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Field-theory renormalization and critical dynamics above T_c: Helium, antiferromagnets, and liquid-gas systems

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The Martin-Siggia-Rose field theories associated with the critical dynamics of mode-coupling systems, are renormalized in a way which decouples statics from dynamics, through the minimal-renormalization procedure. This simplifies considerably the computations which are thus carried out to two-loop order for exponent and Wilson functions. We obtain all dynamic transients for helium, $O(n)$ symmetric antiferromagnets, and liquid-gas systems. We also discuss the relevance for helium of a fixed point where dynamic scaling is weakly violated.

INTRODUCTION AND SUMMARY

The study of the dynamics of critical systems using the Wilson renormalization group[1] has now been developing for several years, both for purely relaxational systems[2,3] and for systems involving reversible mode coupling.[3-8] A complete account of this approach is given in a recent review article by Hohenberg and Halperin.[9] At the same time, field-renormalization techniques have proven successful in describing the critical behavior of static systems. As developed in particular by the Saclay group,[10] they provide a clean way of demonstrating scaling properties, and a relatively easy access to higher-order computations of asymptotic behaviors near T_c or corrections thereof. Their extension is relatively simple for purely relaxational systems[11,12] [models A,B,C of Ref. (8)]. The dynamics produced by the stochastic Langevin equations of motion obeyed by the fields φ_j that describe the system (order parameters, conserved quantities) may be obtained from a simple Lagrangian.[13] Since this Lagrangian only involves the field φ_j itself, its renormalization closely follows the statics. The set-up is more complicated for systems with mode coupling. The Lagrangian now involves both φ_j and a conjugate field $\hat{\varphi}_j$ as introduced by Martin, Rose, and Siggia,[14] and the basic building blocks of the theory (Green's functions of the fields φ_j, $\hat{\varphi}_j$) no longer identify with physical response functions. This is a source of difficulties both: (i) in setting up a properly renormalized theory and (ii) in sorting out a renormalization scheme simple enough to allow for tractable computations. Indeed a straightforward extension of standard renormalization procedures[15-17] leads to a renormalized theory whose static properties depend upon purely dynamic parameters. The discussion of the renormalization-group equations, and in particular of stability, becomes therefore unnecessarily complicated.

In this paper we show how these difficulties are overcome by use of the minimal-renormalization procedure.[18] We apply the technique to helium, $O(n)$ symmetric antiferromagnets, and liquid-gas systems above T_c. For these systems we compute[19] all transient exponents that govern corrections to the critical asymptotic behavior. For helium, besides the dynamic scaling fixed point of Halperin, Hohenberg, and Siggia,[4] we find a new fixed point corresponding to a vanishing value of the ratio transport over kinetic coefficient. We discuss the relevance for helium of that new fixed point which entails a weak violation of dynamic scaling. We compute in particular the thermal conductivity λ, its anomalous asymptotic behavior, and its main correction term which introduces a new universal combination of amplitudes. Precise measurements of λ should be able to tell which fixed point is relevant for helium.

In Sec. II we start from stochastic Langevin equations of motion and briefly derive the associated Martin-Siggia-Rose action for a general probability law of the noise field. We also give relationships between response and correlation functions.

In Sec. III we develop the renormalization program for the symmetric model (model E). The point is to segregate into a few renormalization functions Z all the divergent behavior (divergent when the cutoff Λ goes to infinity in four dimensions, or in the choice we make of the dimensionally regularized theory, when $\epsilon = 4 - d$ goes to zero). For this we list from power counting all one-irreducible vertices that are primitively divergent and introduce as many Z functions as necessary to absorb them. The same is done on response functions which provide the connection with statics. The Z functions contain some arbitrariness which is usually removed by imposing renormalization conditions. Here we choose instead minimal renormalization which decouples statics from dynamics.

In Sec. IV we use the renormalized theory to get the Callan-Symanzik equations that govern the critical behavior, and construct the Wilson and exponent functions. The fixed points (zeros of Wilson functions) are exhibited and their stability is discussed, both for the dynamic scaling fixed point of Halperin, Hohenberg, and Siggia (HHS) and for the weak-scaling fixed point. Transients are given up to ϵ^2. The behavior away from T_c is examined and given for characteristic frequencies and kinetic and transport coefficients. Deviations from the symmetric model are discussed. Thermal conductivity is considered in more detail and the amplitude of the main transient term is obtained.

In Sec. V we give results for the $O(n)$ symmetric antiferromagnet. The liquid-gas system is finally considered and its dynamic transient given to order ϵ^2.

I. MODELS AND LAGRANGIANS

A. Martin-Siggia-Rose Lagrangian

Let us consider systems governed by the stochastic equation

$$\frac{\partial\varphi_j(t)}{\partial t}=-(\Gamma_0)_{jk}\frac{\delta\mathcal{H}\{\varphi\}}{\delta\varphi_k(t)}+V_j\{\varphi(t)\}+\theta_j(t) \quad . \tag{1.1}$$

Here j stands for the component index and space, $\varphi_j(t)$ is either the order-parameter field or any coupled conserved field, $\mathcal{H}\{\varphi\}$, is the Landau-Ginzburg free-energy functional, $(\Gamma_0)_{jk}$ is the kinetic matrix, and $\theta_j(t)$ a stochastic variable governed by a (Gaussian) probability law $\mathcal{P}\{\theta\}$ whose autocorrelation matrix is $2(\Gamma_0)_{jk}\delta(t-t')$. Besides $V_j\{\varphi\}$ is the streaming term which obeys the conservation condition (repeated indices are summed over),[20]

$$\frac{\delta}{\delta\varphi_j}V_j\{\varphi\}e^{-\mathcal{H}\{\varphi\}}=0 \quad , \tag{1.2}$$

insuring that $e^{-\ \{\varphi\}}$ is a stationary probability distribution. In particular, the mode-coupling term of Kawasaki[21] and Kadanoff-Swift[22] has the form

$$V_j\{\varphi\}=(\delta/\delta\varphi_i)Q_{ji}\{\varphi\}-Q_{ji}\{\varphi\}(\delta\mathcal{H}\{\varphi\}/\delta\varphi_i) \quad , \tag{1.3}$$

where Q_{ij} is built from Poisson brackets or commutators of φ_j's and satsifies

$$Q_{ij}=-Q_{ji} \quad . \tag{1.4}$$

Clearly (1.3) and (1.4) imply the conservation condition (1.2) for the probability current $V_j e^{-\mathcal{H}\{\varphi\}}$.

Instead of solving (1.1) for $\varphi_j\{\theta\}$ and computing correlation functions as averages of products of $\varphi_j\{\theta\}$ over the probability weight $\mathcal{P}\{\theta\}$, it is convenient to write an action which generates exactly the same correlation functions by resorting to the set of conjugate variables φ_j, $\hat{\varphi}_j$ of Martin, Siggia, and Rose.[14] Indeed one may replace (1.1) by the stochastic generating functional

$$Z_\theta\{l\}=\int\mathcal{D}\varphi_j(t)\exp\left(\int dt\, l_j(t)\varphi_j(t)\right)\times\prod_{j,t}\delta\left(\frac{\partial\varphi_j(t)}{\partial t}+K_j\{\varphi(t)\}-\theta_j(t)\right)J\{\varphi\} \quad , \tag{1.5}$$

where

$$K_j\{\varphi\}=-(\Gamma_0)_{jk}\frac{\partial\mathcal{H}}{\partial\varphi_k}+V_j\{\varphi\} \quad ,$$

and the δ functions insure that (1.1) is always satisfied. $J\{\varphi\}$ is the Jacobian associated with the argument of the δ function

$$J\{\varphi\}=\exp\left(-\frac{1}{2}\int dt\,\frac{\delta K_j\{\varphi\}}{\delta\varphi_j(t)}\right) \quad . \tag{1.6}$$

That is, we have

$$Z_\theta\{l\}=\int\mathcal{D}\varphi(t)\mathcal{D}\hat{\varphi}(t)\times\exp\left[\int dt\, l_j(t)\varphi_j(t)+i\hat{\varphi}_j\times\left(\frac{\partial\varphi_j(t)}{\partial t}+K_j\{\varphi\}-\theta_j\right)-\frac{1}{2}\frac{\delta K_j}{\delta\varphi_j(t)}\right] \quad .$$

Taking the average over θ_j with the probability law $\mathcal{P}\{\theta\}$ yields the functional which generates the correlation functions

$$Z\{l\}=\int\mathcal{D}\theta_j(t)\mathcal{P}\{\theta\}\frac{Z_\theta\{l\}}{Z_\theta\{0\}} \quad . \tag{1.7}$$

If one remembers that, from (1.5), $Z_\theta\{0\}=1$, the integration over θ_j is readily done to yield

$$\begin{aligned}Z\{l\}&\equiv\int\mathcal{D}\varphi_j(t)\mathcal{D}\hat{\varphi}_j(t)\exp\left(\mathcal{a}\{\varphi,\hat{\varphi}\}+\int dt\, l_j(t)\varphi_j(t)\right)\\&=\int\mathcal{D}\varphi_j(t)\mathcal{D}\hat{\varphi}_j(t)\exp\left\{\int dt\left[l_j(t)\varphi_j(t)+i\hat{\varphi}_j\left(\frac{\partial\varphi_j(t)}{\partial t}+K_j\{\varphi\}\right)-\hat{\varphi}_j(t)(\Gamma_0)_{jk}\hat{\varphi}_k(t)-\frac{1}{2}\frac{\delta K_j\{\varphi\}}{\delta\varphi_j(t)}\right]\right\} \quad .\end{aligned} \tag{1.8}$$

Clearly if $\mathcal{P}\{\theta\}$ is a *non-Gaussian* probability law the quadratic term in $\hat{\varphi}$ involving the first cumulant

$$S_2(j,t_1;k,t_2) \equiv 2(\Gamma_0)_{jk}\delta(t_1-t_2) \tag{1.9}$$

is replaced by the sum

$$\sum_{p=2}^{\infty} \frac{(-i)^p}{p!} \int dt_1 \ldots dt_p \hat{\varphi}_1(t_1) \ldots \hat{\varphi}_p(t_p) \times S_p(1,t_1; \ldots ;p,t) \ , \tag{1.10}$$

where the S_p's are the cumulants of $\mathcal{P}\{\theta\}$.

We shall also need in the following the generating functional for φ and $\hat{\varphi}$ correlations (G functions)

$$Z\{l,\hat{l}\} = \int \mathfrak{D}\varphi(t)\mathfrak{D}\hat{\varphi}(t) \exp\left[\mathfrak{A}\{\varphi,\hat{\varphi}\} + \int dt \left[l_j(t)\varphi_j(t) + \hat{l}_j(t)\hat{\varphi}_j(t)\right]\right] \ . \tag{1.11}$$

B. Correlation and response functions

The *correlation functions* of φ_j's are obtained from the above generating functional (1.8) by taking derivatives with respect to the source l coupled to φ. Thus, we have

$$C_p(1,t_1; \cdots ;p,t_p) \equiv Z^{-1}\{0\} \frac{\delta^{(p)}}{\delta l_1(t_1) \cdots \delta l_p(t_p)} Z\{l\}|_{l=0} = \langle\phi_1(t_1) \cdots \phi_p(t_p)\rangle \ , \tag{1.12}$$

where the bracket means average taken with the weight $Z^{-1}\{0\}\exp\mathfrak{A}$. When $t_1 = t_2 = \ldots = t_p$, C_p becomes the *static* correlation function $C_{p,\mathrm{st}}$.

