with positive real parts in the results (11) for $m = 1$ and 2 indicates that the corresponding eigenoperators are likely to become relevant by four dimensions. The ϵ expansions can therefore not be trusted down to $d = 3$ for Ising and X-Y spin glasses.[6] Since eigenvalues are negative for $m \geq 3$, Heisenberg models may escape this difficulty.

In order to predict reliable exponents for $d = 3$ we require knowledge of the behavior of perturbation theory at high order. Calculations on the lines of Lipatov and Brezin *et al.*[7] to determine such structure, however, run into difficulties. We may only note that, although the series is at best asymptotic, at least to $O(\epsilon^3)$ the signs alternate. One may therefore still entertain the hope that this model can describe the Heisenberg spin glass.

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[6] A. J. Bray, M. A. Moore, and P. Reed [J. Phys. C 11, 1187 (1978)], on the basis of Monte Carlo simulations, suggested that the order parameter for the Edwards-Anderson Ising spin glass does not exist in $d = 3$. Our calculation would lead us to suspect the validity of drawing conclusions from an ϵ expansion extrapolated down to three dimensions for an Edwards-Anderson Ising spin glass. Hence, our treatments are in a sense complementary.

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