The *response functions* are generated from the same functional by taking derivatives with respect to the "physical" external field $h_j(t)$ which occurs as

$$\mathcal{H}\{\varphi,h(t)\} \equiv \mathcal{H}_0\{\varphi\} - \sum_j h_j(t)\varphi_j \ . \tag{1.13}$$

The field $h_j(t)$ is thus coupled to $i[(\Gamma_0)_{kj} + Q_{kj}\{\varphi\}]\hat{\varphi}_k(t)$ in $\mathfrak{A}\{\varphi,\hat{\varphi}\}$. The pth response function is, therefore,

$$R_p(0,t_0|1,t_1; \ldots ;p,t_p) \equiv Z^{-1}\{0\} \frac{\delta^{(p+1)}}{\delta l_0(t_0)\delta h_1(t_1) \ldots \delta h_p(t_p)} Z\{l\}|_{l=h=0} = \frac{\delta^{(p)}}{\delta h_1(t_1) \ldots \delta h_p(t_p)} \langle\varphi_0(t_0)\rangle|_{l=h=0} \ . \tag{1.14}$$

The followng properties of R_p will be useful later (i) $R_p = 0$ if $t_j > t_0$ for any j (causality) and (ii)

$$\int dt_1 \ldots dt_p R_p(0,t_0|1,t_1; \ldots ;p,t_p) = C_{p+1,\mathrm{st}}(0,1, \ldots ,p) \ , \tag{1.15}$$

i.e., the zero frequency limit of the response functions are static correlation functions (Appendix A). (iii) The linear ($p=1$) response function $R_1(0,t_0|1,t_1) \equiv R(t_0-t_1)$ satisfies the fluctuation-dissipation theorem[2,23,24]

$$R(t) - R(-t) = \frac{\partial}{\partial t}C(t) \ , \tag{1.16}$$

where $C(t_0-t_1)$ is the two-point correlation function $C_2(0,t_0;1,t_1)$ and where it is understood that $R(-t)$ is computed with due account of time-reversal properties.

C. Models

We shall not discuss here the difficult question of the choice and the derivation of the appropriate stochastic equations to describe a given physical system. Such equations involve in general a multicomponent field φ_j which is made up of slowly varying fields, namely,[9] (i) the order parameter $\psi_\alpha(xt)$, (ii) the energy field $E(xt)$, when it is relevant, and (iii) the generators of the continuous symmetries of the microscopic Hamiltonian (i.e., a conserved field) $m_i(xt)$.

The fields E and m_i may or may not couple to ψ_α in $\mathcal{H}$. The Q_{ij} function is linear in φ and proportional to the structure constants of the symmetry group of the Hamiltonian

$$Q_{ij} = q_{ij}^k\varphi_k \ .$$

We shall consider explicitly the following models: (a) the planar antiferromagnet and superfluid helium (models E and F), (b) the liquid-gas model (model H), and (c) the $O(n)$ symmetric model of Sasvari, Schwabl, and Szepfalusy.[25] For reasons of simplicity we shall carry out the derivations on model E, a reasonable model for helium and planar antiferromagnets in the critical region. When discussing stability we shall also use its straightforward extension to n components. We shall mention in due time what has to be changed when dealing with the other models. The fields of model E are: (i) A two component or complex order parameter $\psi(x,t)$ which is not conserved. This represents the condensate wave function for He or the staggered magnetization in the easy plane for the planar antiferromagnet. (ii) A conserved scalar field $m(x,t)$ which generates rotations in the order-parameter plane. This represents a linear combination of energy and mass density for He, or normal component of the magnetization for the antiferromagnet. These fields satisfy the stochastic equations[9]

$$\frac{\partial\psi(t)}{\partial t}=-\Gamma_0\frac{\delta\mathcal{H}}{\delta\psi^*(t)}-ig_0\psi(t)\frac{\delta\mathcal{H}}{\delta m(t)}+\theta(t)\ , \tag{1.17}$$

$$\frac{\partial m(t)}{\partial t}=\Lambda_0\nabla^2\frac{\delta\mathcal{H}}{\delta m(t)}+ig_0\left[\psi^*(t)\frac{\delta\mathcal{H}}{\delta\psi^*(t)}-\psi(t)\frac{\delta\mathcal{H}}{\delta\psi(t)}\right]+\zeta(t)\ , \tag{1.18}$$

where

$$\mathcal{H}=\mathcal{H}_0-\int d^dx\,(h_m m+h\psi^*+h^*\psi)\ , \tag{1.19}$$

$$\mathcal{H}_0=\int d^dx\left[|\nabla\psi|^2+r_0|\psi|^2+\frac{u_0}{3!}|\psi|^4+\tfrac{1}{2}m^2\right] \tag{1.20}$$

is the Landau-Ginzburg functional. The Gaussian noises θ and ζ have zero mean value and correlations given by

$$\langle\theta^*(xt)\theta(x't')\rangle=2\Gamma_0\delta(x-x')\delta(t-t')\ , \tag{1.21}$$

$$\langle\zeta(xt)\zeta(x't')\rangle=-2\Lambda_0\nabla^2\delta(x-x')\delta(t-t')\ . \tag{1.22}$$

These equations induce the Martin-Siggia-Rose (MSR) action

$$\mathcal{A}=\int dt\,d^dx\left\{-\hat\psi^*2\Gamma_0\hat\psi+\left[i\hat\psi\left(\frac{\partial}{\partial t}\psi+\Gamma_0\frac{\delta\mathcal{H}}{\delta\psi^*}+ig_0\psi\frac{\delta\mathcal{H}}{\delta m}\right)-\text{c.c.}\right]+\hat m\Lambda_0\nabla^2\hat m+i\hat m\left[\frac{\partial}{\partial t}m-\Lambda_0\nabla^2\frac{\delta\mathcal{H}}{\delta m}+ig_0\left(\psi^*\frac{\delta\mathcal{H}}{\delta\psi^*}-\psi\frac{\delta\mathcal{H}}{\delta\psi}\right)\right]+\ln J\{\psi,\psi^*,m\}\right\}\ . \tag{1.23}$$

The perturbation expansion generated from $\mathcal{A}$ is represented[26] by a well-known[4] diagram series. Its construction rules are summarized in Appendix B where the effect of the Jacobian term $\ln J$ is also discussed.

We only list here the properties[14] needed for the discussion below. (i) Diagrams are built with two types of propagators for each field

$$\langle\hat\psi\psi\rangle\equiv G_{\hat\psi\psi}\ , \tag{1.24}$$

$$\langle\psi\psi^*\rangle\equiv G_{\psi\psi^*}\ , \tag{1.25}$$

the last one being the correlation function. For the third type, we have

$$\langle\hat\psi\hat\psi^*\rangle\equiv G_{\hat\psi\hat\psi^*}=0 \tag{1.26}$$

Similar properties hold for the fields $m,\hat m$. (ii) Causality implies that the last time argument in any averages must appear in a ψ or m field. In particular, all averages built with only hatted fields vanish.

II. RENORMALIZATION

In this section we show that the action (1.23) may be renormalized by a simple redefinition of its fields and parameters. This is not obvious since there are fewer parameters than vertices in (1.23). Besides it is not clear that the (static) parameters appearing in $\mathcal{H}$ renormalize just as they would in a purely static theory. Standard renormalization procedures take as basic functions, not the G averages, but the one-irreducible vertex functions Γ, since the G functions are build from the Γ's without further integrations in the frequency wave-vector representation. More precisely, let $Z(l,\hat l)$ be the generating functional for the G functions defined in (1.11). The generating functional $W\{\langle\varphi\rangle,\langle\hat\varphi\rangle\}$ for the one-irreducible vertex functions Γ is given by

$$W\{\langle\varphi\rangle,\langle\hat\varphi\rangle\}+\int dt\left[l_j(t)\langle\varphi_j(t)\rangle+\hat l_j(t)\langle\hat\varphi_j(t)\rangle\right]=\ln Z\{l,\hat l\}\ , \tag{2.1}$$

where $\varphi_i=\psi,\psi^*,m$, etc. In (2.1) $l,\hat l$ are considered as functionals of $\langle\varphi\rangle$, $\langle\hat\varphi\rangle$ satisfying the implicit equations

$$\frac{\delta\ln Z}{\delta l}=\langle\varphi\rangle\ ,\quad\frac{\delta\ln Z}{\delta\hat l}=\langle\hat\varphi\rangle\ . \tag{2.2}$$

Assuming $Z\{l,\hat l\}$ is known through its (regularized) perturbation expansion, (2.1), (2.2) allow to build W whose Taylor expansion around $\langle\varphi\rangle=\langle\hat\varphi\rangle=0$ yields the one-irreducible vertex functions. At zero-loop order one recovers the Lagrangian density of (1.23). Beyond, ultraviolet divergences appear as powers of the cutoff Λ or if the theory is dimensionally regularized in d dimensions, $d=4-\epsilon$, as poles at $\epsilon=0$, since the critical dimension is dimension four. Redefining fields and parameters introduces renormalization Z functions, whose purpose is to absorb all infinities (singular terms in ϵ) that appear in the loop expansion of the bare one-irreducible Γ functions.[27] This is done by making sure that Z functions remove all primitive divergences,[28] since it is then a matter of (complicated) combinatorics to verify that all other divergences are also removed. Note that property (1.26) of the G functions implies

$$\Gamma_{\psi\psi^*}=\Gamma_{mm}=0\ , \tag{2.3}$$

for the Γ functions (more generally, causality entails that all Γ's built with only nonhatted operators vanish).

A. Dimensional considerations and primitively divergent diagrams

We here identify all vertices containing primitive divergences. By comparing various terms of (1.23) and taking into account $[\mathcal{H}]=0$ (dimensionality is expressed in wave-vector units) one gets

$$[\hat{\psi}]=[k^2\psi]=\tfrac{1}{2}d+1\ ,$$

$$[\hat{m}]=[m]=\tfrac{1}{2}d\ ,$$

$$[\omega]=[\Lambda k^2]=[\Gamma k^2]\ ,$$

where ω and k are frequency and wave vector. One has also, as usual

$$[u_0]=4-d\equiv\epsilon\ .$$

From the point of view of power counting $[\omega]=2$ leaving a free choice for the frequency unit. It follows that:

$$[g_0]=2-\tfrac{1}{2}d=\tfrac{1}{2}\epsilon\ ,$$

i.e., both couplings u_0 and g_0 are relevant for $d<d_c=4$.[29]

Leaving apart for the time being composite operators, we consider one-irreducible vertices Γ (taken at nonexceptional momenta) and exhibit those which may show primitive divergences, i.e., those with positive dimensionality, that is, (i) vertices involving no hatted operators $\hat{\psi}$ or $\hat{m}$ (these vanish) and (ii) vertices with only *one* hatted operator:

case (a): $\Gamma_{\hat{\psi}\psi}\sim\Lambda^2$

case (b): $\Gamma_{\hat{\psi}\psi m}\sim\ln\Lambda$

case (c): $\Gamma_{\hat{\psi}\psi\psi\psi^*}\sim\ln\Lambda$

We have listed here the (Λ) cutoff dependence in four dimensions. Furthermore, since m is a *conserved* field, each $\hat{m}$ operator is always accompanied by a power of its wave number, i.e., when $\hat{m}$ is an external operator, it decreases by at least one the degree of divergence. Therefore, the only divergent Γ's involving $\hat{m}$ are

case (e′): $\Gamma_{\hat{m}m}\sim\ln\Lambda\ ,$

case (d): $\Gamma_{\hat{m}\psi\psi^*}\sim\ln\Lambda\ ,$

each one of these vertices being at least proportional to the wave number of $\hat{m}$. In fact $\Gamma_{\hat{m}m}$ is proportional to k^2 so that case (d) may be replaced by

case (e): $\dfrac{\partial}{\partial k^2}\Gamma_{\hat{m}m}\sim\ln\Lambda\ ,$

whereas

$$\frac{\partial}{\partial i\omega}\Gamma_{\hat{m}m}$$

cannot contain primitive divergences. For the same reason, even though the composite operator $(\hat{m}(x)\psi^*(x))$ has the same dimensionality as $\hat{\psi}(x)$ in dimension four, it will not be necessary to take it into account when renormalizing $\hat{\psi}$ (it does not mix with $\hat{\psi}$). (iii) Vertices involving *two* hatted operators

case (f): $\Gamma_{\hat{\psi}\hat{\psi}^*}\sim\ln\Lambda\ ,$

case (g): $\Gamma_{\hat{m}\hat{m}}\sim\ln\Lambda\ ,$

where for (2.4i) we have used the above remark on m conservation. All other vertices, beyond the seven vertices (a)–(g), are *not* primitively divergent near four dimensions. From case (a) it also follows that

case (h): $\dfrac{\partial}{\partial i\omega}\Gamma_{\hat{\psi}\psi}(-i\omega,k)\sim\ln\Lambda\ ,$

case (i): $\dfrac{\partial}{\partial k^2}\Gamma_{\hat{\psi}\psi}(-i\omega,k)\sim\ln\Lambda\ ,$

show independent primitive divergences.

B. Renormalization of vertex functions

We have altogether nine divergent vertices, cases (a)–(i). Each of these vertices corresponds to a term present in the action (1.23) which is also its zero-loop approximation. The action also contains a tenth vertex $\hat{m}i\omega m$ corresponding to the nonprimitively divergent function (k). To remove the divergences, we introduce nine corresponding renormalization functions $Z_a - Z_i$ which we choose to parametrize as follows:

(i) wave-function renormalization

$$\psi=Z_\psi^{1/2}\psi_R\ , \tag{2.4}$$

$$\hat{\psi}=(Z_\psi^{1/2}/\hat{Z}_\psi)\hat{\psi}_R\ , \tag{2.5}$$

$$m=Z_m^{1/2}m_R\ , \tag{2.6}$$

$$\hat{m}=(Z_m^{1/2}/\hat{Z}_m)\hat{m}_R\ , \tag{2.7}$$

(ii) kinetic coefficient renormalization

$$\Gamma_0=\Gamma/Z_\Gamma\ , \tag{2.8}$$

$$\Lambda_0=\Lambda/Z_\Lambda\ , \tag{2.9}$$

(iii) vertex renormalization

$$K_d u_0=\mu^\epsilon u(Z_u/Z_\psi^2)\ , \tag{2.10}$$

$$K_d^{1/2}g_0=\mu^{\epsilon/2}gZ_g\ , \tag{2.11}$$

where μ is a reference wave number and the factor

$$K_d=2(\pi)^{d/2}(2\pi)^{-d}/\Gamma(\tfrac{1}{2}d) \tag{2.12}$$

is introduced for convenience.

For simplicity we choose to work at the *critical temperature* T_c, i.e., for the particular value r_{0c} of r_0 for which the zero-frequency, zero-wave-vector linear response $\delta\langle\psi\rangle/\delta h$ is infinite. The extension away from T_c is done below by treating the mass term $(r_0-r_{0c})\hat{\psi}\psi$ as a (soft) insertion.[30] Besides, the parametrization (2.4 −2.11) implies renormalizing a nondivergent vertex, namely, $(\partial/\partial i\omega)\Gamma_{\hat{m}m}$ or in the Lagrangian density

$$\hat{m}i\omega m=(Z_m/\hat{Z}_m)\hat{m}_R i\omega m_R \quad . \tag{2.13}$$

One has, therefore,

$$Z_m=\hat{Z}_m \quad . \tag{2.14}$$

This leaves us with eight Z functions, and we still miss one which can be associated with the vertex $\Gamma_{\hat{m}\psi\psi^*}$. It will be seen (Appendix C) that the primitive divergences of $\Gamma_{\hat{m}\psi\psi^*}$ are not independent of those of other vertices which allow us to do with just the above introduced Z functions. More generally the Martin-Siggia-Rose (MSR) field theories associate a whole class of vertices with one given coupling constant [e.g., in model E (1.23) $\hat{\psi}\psi m$, $\hat{\psi}^*\psi^*m$, and $\hat{m}\psi\psi^*$ with g_0]. The renormalization procedure introduced by (2.4)−(2.12) provides with only one Z function for each coupling constant (Z_g, Z_u). This, *a priori* is not sufficient to remove all divergences arising in the class of vertices that go along for example with g_0. That this is indeed sufficient is shown in Appendix C.

Finally the Z functions are not uniquely determined by the fact that they remove all divergences from the perturbation expansion of the vertex functions Γ. They become completely defined either (i) by fixing normalization conditions on the nine vertices for prefixed values of the arguments (usually taken in simple relation to μ) or (ii) by deciding that the Z functions just remove the poles in ϵ of the divergent vertex functions Γ. This is the minimal-renormalization procedure.[18,27]

C. Renormalization of response functions and static limit

We first consider linear responses

$$R_\psi \equiv Z^{-1}\frac{\delta}{\delta l\,\delta h}Z\,|_{l=h=0}=\langle\psi,(\Gamma_0\hat{\psi}+ig_0\hat{m}\psi^*)\rangle \quad , \tag{2.15}$$

$$\begin{aligned}R_m &\equiv Z^{-1}\frac{\delta}{\delta l_m\,\delta h_m}Z\,|_{l_m=h_m=0}\\ &=\langle m,[\Lambda_0 k^2\hat{m}+ig_0(\hat{\psi}\psi-\hat{\psi}^*\psi^*)]\rangle \quad .\end{aligned} \tag{2.16}$$

One may remark that the composite operator $(\hat{m}\psi^*)$ appears in (2.15) via coupling with the external source h, and, therefore, the operator $\hat{m}$ is not accompanied by a power of its wave number (as it was required above by m conservation).

Renormalization of R and R_m requires knowledge of how composite operators such as $(\hat{m}\psi^*)$ or $(\hat{\psi}\psi-\hat{\psi}^*\psi^*)$ (taken at coinciding space-time points) renormalize. This is made through the fluctuation dissipation theorem (1.16) which linearly relates response and correlation functions. It follows that bare and renormalized responses are in the same relation as the corresponding correlation functions,

$$R_\psi=Z_\psi R_\psi^R \quad , \tag{2.17}$$

$$R_m=Z_m R_m^R \quad . \tag{2.18}$$

Their zero-frequency limit are the static (equal time) correlation functions. At this point the statics renormalizes using the Z functions introduced for the full dynamical problem, i.e., the Z_ψ and Z_m functions do not necessarily depend only upon static parameters but may also depend on dynamic ones.

We analyze now the R functions in more detail, rewriting them in frequency wave-vector representation as

$$R_\psi=G_{\psi\hat{\psi}}\Gamma_0(1+S) \quad , \tag{2.19}$$

$$R_m=G_{m\hat{m}}\Lambda_0 k^2(1+P) \quad . \tag{2.20}$$

By inspection on the diagram expansion (Appendix B) it is seen that the functions S and P are represented by one-particle irreducible diagrams and are, respectively, equal to $(ig_0/\Gamma_0)\Gamma_{\hat{\psi}(\hat{m}\psi^*)}$ and $(ig_0/\Lambda_0 k^2)\Gamma_{\hat{m}(\hat{\psi}\psi-\hat{\psi}^*\psi^*)}$. We also rewrite correlation functions as

$$\langle\psi\psi^*\rangle\equiv G_{\psi\psi^*}=|G_{\psi\hat{\psi}}|^2\Gamma_{\hat{\psi}\hat{\psi}^*} \quad , \tag{2.21}$$

$$\langle mm\rangle\equiv G_{mm}=|G_{m\hat{m}}|^2\Gamma_{\hat{m}\hat{m}} \quad . \tag{2.22}$$

Then, fluctuation dissipation yields

$$\Gamma_{\hat{\psi}\hat{\psi}^*}=(2\Gamma_0/\omega)\,\mathrm{Im}[\Gamma_{\psi\hat{\psi}}(1+S)] \quad , \tag{2.23}$$

$$\Gamma_{\hat{m}\hat{m}}=(2\Lambda_0 k^2/\omega)\,\mathrm{Im}[\Gamma_{m\hat{m}}(1+P)] \quad . \tag{2.24}$$

The standard vertex functions exhibited in (2.23)−(2.24) are multiplicatively renormalized according to (2.4)−(2.7), the kinetic coefficients according to (2.8), (2.9). It follows that S^R and P^R defined by

$$1+S^R\equiv(1+S)(Z_\Gamma\hat{Z}_\psi)^{-1} \quad , \tag{2.25}$$

$$1+P^R\equiv(1+P)(Z_\Lambda\hat{Z}_m)^{-1} \quad , \tag{2.26}$$

are finite. This renormalization corresponds to a rather complicated way of renormalizing the composite fields.

D. Determination of the Z functions

In order to obtain for Z_ψ and Z_m a simple form involving only static parameters, it is necessary to choose *normalization conditions* compatible with those that appear in the purely static problem. It follows (because the frequency wave-vector representation is privileged in the whole renormalization procedure) that those normalization conditions have to be imposed on response functions R, not on correlation functions, or on basic vertex functions Γ. Expressed in terms of Γ functions they become complicated, although tractable, conditions.

A way to avoid such complications is to use the minimal-renormalization[18,27] procedure. In effect the Z functions are then *uniquely* determined by the property that they only subtract poles in ϵ of the bare vertex functions Γ. Equations (2.17)–(2.18) relate bare and renormalized responses. Their zero-frequency limit relates bare and renormalized static correlation functions and allows one to identify Z_ψ, Z_m as just the same functions that emerge from a minimal-renormalization procedure applied to a purely static theory. The same holds for Z_u, by consideration of the four-point response. Note finally that for the model chosen, the absence of coupling between m and ψ in $\mathcal{H}$ implies

$$Z_m = 1 \quad . \tag{2.27}$$

Some simplifications occur, due to constraints obeyed by vertex functions of the model considered. (i) We have used such a constraint above to obtain relation (2.14) [*conservation of m* implying the nondivergence of $(\partial/\partial i\omega)\Gamma_{\hat{m}m}$] which together with (2.27) yields

$$Z_m = \hat{Z}_m = 1 \quad . \tag{2.28}$$

(ii) Likewise, the fact that m is the *rotation generator* for the order parameter yields a Ward identity[4,9] between response functions

$$\int d^dy \frac{\delta}{\delta h_m(y,t')} R_\psi(x,t;0,0) = \begin{cases} ig_0 R_\psi(x,t;0,0) \ , & 0 < t' < t \\ 0 \ \text{otherwise} \ . \end{cases} \tag{2.29}$$

Since response functions renormalize multiplicatively with Z_ψ, Z_m (2.29) entails immediately the basic relationship

$$Z_g = Z_m^{1/2} \quad . \tag{2.30}$$

(iii) Finally, the *decoupling* of m and the order parameter in $\mathcal{H}$ implies a relationship between primitively divergent parts of $\Gamma_{\hat{m}\psi\psi^*}$ and the response R_ψ, as is shown in Appendix C. This relationship rids us of the missing ninth Z function. When instead, m couples to the order parameter through a term $\gamma_0 m |\psi|^2$ in $\mathcal{H}_0$, the corresponding relationship shows that the renormalization function Z_γ associated with the coupling γ_0 is uniquely determined in terms of static parameters (Appendix C).

Besides the two static functions Z_ψ, Z_m, and the ones given by (2.28)–(2.30) we are left with three dynamic functions $\hat{Z}_\psi$, Z_Γ, Z_Λ. From a computational point of view the simplest way to determine them is the following. (a) Fix $(Z_\Gamma \hat{Z}_\psi)$ by requiring that S^R defined by (2.23) be finite, that is requiring that

$$[1 + (ig_0/\Gamma_0)\Gamma_{\hat{\psi}(\hat{m}\psi^*)}]/Z_\Gamma \hat{Z}_\psi \tag{2.31}$$

contain no poles in ϵ. (b) Fix Z_Λ by requiring that P^R defined by (2.24) be finite, that is, that

$$[1 + (ig_0/\Lambda_0 k^2)\Gamma_{\hat{m}(\hat{\psi}\psi - \hat{\psi}^*\psi^*)}]/Z_\Lambda \tag{2.32}$$

be also finite. (c) Fix $Z_\psi/\hat{Z}_\psi$ by requiring finiteness for

$$(Z_\psi/\hat{Z}_\psi)[\partial/\partial(-i\omega)]\Gamma_{\hat{\psi}\psi} \quad . \tag{2.33}$$

The *bare* vertex functions appearing in (2.31)–(2.33) are computed by perturbation theory, in terms of the bare parameters. Using (2.8)–(2.12) these are replaced by renormalized parameters and Z functions. Expressing that (2.31)–(2.33) contain no poles in ϵ, determines the Z functions at each order in the loop expansion, if we specify that the Z's solely contain poles in ϵ,

$$Z_i(\epsilon) = 1 + \sum_{k=1}^{\infty} C_i^{(k)} \epsilon^{-k} \quad . \tag{2.34}$$

The result is given in Appendix D. By inspection it is seen that the renormalized vertex functions and the Z functions are power series in u, $f \equiv g^2/\Lambda\Gamma$, whose coefficients remain functions of the ratio $w \equiv \Gamma/\Lambda$.

III. RENORMALIZATION GROUP AND CRITICAL PROPERTIES OF He

We exploit in this section the existence of a renormalized theory to investigate dynamic critical behavior in the same way as it was done in the statics.[10] We first consider the behavior at exactly T_c. We derive a Callan-Symanzik equation, find its fixed points, and discuss their stability. The behavior away from T_c is obtained as in (10) by considering the mass term as an insertion.

A. Callan-Symanzik equations and Wilson functions

1. Response or correlation functions

Equation (2.17) relating the bare linear response for the order parameter to the renormalized one may be written

$$R_\psi(-i\omega/\Gamma_0, k; u_0, g_0, \Gamma_0/\Lambda_0; \Lambda) \equiv (\delta/\delta h)\langle\psi_{k\omega}\rangle = Z_\psi R_\psi^R\left(-\frac{i\omega}{\Gamma}, k; u, f, w; \mu\right) \quad . \tag{3.1}$$

Similar relationships hold for various correlation or Green's functions. The associated Callan-Symanzik equation results[10] in varying μ at fixed bare parameters

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_l W_l\frac{\partial}{\partial l}-\eta_\Gamma\omega\frac{\partial}{\partial\omega}-\eta_\psi\right]\times R_\psi^R\left[-\frac{i\omega}{\Gamma},k;u,f,w;\mu\right]=0 \quad . \tag{3.2}$$

Here the *exponent functions* η_j ($j=\psi,u,\Gamma,\Lambda$),

$$\eta_j=\mu\frac{d}{d\mu}\ln Z_j|_0 \tag{3.3}$$

and the *Wilson functions* W_l ($l=u,f,w$),

$$W_l=\mu\frac{d}{d\mu}l\bigg|_0 \ , \tag{3.4}$$

whose zeroes define the fixed points of the theory, are obtained by differentiating at fixed bare parameters.

Taking into account the definitions of u, f, w, namely,

$$K_d u_0\equiv\mu^\epsilon u(Z_u/Z_\psi^2) \ , \tag{3.5}$$

$$K_d(g_0^2/\Lambda_0\Gamma_0)\equiv\mu^\epsilon fZ_\Gamma Z_\Lambda Z_g^2=\mu^\epsilon fZ_\Gamma Z_\Lambda \ , \tag{3.6}$$

$$\Gamma_0/\Lambda_0\equiv w(Z_\Lambda/Z_\Gamma) \ , \tag{3.7}$$

and using (2.28), (2.30), we obtain for the Wilson functions

$$W_u=-u(\epsilon+\eta_u-2\eta_\psi) \ , \tag{3.8}$$

$$W_f=-f(\epsilon+\eta_\Gamma+\eta_\Lambda) \ , \tag{3.9}$$

$$W_w=+w(\eta_\Gamma-\eta_\Lambda) \ . \tag{3.10}$$

The minimal-renormalization procedure used leads to ϵ-*independent* exponent functions: the only ϵ dependence in the coefficients of the Callan-Symanzik equation is explicitly shown in Eqs. (3.8) and (3.9).

Integration of (3.2) with the homogeneity property gives

$$R_\psi^R\left[-\frac{i\omega}{\Gamma},k;u,f,w;\mu\right]=\exp\left[\int_1^\rho\frac{d\rho'}{\rho'}\eta_\psi[u(\rho')]\right]\rho^{-2}R_\psi^R\left[-\frac{i\omega}{\rho^2\Gamma(\rho)},\frac{k}{\rho};u(\rho),f(\rho),w(\rho);\mu\right] \ . \tag{3.11}$$

The parameters $l(\rho)$, $l=u$, f, w, and ρ an arbitrary dilation parameter, are determined by the flow equations

$$\rho\frac{d}{d\rho}l(\rho)=W_l(u(\rho),f(\rho),w(\rho)), \quad l=u,f,w \ , \tag{3.12}$$

$$\rho\frac{d}{d\rho}\ln\Gamma(\rho)=\eta_\Gamma(u(\rho),f(\rho),w(\rho)) \ , \tag{3.13}$$

and the corresponding initial conditions $l(1)=l$ and $\Gamma(1)=\Gamma$.

The *critical region* $k\ll\mu$ is studied by choosing $\rho=k/\mu\ll1$. Solutions of Eq. (3.12) in this limit are governed by fixed points values l^* for which all W_l vanish, and by the stability matrix $W_{ll'}\equiv\partial W_l/\partial l'$ whose eigenvalues ω_l at l^* are the *transient exponents*. If $l^*(u^*,f^*,w^*)$ is such a fixed point, with $\eta_\psi{}^*$, $\eta_\Gamma{}^*$, $\eta_\Lambda{}^*$ the corresponding exponents, (3.11) becomes then asymptotically

$$R_\psi^R\left[-\frac{i\omega}{\Gamma},k;u,f,w;\mu\right]\simeq\left[\frac{k}{\mu}\right]^{-2+\eta_\psi{}^*}R_\psi^R\left[-\frac{i\omega}{\Gamma}\left[\frac{\mu}{k}\right]^{2+\eta_\Gamma{}^*},\mu;u^*,f^*,w^*;\mu\right] \ , \tag{3.14}$$

where we see that the response function depends upon the ratio $i\omega/k^z$, z being the dynamic exponent

$$z=2+\eta_\Gamma{}^* \tag{3.15}$$

Corrections,[10,31] are easily obtained by Taylor expanding (3.11) in powers of $l(\rho)-l^*$, and by writing it in terms of $(k/\mu)^{\omega_l}$.

2. *Kinetic and transport coefficients*

They may be defined for $T>T_c$ and in the hydrodynamic region, as

$$\Gamma_{\text{eff}}^{-1}=\frac{\partial}{\partial(-i\omega)}R_\psi^{-1}\bigg|_{\omega=0} \ , \tag{3.16}$$

$$k^{-2}\Lambda_{\text{eff}}^{-1}=\frac{\partial}{\partial(-i\omega)}R_m^{-1}\bigg|_{\omega=0} \ . \tag{3.17}$$

The kinetic and transport coefficients Γ_{eff}^R, Λ_{eff}^R defined from the renormalized response functions R_ψ^R, R_m^R, are related to the bare ones (3.16), (3.17) by the multiplicative *constant* (for fixed μ) factors Z_ψ^{-1}, Z_m^{-1}, respectively.

From (3.14) we then obtain

$$\frac{1}{\Gamma_{\text{eff}}^R(k;l;\mu)}=\frac{1}{\Gamma}\left[\frac{k}{\mu}\right]^{\eta_\psi{}^*+\eta_\Gamma{}^*}\frac{\partial}{\partial\zeta}[R^R(\zeta,\mu;l^*;\mu)]_{\zeta=0}^{-1} \ , \tag{3.18}$$

where the last factor is regular. The same asymptotic behavior holds for Λ_{eff},

$$\frac{1}{\Lambda_{\text{eff}}^R(k;l;\mu)}=\frac{1}{\Gamma}\left[\frac{k}{\mu}\right]^{\eta_\Lambda{}^*}\frac{\partial}{\partial\zeta}[R_m^R(\zeta,\mu;l^*;\mu)]_{\zeta=0}^{-1} \ , \tag{3.19}$$

where the last factor is *proportional to w*. Therefore, (3.19) is not valid when $w(\rho)$ vanishes with ρ. When, as is the case below, $w(\rho)$ behaves like $w\rho^{\omega_w}$ then (3.19) must be replaced by

$$\frac{1}{\Lambda_{\rm eff}^R(k;l;\mu)} = \frac{1}{\Gamma}\left(\frac{k}{\mu}\right)^{\eta_\Gamma^*+\omega_w} w\frac{\partial}{\partial\zeta}[wR_m^R(\zeta,\mu;l;\mu)]^{-1}_{\zeta=0,l=l^*} , \tag{3.20}$$

where the last factor is now regular.

B. Fixed points and stability regions

The minimal-renormalization procedure yields static functions (W_u, η_ψ) which only depend upon u and decouple from the dynamic ones. In practical terms it means that the fixed point value u^* is obtained from (3.8) alone $[u^*=\frac{3}{5}\epsilon+O(\epsilon^2)]$ and the stability matrix has the form

$$W=\begin{vmatrix} W_{uu} & 0 & 0 \\ W_{fu} & W_{ff} & W_{fw} \\ W_{wu} & W_{wf} & W_{ww}\end{vmatrix} , \tag{3.21}$$

exhibiting one transient[10,31]

$$\omega_u = W_{uu} = \epsilon + \tfrac{3}{5}\epsilon^2 + O(\epsilon^3) . \tag{3.22}$$

For the dynamic part of the problem f^*, w^*, are given by the zeroes of (3.9), (3.10) and the transient ω_f, ω_w by the eigenvalues of the 2×2 submatrix of (3.21) at the fixed point.

One sees from (3.9), (3.10) that the following fixed points exist (i) $f^*=0$ implying $\eta_\Lambda{}^*=0$ and $\eta_\Gamma{}^*$ equals its model A value,[2,9] $\eta_\Gamma^A>0$. With $W_{fw}=0$ at $f^*=0$, one gets

$$\omega_f = W_{ff} = -(\epsilon+\eta_\Gamma^A) < 0 , \tag{3.23}$$

showing that model A is unstable with respect to the introduction of mode coupling terms. (ii) $f^*\neq 0$ implying with (3.9)

$$\epsilon + \eta_\Gamma{}^* + \eta_\Lambda{}^* = 0 . \tag{3.24}$$

Equation (3.10) allows for *three possibilities* of fixed point values for w,

$$w^* = +\infty , \tag{3.25a}$$

$$w^* = 0 , \tag{3.25b}$$

or $w^*\neq 0$, ∞ leading to the dynamic scaling relation

$$\eta_\Gamma{}^* = \eta_\Lambda{}^* . \tag{3.25c}$$

Computing the dynamic exponents in the *two-loop* approximation yields

$$\eta_\Gamma = -f/(1+w) + f^2 G(w) + \eta_\Gamma^A(u) , \tag{3.26}$$

$$\eta_\Lambda = -fn/4 + f^2 L(w) , \tag{3.27}$$

where

$$\eta_\Gamma^A = [(n+2)/2(n+8)^2](6\ln\tfrac{4}{3}-1)\epsilon^2 , \tag{3.28}$$

is the two-loop result for model A,[2,9] and

$$G(w) = \frac{1}{8(1+w)^3}\left[4(1+2w)\ln\frac{(1+w)^2}{1+2w} + 9(4+n)(\ln\tfrac{4}{3})(1+w) - (4+2n)w - (8+2n)\right] , \tag{3.29}$$

$$L(w) = \frac{n}{8(1+w)}\left[w^2(2+w)\ln\frac{(1+w)^2}{w(2+w)} - w - \tfrac{1}{2}\right] . \tag{3.30}$$

The result is valid for the extension to the n component order parameter ψ_α, $\psi_\alpha{}^*$, $\alpha=1,\ldots,\frac{1}{2}n$. With (3.26)–(3.30) the two eigenvalues ω_f, ω_w of $W_{ll'}$ are easily computed.

(a) The fixed point value $w^*=+\infty$ leads to transients

$$\omega_f = \tfrac{1}{4}nf^* , \quad \omega_w = -\eta_\Gamma^A - \omega_f \tag{3.31}$$

displaying its unstability.

(b) The dynamic scaling solution,[4,32,33] with fixed point values

$$w^* = \frac{4-n}{n} + \frac{32}{n^3}\left[L\left(\frac{4-n}{n}\right) - G\left(\frac{4-n}{n}\right)\right]\epsilon - \frac{8}{n\epsilon}\eta^A(u^*) , \tag{3.32}$$

$$f^* = \frac{2\epsilon}{n} + \frac{16}{n^3}L\left(\frac{4-n}{n}\right)\epsilon^2 , \tag{3.33}$$

yields

$$\eta_\Gamma{}^* = \eta_\Lambda{}^* = -\frac{\epsilon}{2} , \tag{3.34}$$

i.e., for the dynamic exponent (3.15)

$$z = \tfrac{1}{2}d . \tag{3.35}$$

The associated transients are, to two-loop order,

$$\omega_f = \epsilon - \frac{8\epsilon^2}{n(4+n)}L\left(\frac{4-n}{n}\right) - \frac{32\epsilon^2}{n^2(4+n)}G\left(\frac{4-n}{n}\right) + \frac{8}{4+n}\eta_\Gamma^A(u^*) \tag{3.36}$$

$$\omega_w = \frac{4-n}{8}\epsilon + \frac{n^2+8n-16}{n^2(4+n)}\epsilon^2 L\left[\frac{4-n}{n}\right] - \frac{2(n^2+4n-16)}{n^2(4+n)}\epsilon^2 G\left[\frac{4-n}{n}\right] + \frac{4(4-n)}{n^2} \times \epsilon^2\left[\dot{G}\left[\frac{4-n}{n}\right] - \dot{L}\left[\frac{4-n}{n}\right]\right] - \frac{n^2}{2(4+n)}\eta_\Gamma^A(u^*) \ . \quad (3.37)$$

The solution remains stable ($\omega_w > 0$) for $n < n_c(\epsilon)$ *(region I)*, where

$$n_c(\epsilon) = 4 - (19\ln\tfrac{4}{3} - \tfrac{11}{3})\epsilon + O(\epsilon^2) \ . \quad (3.38)$$

For $n > n_c(\epsilon)$ *(region II)* this solution looses meaning ($w^* < 0$).

(c) The solution with fixed point value $w^* = 0$ leads to transients

$$\omega_f = -f^*\frac{\partial}{\partial f^*}(\eta_\Gamma + \eta_\Lambda) \ , \quad (3.39)$$

$$\omega_w = \eta_\Gamma{}^* - \eta_\Lambda{}^* \ , \quad (3.40)$$

and to a dynamic exponent

$$z = \tfrac{1}{2}(d + \omega_w) \ , \quad (3.41)$$

computed at the fixed point values

$$w^* = 0 \quad (3.25b)$$

$$f^* = \frac{4\epsilon}{4+n} + \frac{64}{(4+n)^3}\epsilon^2[G(0) + L(0)] + \frac{4}{4+n}\eta_\Gamma^A(u^*) \ . \quad (3.42)$$

Solving the flow equation for $w(\rho)$ yields

$$w(\rho) \simeq \rho^{\omega_w} \ , \quad (3.43)$$

with the induced asymptotic behavior (3.18), (3.20),

$$\Gamma_{\text{eff}} \simeq \left[\frac{k}{\mu}\right]^{\eta_\psi{}^* + \eta_\Gamma{}^*} \ , \quad (3.44)$$

$$\Lambda_{\text{eff}} \simeq \left[\frac{k}{\mu}\right]^{\eta_\Gamma{}^* - \omega_w} \ , \quad (3.45)$$

showing off a violation of extended dynamic scaling. For this reason we may call the $w^* = 0$ solution, the *weak-scaling fixed point.*

The transient ω_w vanishes along the same line $n = n_c(\epsilon)$, where the dynamic scaling fixed point takes over, ω_w is positive in region II, ω_f remains positive throughout the separatrix $n_c(\epsilon)$. One gets

$$\omega_f = \epsilon - \frac{16\epsilon^2}{(4+n)^2}[G(0) + L(0)] + \eta_\Gamma^A(u^*) \ , \quad (3.46)$$

$$\omega_w = \frac{n-4}{n+4}\epsilon + \frac{32n\epsilon^2}{(4+n)^3}G(0) - \frac{128\epsilon^2}{(4+n)^3}L(0) + \frac{2n}{4+n}\eta_\Gamma^A(u^*) \ . \quad (3.47)$$

The physical point for helium in the (ϵ,n) plane lies at $\epsilon = 1$, $n = 2$, i.e., nearly on the tangent (at $\epsilon = 0$) to the separatrix. However, the ϵ expansion cannot tell us whether it lies in region I or II. If we compute the transients at the physical point ($\epsilon = 1$, $n + 2$) we obtain

$$\omega_f = \epsilon - 0.230\epsilon^2 \ , \quad (3.48)$$

$$\omega_w = \tfrac{1}{4}\epsilon - 0.139\epsilon^2 \ , \quad (3.49)$$

for the *dynamic scaling* fixed point.

For the *weak-scaling* fixed point we get

$$\omega_f = \epsilon - 0.126\epsilon^2 \ , \quad (3.50)$$

$$\omega_w = -\tfrac{1}{3}\epsilon + 0.214\epsilon^2 \ . \quad (3.51)$$

Taken at their face value the dynamic scaling solution appears as the *stable* one. However, one of its associated transients (ω_w) is small, largely shrinking, therefore, the asymptotic region.

On the other hand the ϵ expansion result for ω_w cannot rule out, without additional information, the possibility that the physical point lies in region II. If this alternative solution were the physical one, it would lead to a series of consequences that we discuss later.

C. Renormalization away from the critical temperature

We have only worked so far at the critical temperature.[10] Away from T_c, we have an extra term in the Landau-Ginzburg functional (1.20),

$$\int d^dx\,(r_0 - r_{0c})|\psi|^2 \ ,$$

which in the action (1.23) generates the extra terms

$$\Gamma_0(r_0 - r_{0c})\int dt\,d^dx\,i(\hat{\psi}\psi + \hat{\psi}^*\psi^*) \ . \quad (3.52)$$

In the renormalized version of the action these terms become

$$t\Gamma Z_\psi(Z_\Gamma \hat{Z}_\psi)^{-1}Z_{\psi^2}\int dt\,d^dx\,i(\hat{\psi}_R\psi_R + \hat{\psi}_R{}^*\psi_R{}^*) \ , \quad (3.53)$$

where t (no confusion with time) is

$$t = (r_0 - r_{0c})(Z_{\psi^2})^{-1} \ , \quad (3.54)$$

a measure of $T - T_c$. These terms are treated as insertions.[30] The new renormalization function Z_{ψ^2} takes care of the only new primitively divergent vertex, namely $(\partial/\partial r_0)\Gamma_{\hat{\psi}\psi}$. Note that Z_{ψ^2} is a static quantity, as may be seen by computing $(\partial/\partial r_0)\Gamma_{\hat{\psi}\psi}$ at zero frequency from (2.19). The Callan-Symanzik Eq. (3.2) takes now the form

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_l W_l\frac{\partial}{\partial l}-\eta_\Gamma\omega\frac{\partial}{\partial\omega}-\eta_{\psi^2}t\frac{\partial}{\partial t}-\eta_\psi\right]\times R^R\left[\frac{i\omega}{\Gamma},k;t;u,f,w;\mu\right]=0 \quad (3.55)$$

and under dilation, $t(\rho)$ obeys

$$\rho\frac{d}{d\rho}t(\rho)=-\eta_{\psi^2}(u(\rho))t(\rho) \quad . \quad (3.56)$$

Asymptotically, we have

$$t(\rho)\simeq t\rho^{-\eta_{\psi^2}^*} \quad . \quad (3.57)$$

The asymptotic form (3.14) is now replaced by

$$R^R\left[\frac{i\omega}{\Gamma},k;t;u,f,w;\mu\right]\simeq\left[\frac{k}{\mu}\right]^{-2+\eta_\psi}R^R\left[\frac{i\omega}{\Gamma}\left[\frac{\mu}{k}\right]^z,\mu;t\left[\frac{\mu}{k}\right]^{1/\nu};u^*,f^*,w^*;\mu\right] , \quad (3.58)$$

where the standard critical exponent ν is related to $\eta_{\psi^2}{}^*$ by

$$1/\nu=2-\eta_{\psi^2}{}^* \quad . \quad (3.59)$$

The response function thus takes the form

$$R^R\simeq k^{-2}\left[\frac{k}{\mu}\right]^{\eta_\psi{}^*}\phi\left[\frac{i\omega}{\Gamma}\left[\frac{\mu}{k}\right]^z,k\xi\right] , \quad (3.60)$$

where ξ is the correlation length $\xi\simeq t^{-\nu}$.

D. Critical behavior of characteristic frequencies and kinetic coefficients

The characteristic frequency ω_ψ of the order parameter is defined by

$$\omega_\psi^{-1}\equiv\frac{\partial}{\partial i\omega}\ln R_\psi|_{\omega=0}=R_\psi(\omega=0)\Gamma_{\rm eff}^{-1} \quad (3.61)$$

and with (3.58) it takes the form

$$\omega_\psi\simeq(k/\mu)^z\Omega_\psi(k\xi) \quad , \quad (3.62)$$

$$\Omega_\psi^{-1}=\Gamma^{-1}\frac{\partial}{\partial x}\ln\phi(x,k\xi)|_{x=0} \quad . \quad (3.63)$$

Likewise ω_m defined through

$$\omega_m^{-1}\equiv\frac{\partial}{\partial i\omega}\ln R_m|_{\omega=0}=R_m(\omega=0)\Lambda_{\rm eff}^{-1}k^{-2} \quad , \quad (3.64)$$

takes the form

$$\omega_m\simeq(k/\mu)^z\Omega_m(k\xi) \quad , \quad (3.65)$$

for the dynamic scaling fixed point. Equations (3.62)–(3.65) exhibit the dynamic scaling form[4,32,33] of characteristic frequencies. On the other hand, for the weak scaling point, (3.65) is replaced by

$$\omega_m\simeq(k/\mu)^{z-\omega_w}\Omega_m(k\xi) \quad , \quad (3.66)$$

where the dynamic exponent z is given by (3.41).

The *thermal conductivity* λ is the $k=0$ limit of $\Lambda_{\rm eff}$,

$$\lambda\equiv\Lambda_{\rm eff}(k=0) \quad . \quad (3.67)$$

From the definition (3.64) or (3.17) and from (3.65), (3.66) it follows that the thermal conductivity λ behaves like

$$\lambda\simeq\xi^{\epsilon/2} \quad , \quad (3.68)$$

for the dynamic scaling fixed point, and

$$\lambda\simeq\xi^{\epsilon/2+\omega_w/2} \quad , \quad (3.69)$$

for the weak scaling fixed point.

E. Deviations from the symmetric model

A more realistic model[4] for He takes into account the coupling between the m field and the local changes in temperature, implying a coupling of the form $\gamma_0 m\psi\psi^*$ in the Landau-Ginzburg functional $\mathcal{H}_0$. This asymmetric coupling renders complex the divergence of $(\partial/\partial i\omega)\Gamma_{\hat\psi\psi}$, forcing the introduction of a complex kinetic coefficient $\Gamma_0(1+ib_0)$, Γ_0 and b_0 real, in the equation of motion (1.17), the noise correlation being still given by $2\Gamma_0\delta(x-x')\delta(t-t')$ [Eqs. (1.21), (1.22) unchanged]. This is model F of Halperin, Hohenberg, and Siggia.[4,9]

1. Renormalization

The divergent part of Γ vertices now becomes complex making it necessary to introduce complex Z functions. It is sufficient to introduce a complex $\hat Z_\psi$, and a complex Z function for the kinetic coefficient Γ which we parametrize with *real* Z_Γ and Z_b defined by

$$\Gamma_0(1+ib_0)=\Gamma(1+ib)\frac{1+ibZ_b}{Z_\Gamma(1+ib)} \quad . \quad (3.70)$$

This implies $\Gamma_0=\Gamma/Z_\Gamma$ as in (3.9) and

$$b_0=bZ_b \quad . \quad (3.71)$$

Besides the γ_0 coupling in $\mathcal{H}_0$ leads to a static vertex renormalization[12] Z_γ defined by

$$K_d^{1/2}\gamma_0=\mu^{\epsilon/2}\frac{\gamma Z_\gamma}{Z_\psi Z_m^{1/2}} \quad , \quad (3.72)$$

and a nontrivial Z_m (still equal to $\hat Z_m$).

The two complex Z functions $\hat Z_\psi$ and (Z_Γ,Z_b) take care of complex divergences of $(\partial/\partial i\omega)\Gamma_{\hat\psi\psi}$ and

$(\partial/\partial k^2)\Gamma_{\tilde{\psi}\psi}$. That all other primitively divergent Γ vertices are taken care of with the above set of Z functions follows from static renormalizability and relationships between responses and Γ vertices displayed in Appendix C.

2. Callan-Symanzik equation

In (3.2), the sum $\sum_l W_l \partial/\partial l$ now runs over five terms $l = u, \gamma, f, w, b$. We have, successively,

$$W_\gamma = -\tfrac{1}{2}\gamma[\epsilon + 2\eta_{\psi^2}(\tilde{u}) + \eta_m(\tilde{u}, \gamma)] \ , \tag{3.73}$$

where $\tilde{u} = u - 3\gamma^2$, and η_m defined by (3.3) has the form

$$\eta_m = -\gamma^2 B(\tilde{u}) \ , \tag{3.74}$$

$W_u(\tilde{u})$ unchanged, as derived from the statics. Besides from the Ward identity (2.32) and definition (3.6) one gets

$$W_f = -f(\epsilon + \eta_\Gamma + \eta_\Lambda + \eta_m) \ , \tag{3.75}$$

instead of (3.9).

3. *Fixed points and stability*

In principle, one would have to study five coupled Wilson functions and therefore a 5×5 stability matrix. However, since our renormalization procedure decouples statics from dynamics, $W_u(\tilde{u})$ and $W(\tilde{u}, \gamma)$ do not depend on dynamical parameters. The discussion for fixed points and stability for $\tilde{u}$, γ separates and may be carried out independently as in (12). Briefly one gets (a) the *symmetric* fixed point $\gamma^* = 0$, stable if the transient exponent

$$\omega_\gamma^{(s)} = d - \frac{2}{\nu} \equiv -\frac{\alpha}{\nu} \ . \tag{3.76}$$

is positive. At this fixed point,

$$\eta_m{}^* = 0 \ . \tag{3.77}$$

(b) One also obtains the *asymmetric* fixed point $\gamma^* = -\alpha/\gamma B(\tilde{u}^*)$, stable if

$$\omega_\gamma^{(a)} = 2/\nu - d = +\alpha/\nu \tag{3.78}$$

is positive. At this fixed point we have,

$$\eta_m{}^* = -\alpha/\nu \tag{3.79}$$

corresponding to a divergent specific heat.

The stability regions of these two fixed points are separated by the line $n = n_c(\epsilon)$ defined by

$$\alpha = 0 \tag{3.80}$$

where the transients ω_γ vanish. To lowest order we get

$$n_c(\epsilon) = 4 - 4\epsilon + O(\epsilon^2) \ . \tag{3.81}$$

The physical point ($\epsilon = 1, n = 2$) is on the symmetric side of the tangent at $\epsilon = 0$ to the separatrix. The ϵ expansion to two loop for the transient in the symmetric region, at $n = 2$, gives, however,

$$\omega_\gamma^{(s)} = -\tfrac{1}{5}\epsilon + \tfrac{1}{25}7\epsilon^2 \ . \tag{3.82}$$

As is the case for ω_w [Eqs. (3.38), (3.51)] it is difficult without extra information to decide on the basis of the ϵ expansion, whether the physical point lies in the symmetric or asymmetric region. On the basis of experimental results[34] one knows that $\alpha \simeq -0.02$ and the fixed point is symmetrical. In view of the smallness of the transient $\omega_\gamma^{(s)} = -\alpha/\nu$, $\gamma(\rho)$ reaches the origin so slowly that a special treatment is needed for solving the flow equations, for which we refer to (2,4).

If one computes W_b to first order in b, for the symmetric fixed point $\gamma^* = 0$, one gets

$$\frac{W_b}{b} = f\frac{1+2w}{(1+w)^2} - \tfrac{1}{6}uf(n+2)(1 - 3\ln\tfrac{4}{3}) + \tfrac{1}{18}u^2(n+2)\ln\tfrac{4}{3} - f^2K(w) \tag{3.83}$$

$$\begin{aligned} K(w) = \tfrac{1}{2}(1+w)^{-4}\Bigg[& w\left(1 + 2w + (1+4w)\ln\tfrac{1}{2}(1+w) - (1+4w+6w^2)\ln\frac{2(1+w)}{1+2w}\right) \\ & + \tfrac{1}{4}(1+w)n[2 - 3\ln\tfrac{4}{3} + w(-2 + 15\ln\tfrac{4}{3})] + 4(1 - 3\ln\tfrac{4}{3}) + 6w\ln\tfrac{4}{3} \\ & + (1+w)\Bigg(4(1 - 3\ln\tfrac{4}{3}) + 6w\ln\tfrac{4}{3} + (1+3w)\ln\tfrac{1}{2}(1+w) + (1+w)\ln\frac{(1+w)^2}{1+2w} \\ & + (1+2w)(1+3w)\ln\frac{2(1+w)}{1+2w}\Bigg)\Bigg] \ , \end{aligned} \tag{3.84}$$

i.e., $b^*=0$ is a stable fixed point with the associated transient

$$\omega_b = b^{-1} W_b(u^*,f^*,w^*)|_{b=0} \ . \tag{3.85}$$

For $n=2$, we get

$$w^* \neq 0 : \omega_b = \tfrac{3}{4}\epsilon - 0.299\epsilon^2 \ , \tag{3.86}$$

$$w^* = 0 : \omega_b = \tfrac{2}{3}\epsilon - 0.159\epsilon^2 \ , \tag{3.87}$$

for the dynamic scaling and weak-scaling fixed point, respectively.

F. Thermal conductivity in the critical region

We return now to thermal conductivity for which we have the asymptotic behavior (3.67), (3.68) and corrections determined by transients ω_u (3.22), ω_γ (3.78), ω_f [(3.48), (3.50)], ω_w [(3.49), (3.51)], ω_b [(3.86), (3.87)]. If we express it in terms of renormalized quantities through (3.67), (2.20), (2.26), (3.17) we obtain

$$\lambda = Z_m \Lambda^R_{\text{eff}}(k=0) = Z_m \frac{\Gamma}{w}(1+P^R)\bigg|_{\omega=k=0} \ . \tag{3.88}$$

The renormalized kinetic coefficient Γ may in turn, using (3.6) and (3.7) be written

$$\Gamma = (K_d g_0^2 w / Z_m f \mu^\epsilon)^{1/2} \ , \tag{3.89}$$

where both g_0 and Z_m are related to independently known static quantities[4]

$$Z_m = C_p \tag{3.90}$$

$$g_0 = T\sigma \tag{3.91}$$

C_p is the specific heat, σ the entropy density per unit mass (we use units such that $\hbar = k_B = 1$).

Using the renormalization group on Λ^R_{eff} we obtain the results (i) if $w^* \neq 0$ *(dynamic scaling* fixed point):

$$\lambda = (K_d T^2 \sigma^2 C_p)^{1/2} (w^* f^*)^{-1/2} \times \xi^{\epsilon/2}(1 - \tfrac{1}{4} f^*) \times \left[1 + \sum_l c_l \left(\frac{\xi}{\xi_0} \right)^{\omega_l} \right] \ , \tag{3.92}$$

where the fixed point values w^*, f^*, are given by (3.32), (3.33). The factor $1 - \frac{1}{4} f^*$ is the one-loop result for $1 + P^R(l^*)$, the sum over l runs over the five parameters $l = u$, γ, f, w, b, and the six constants c_l, ξ_0 are nonuniversal. As mentioned above the γ correction needs a special treatment going beyond the linear order.[2,4] Its net effect is to change the last bracket as follows

$$1 + \sum_l c_l \left(\frac{\xi}{\xi_0} \right)^{\omega_l} \to \varphi(\alpha_e) \left[1 + \sum_l{}' c_l \left(\frac{\xi}{\xi_0} \right)^{\omega_l} \right] \ , \tag{3.93}$$

where φ is a universal function of the effective specific-heat exponent

$$\alpha_e = \partial \ln C_p / \partial \ln |T - T_c| \tag{3.94}$$

and the sum $\sum'$ now runs over only four parameters $l = u$, f, w, b. The dominant term in $\sum'$ is the w term, since ω_w is the smallest transient. (ii) If $w^* = 0$ *(weak-scaling* fixed point), the renormalization group tells us that $w(\rho) \simeq w \rho^{\omega_w}$, thus leading to

$$\lambda = (K_d T^2 \sigma^2 C_p)^{1/2} (f^*)^{-1/2} w^{-1/2} \xi^{\epsilon/2} \times \left(\frac{\xi}{\xi_0} \right)^{\omega_w/2} (1 - f^*/4) \overline{\varphi}(\alpha_e) \left[1 + \sum_l{}' \overline{c}_l \left(\frac{\xi}{\xi_0} \right)^{\omega_l} \right] \tag{3.95}$$

where f^* is computed at the new fixed point (3.42). In (3.95) there appears a new nonuniversal quantity w, and the extra exponent ω_w in the leading term. Corrections are again dominated by the w term in $\sum'$. Its coefficient $\overline{c}_w$ takes the form

$$\overline{c}_w = A(\epsilon) w \ . \tag{3.96}$$

The important fact is that A is *universal,* and w is the same quantity as in (3.95). One obtains therefore a universal combination of amplitudes as it occurs[35,36] whenever there exists an exact scaling law among exponents. The quantity $A(\epsilon)$ is given by

$$A(\epsilon) = \frac{\partial}{\partial w} \ln(1 + P^R) \bigg|_{\substack{w=0 \\ f=f^*}} \ . \tag{3.97}$$

The only diagrams contributing to (3.97) are the two-loop graphs 7_b, 7_c and Appendix D which for small w take the form

$$(7_b) = \tfrac{1}{16} f^2 [\tfrac{1}{6}\pi^2 - \tfrac{9}{4} + w(\tfrac{1}{6}\pi^2 + \tfrac{1}{4}) + 5w \ln 2w + w \ln^2 2w] \ , \tag{3.98}$$

$$(7_c) = \tfrac{1}{16} f^2 [-\tfrac{5}{2} + w(3 - \tfrac{1}{3}\pi^2) - 5w \ln 2w - w \ln^2 2w] \ . \tag{3.99}$$

The appearance of logarithmic terms as in (3.98), (3.99) would have rendered complicated the analysis of correction terms. It turns out, however, that they exactly cancel (to two-loop order) in P^R. It is hard to tell whether this property is preserved to higher orders. One obtains nevertheless to order ϵ^2,

$$A(\epsilon) = \tfrac{1}{144}\epsilon^2 (13 - \tfrac{2}{3}\pi^2) = 0.045\epsilon^2 \ . \tag{3.100}$$

When comparing with experimental results[37] it is convenient to consider the ratio

$$R_\lambda = \lambda (T^2 \sigma^2 C_p)^{-1/2} \xi^{-1/2} \varphi^{-1} \ , \tag{3.101}$$

which tends to the Halperin-Hohnberg-Siggia (HHS) universal value,[4]

$$R_\lambda{}^* = (K_d/w^*f^*)^{1/2}(1-\tfrac{1}{4}f^*) \quad . \tag{3.102}$$

At the dynamic fixed point we obtain

$$R_\lambda{}^* = (K_d/\epsilon)^{1/2}(1+0.47\epsilon) \quad , \tag{3.103}$$

At the weak-scaling fixed point the corresponding ratio $\overline{R}_\lambda$ defined as (3.101), with $\overline{\varphi}^{-1}$ instead of φ^{-1}, has the nonuniversal asymptotic behavior,

$$\overline{R}_\lambda \simeq (K_d/f^*)^{1/2}(1-\tfrac{1}{4}f^*)\, w^{-1/2}(\xi/\xi_0)^{\omega_w/2} \tag{3.104}$$

$$= (3K_d/2\epsilon)^{1/2}(1-0.24\epsilon)\, w^{-1/2}(\xi/\xi_0)^{\omega_w/2} \tag{3.105}$$

However, since w also appears in the main correction term, one has

$$\overline{R}_\lambda^2 \simeq C_1(\xi/\xi_0)^{\omega_w} + 0.135K_d\epsilon \quad , \tag{3.106}$$

up to correction terms (involving ω_u, ω_f, ω_b) that vanish at the critical point. It remains to be seen whether the weak-scaling fixed point, and in particular (3.106) may give a better account of experimental data than (3.101)–(3.103).

IV. ANTIFERROMAGNETS AND LIQUID-GAS SYSTEMS

We apply in this section the previous renormalization technique to other mode coupling systems: (i) the symmetric $O(n)$ model of Sasvari, Schwabl, and Szepfalusy[25] (SSS) which for $n=3$ reduces to HHS model G,[4,9] and for $n=2$ to HHS model E,[4,9] (ii) the liquid-gas model, i.e., HHS model H.[5,9,21] We confine ourselves to pointing out differences from the previous discussion and to exhibiting results.

A. SSS $O(n)$ symmetric model

The order parameter ψ is a *real n* component vector field. The conserved field m is now an antisymmetric tensor whose $m_{\alpha\beta}$ component generates rotations of ψ in the $\alpha\beta$-plane. The equations of motion for this model read

$$\frac{\partial\psi_\alpha}{\partial t} = -\Gamma_0\frac{\delta\mathcal{H}}{\delta\psi_\alpha} + g_0\psi_\beta\frac{\delta\mathcal{H}}{\delta m_{\alpha\beta}} + \theta_\alpha \quad , \tag{4.1}$$

$$\frac{\partial m_{\alpha\beta}}{\partial t} = \Lambda_0\nabla^2\frac{\delta\mathcal{H}}{\delta m_{\alpha\beta}} + g_0\left[\psi_\alpha\frac{\delta\mathcal{H}}{\delta\psi_\beta} - \psi_\beta\frac{\delta\mathcal{H}}{\delta\psi_\alpha}\right] + \zeta_{\alpha\beta} \quad , \tag{4.2}$$

with the Landau-Ginzburg functional,

$$\mathcal{H} = \mathcal{H}_0 - \int d^dx\,(h_{\alpha\beta}m_{\alpha\beta} + h_\alpha\psi_\alpha) \quad , \tag{4.3}$$

$$\mathcal{H}_0 = \int d^dx\left(\tfrac{1}{2}\nabla\psi_\alpha\nabla\psi_\alpha + \tfrac{1}{2}r_0\psi_\alpha\psi_\alpha + \frac{1}{4!}u_0(\psi_\alpha\psi_\alpha)^2 + \tfrac{1}{4}m_{\alpha\beta}m_{\alpha\beta}\right) . \tag{4.4}$$

Repeated indices are summed over, $h_{\alpha\beta}$ and $\zeta_{\alpha\beta}$ are antisymmetric tensors. The Gaussian noises θ, ζ are governed by the correlation functions

$$\langle\theta_\alpha(xt)\theta_\beta(x't')\rangle = 2\Gamma_0\delta_{\alpha\beta}\delta(x-x')\delta(t-t') \tag{4.5}$$

$$\langle\zeta_{\alpha\beta}(xt)\zeta_{\alpha'\beta'}(x't')\rangle = -2\Lambda_0(\delta_{\alpha\alpha'}\delta_{\beta\beta'} - \delta_{\alpha\beta'}\delta_{\beta\alpha'})\nabla^2\delta(x-x')\delta(t-t') \tag{4.6}$$

The MSR action is obtained as above. It generates a perturbation expansion which only differs from the one of Appendix B by graph multiplicity.

The renormalization scheme is unchanged, so is the Wilson function expression (3.8)–(3.10) given in terms of the exponent functions η_Γ, η_Λ which now become

$$\eta_\Gamma = -f(n-1)/(1+w) + f^2G_{\rm AF}(w) + \eta_\Gamma^A(u) \quad , \tag{4.7}$$

$$\eta_\Lambda = -\tfrac{1}{2}f + f^2L_{\rm AF}(w) \quad . \tag{4.8}$$

These equations replace (3.26), (3.27), η_Γ^A is given by (3.28) and we have

$$G_{\rm AF}(w) = \frac{n-1}{2(1+w)^3}\left[[1+(2-n)w](1+2w)\ln\frac{2(1+w)}{1+2w} + (1+nw)\ln\tfrac{1}{2}(1+w) + \tfrac{1}{2}(27\ln\tfrac{4}{3}-6)(1+w) + (n-1)w\right] \tag{4.9}$$

$$L_{\rm AF}(w) = -\frac{1}{4(1+w)}\left[w(2+w)(n-2+w)\ln\frac{(1+w)^2}{w(2+w)} + w + \tfrac{3}{2} - \frac{n}{2}\right] \quad , \tag{4.10}$$

instead of (3.29), (3.30). The separatrix where ω_w vanishes, takes the form

$$n_c(\epsilon) = \tfrac{3}{2} + p\epsilon + O(\epsilon^2) \quad , \tag{4.11}$$

$$p = (\tfrac{27}{8} + \tfrac{42}{361})\ln\tfrac{4}{3} - (\tfrac{9}{16} + \tfrac{7}{361}) = 0.42 \quad . \tag{4.12}$$

The physical point ($\epsilon=1$, $n=3$) is far to the right of the tangent (at $\epsilon=0$) to the separatrix, i.e., well in the stability region of the dynamic scaling fixed point

$$w^* = (2n-3) + 4(n-1)\,\epsilon[L_{AF}(2n-3) - G_{AF}(2n-3)] - \frac{4(n-1)}{\epsilon}\eta_\Gamma^A(u^*) \ , \tag{4.13}$$

$$f^* = \epsilon + 2\epsilon^2 L_{AF}(2n-3) \tag{4.14}$$

The transients computed at this fixed point become

$$\omega_f = \epsilon + \epsilon^2 F + \frac{4(n-1)}{2n-1}\eta_\Gamma^A(u^*) \ , \tag{4.15}$$

$$\omega_w = \frac{2n-3}{4(n-1)}\epsilon + \epsilon^2 W - \frac{1}{(2n-1)(n-1)}\eta_\Gamma^A(u^*) \ , \tag{4.16}$$

with

$$F = -\frac{4(n-1)}{2n-1}G_{AF}(2n-3) - \frac{2}{2n-1}L_{AF}(2n-3) \ , \tag{4.17}$$

$$W = \frac{4n^2-10n+5}{(2n-1)(n-1)}G_{AF}(2n-3) - \frac{4n^2-12n+7}{2(2n-1)(n-1)}L_{AF}(2n-3) + (2n-3)[\dot{G}_{AF}(2n-3) - \dot{L}_{AF}(2n-3)] \ . \tag{4.18}$$

For $n = 3$, we have

$$\omega_f = \epsilon - 0.314\epsilon^2 \ , \tag{4.19}$$

$$\omega_w = \tfrac{3}{8}\epsilon - 0.103\epsilon^2 \ , \tag{4.20}$$

confirming that the physical point lies in the stability region of the dynamic scaling fixed point.

The unstable $w^* = 0$ fixed point had already been noticed by Gunton and Kawasaki.[16]

B. Liquid-gas model

The order parameter ψ, a linear combination of energy and mass density, is a conserved real scalar ($n = 1$). The conserved field $\vec{j}$, coupled to ψ in equations of motion, is the transverse part of the momentum density.

1. Equations of motion

$$\frac{\partial\psi}{\partial t} = \Lambda_0\nabla^2\frac{\delta\mathcal{H}}{\delta\psi} - g_0(\vec{\nabla}_\psi)\frac{\delta\mathcal{H}}{\delta\vec{j}} + \theta \ , \tag{4.21}$$

$$\frac{\partial\vec{j}}{\partial t} = \mathfrak{C}\left[\eta_0\nabla^2\frac{\delta\mathcal{H}}{\delta\vec{j}} + g_0(\vec{\nabla}\psi)\frac{\delta\mathcal{H}}{\delta\psi} + \vec{\zeta}\right] \ , \tag{4.22}$$

$$\mathcal{H} = \mathcal{H}_0 - \int d^dx\,(h\psi + \vec{h}_j\cdot\vec{j}) \ , \tag{4.23}$$

$$\mathcal{H}_0 = \int d^dx\left[\tfrac{1}{2}r_0\psi^2 + \tfrac{1}{2}(\vec{\nabla}\psi)^2 + \frac{u_0}{4!}\psi^4 + \tfrac{1}{2}j^2\right] \ . \tag{4.24}$$

The projector $\mathfrak{C}$ selects the transverse part of the vector it applies to

$$\mathfrak{C}_{\alpha\beta}(k) = \delta_{\alpha\beta} - k_\alpha k_\beta/k^2 \tag{4.25}$$

θ, $\vec{\zeta}$ are corresponding Gaussian noises.

2. Renormalization

The MSR action is constructed from (4.21)–(4.25) as above. It generates a perturbation expansion where parameters appear in combinations with the following dimensions

$$g_0^2/\Lambda_0\eta_0 = \epsilon \ , \tag{4.26}$$

$$\Lambda_0/\eta_0 = -2 \ . \tag{4.27}$$

It is clear from (4.27) that

$$w_0 \equiv \Lambda_0/\eta_0 \tag{4.28}$$

is an irrelevant parameter which must vanish at the fixed point. We are therefore interested in perturbation expansion for very small values of this parameter ($w_0\mu^2 \ll 1$). This forces us to compute regularized perturbation theory for zero value of w_0 and treat its effect as an "insertion" (in effect it is not an insertion in the field-theoretical sense, since w_0 is not coupled to a local operator). The situation is analogous to that encountered in the discussion of critical dynamics above dimension two where[38] one had to compute in the limit $u = \infty$ (hard-sphere limit) without being able to treat u^{-1} as an insertion. The renormalization scheme is then fairly obvious. Besides the static renormalization functions Z_ψ, Z_u, $Z_j = 1$, one is left with $\hat{Z}_\psi = 1$ (order-parameter conservation), $Z_g = 1$ [Ward identity corresponding to Galilean invariance of (4.21), (4.22)], and Z_η, Z_Λ. With the standard definitions

$$K_d(g_0^2/\Lambda_0\eta_0) \equiv f\mu^\epsilon Z_\Lambda Z_\eta \ , \tag{4.29}$$

$$w_0 = w\mu^{-2}Z_\eta Z_\Lambda^{-1} \ , \tag{4.30}$$

and (3.3), (3.4), we get

$$W_f = -f(\epsilon + \eta_\Lambda + \eta_\eta) \ , \tag{4.31}$$

$$W_w = w(2 + \eta_\Lambda - \eta_\eta) \ . \tag{4.32}$$

From (4.32) we see that the only possible fixed point value for w in perturbation expansion is $w^* = 0$. The exponent functions are obtained, up to two loops as

$$\eta_\Lambda = \tfrac{3}{4}f + 0.141f^2 - \eta_\psi^*(u^*) \ , \tag{4.33}$$

$$\eta_\eta = \tfrac{1}{24}f + 0.029f^2 \ . \tag{4.34}$$

The stable fixed point for W_f is at $f^* \neq 0$ which entails

$$\epsilon + \eta_\Lambda{}^* + \eta_\eta{}^* = 0 \quad , \tag{4.35}$$

corresponding to the Kadanoff-Swift[22] scaling law. The transient exponents are derived from (4.31), (4.32) as

$$\omega_f = -f\frac{\partial}{\partial f}(\eta_\Lambda + \eta_\eta)\big|_{f=f^*} \quad , \tag{4.36}$$

$$\omega_w = 2 + \eta_\Lambda{}^* - \eta_\eta{}^* \quad , \tag{4.37}$$

yielding to order ϵ^2,

$$\omega_f = \epsilon + 0.121\epsilon^2 \quad , \tag{4.38}$$

$$\omega_w = 2 - \tfrac{17}{19}\epsilon + 0.163\epsilon^2 \quad . \tag{4.39}$$

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APPENDIX A: ZERO-FREQUENCY LIMIT OF MULTILINEAR RESPONSE FUNCTIONS

We prove in this Appendix that the zero-frequency limit of a multilinear response function is equal to the corresponding static correlation function [Eq. (1.15)]. It is convenient to use in this case the Fokker-Planck formalism as used, e.g., by Ma and Mazenko[6] to whom we refer for more detail.

The zero-frequency pth response function (1.15) may be written[6]

$$\frac{\delta^{(p)}\langle\varphi_0\rangle}{\delta h_1 \cdots \delta h_p} = \int \mathfrak{D}\varphi_j[\varphi_0 D^{-1}A_1 D^{-1}A_2 \cdots D^{-1}A_p e^{-\mathcal{H}} + \mathrm{Perm}(1,2,\ldots,p)] \quad , \tag{A1}$$

where D is the Fokker-Planck operator corresponding to Eq. (1.1)

$$D = \frac{\delta}{\delta\varphi_l}\left[-V_l + (\Gamma_0)_{lm}\left(\frac{\delta}{\delta\varphi_m} + \frac{\delta\mathcal{H}}{\delta\varphi_m}\right)\right] \tag{A2}$$

and A_j is the operator that couples to the external field h_j

$$A_j = \frac{\delta}{\delta\varphi_k}[(\Gamma_0)_{jk} - Q_{jk}] \quad . \tag{A3}$$

The streaming velocity V_j is given by (1.3) and $\delta/\delta\varphi_k$ plays a role analogous to that of $\hat{\varphi}_k$.

We wish to show that (A1) is equal to the static response, given by

$$\langle\varphi_0\varphi_1\varphi_2 \cdots \varphi_p\rangle = \int \mathfrak{D}\varphi_j\varphi_0\varphi_1 \cdots \varphi_p e^{-\mathcal{H}} \quad . \tag{A4}$$

It will be obviously sufficient to prove for any number p of operators the identity

$$\mathcal{O}_p \equiv A_1 D^{-1} A_2 \cdots D^{-1}A_p e^{-\mathcal{H}} + \mathrm{Perm}(1,2,\ldots,p)$$
$$= D\varphi_1\varphi_2 \cdots \varphi_p e^{-\mathcal{H}} \quad . \tag{A5}$$

For $p=1$ (A5) reduces to

$$A_i e^{-\mathcal{H}} = D\varphi_i e^{-\mathcal{H}} \quad , \tag{A6}$$

which is in our notation relation (2.34) of Ref. 6 and may be directly verified. We can now prove (A5) by induction on p. If we assume that (A5) is satisfied for $(p-1)$ we may write the left-hand side as follows:

$$\mathcal{O}_p = \sum_j A_j\varphi_1\varphi_2 \cdots (\varphi_j) \cdots \varphi_p e^{-\mathcal{H}} \quad , \tag{A7}$$

where (φ_j) indicates that φ_j does *not* appear in the product. On the other hand, one obtains from (A3),

$$[A_j, \varphi_l] = (\Gamma_0)_{jl} - Q_{jl} \quad . \tag{A8}$$

Since the commutator in (A8) commutes with all φ's we may commute the A_j's to the right and apply (A6) to obtain

$$\mathcal{O}_p = \sum_j \sum_{l\neq j}[A_j, \varphi_l]\varphi_1 \cdots (\varphi_j) \cdots (\varphi_l) \cdots \varphi_p e^{-\mathcal{H}}$$
$$+ \sum_j \varphi_1 \cdots (\varphi_j) \cdots \varphi_p D\varphi_j e^{-\mathcal{H}} \quad , \tag{A9}$$

and by use of (A8) and of the antisymmetry of Q_{jl} we obtain

$$\mathcal{O}_p = 2\sum_j \sum_{l>j}(\Gamma_0)_{jl}\varphi_1 \cdots (\varphi_j) \cdots (\varphi_l) \cdots \varphi_p e^{-\mathcal{H}}$$
$$+ \sum_j \varphi_1 \cdots (\varphi_j) \cdots \varphi_p D\varphi_l e^{-\mathcal{H}} \quad . \tag{A10}$$

We have, on the other hand,

$$[D, \varphi_j] = -V_j + (\Gamma_0)_{jl}\frac{\delta\mathcal{H}}{\delta\varphi_l} + 2(\Gamma_0)_{jl}\frac{\delta}{\delta\varphi_l} \tag{A11}$$

and

$$[[D,\varphi_j],\varphi_l] = 2(\Gamma_0)_{jl} \quad , \tag{A12}$$

which commutes with all the rest. Since $e^{-\mathcal{H}}$ is the equilibrium distribution we have $De^{-\mathcal{H}} = 0$ and

$$[D, \varphi_j]e^{-\mathcal{H}} = D\varphi_j e^{-\mathcal{H}} \quad . \tag{A13}$$

We may write, therefore,

$$D\varphi_1\varphi_2 \cdots \varphi_p e^{-\mathcal{H}} = [D,\varphi_1]\varphi_2 \cdots \varphi_p e^{-\mathcal{H}} + \varphi_1[D,\varphi_2] \cdots \varphi_p e^{-\mathcal{H}} + \cdots + \varphi_1 \cdots \varphi_{p-1}[D,\varphi_p]e^{-\mathcal{H}}$$
$$= \sum_j \sum_{l>j}[[D,\varphi_j],\varphi_l]\varphi_1 \cdots (\varphi_j) \cdots (\varphi_l) \cdots \varphi_p e^{-\mathcal{H}} + \sum_j \varphi_1 \cdots (\varphi_j) \cdots \varphi_p D\varphi_j e^{-\mathcal{H}} \quad , \tag{A14}$$

where we have used (A13). It is clear from (A12) that (A14) is equal to (A10).

APPENDIX B: PERTURBATION EXPANSION

A. Diagrammatic rules for model E

We understand that the change of variables mentioned in Ref. 26 has been performed. We indicate by $\langle \cdots \rangle_0$ the average over the quadratic part of (1.23) and consider a complex Γ_0 for generality.

1. Propagator lines (Fig. 1)

$$\text{(a)}:\ G^0_{\psi\hat{\psi}}(1,2) \equiv \langle \psi(1)\hat{\psi}(2)\rangle_0 = -\theta(t_1-t_2)\int \frac{d^dk}{(2\pi)^d}\exp\{-[ik(x_1-x_2)+\Gamma_0(1+ib_0)(k^2+r_0)(t_1-t_2)]\}\ , \tag{B1}$$

$$G^0_{\psi\hat{\psi}}(k,\omega) = -[-i\omega+\Gamma_0(1+ib_0)(k^2+r_0)]^{-1}\ , \tag{B2}$$

$$\text{(b)}:\ G^0_{\psi^*\hat{\psi}^*}(1,2) = [G^0_{\psi\hat{\psi}}(1,2)]^* = -\theta(t_1-t_2)\int \frac{d^dk}{(2\pi)^d}\exp\{-[ik(x_1-x_2)+\Gamma_0(1-ib_0)(k^2+r_0)(t_1-t_2)]\}\ , \tag{B3}$$

$$G^0_{\psi^*\hat{\psi}^*}(k,\omega) = -[-i\omega+\Gamma_0(1-ib_0)(k^2+r_0)]^{-1}\ , \tag{B4}$$

$$\text{(c)}:\ G^0_{m\hat{m}}(1,2) \equiv \langle m(1)\hat{m}(2)\rangle_0 = -\theta(t_1-t_2)\int \frac{d^dk}{(2\pi)^d}\exp\{-[ik(x_1-x_2)+\Lambda_0 k^2(t_1-t_2)]\}\ , \tag{B5}$$

$$G^0_{m\hat{m}}(k,\omega) = -(-i\omega+\Lambda_0 k^2)^{-1}\ . \tag{B6}$$

Here $\theta(t)$ is the unit step function.

2. Correlation lines (Fig. 2)

$$\text{(a)}:\ G^0_{\psi\psi^*}(1,2) \equiv \langle \psi(1)\psi^*(2)\rangle = \int \frac{d^dk}{(2\pi)^d}\exp\{-[ik(x_1-x_2)+\Gamma_0(k^2+r_0)|t_1-t_2|]\}(k^2+r_0)^{-1} \tag{B7}$$

$$G^0_{\psi\psi^*}(k,\omega) = 2\Gamma_0/|-i\omega+\Gamma_0(1+ib_0)(k^2+r_0)|^2\ , \tag{B8}$$

$$\text{(b)}:\ G_{mm}(1,2) \equiv \langle m(1)m(2)\rangle_0 = \int \frac{d^dk}{(2\pi)^d}\exp\{-[ik(x_1-x_2)+\Lambda_0 k^2]\}\ , \tag{B9}$$

$$G_{mm}(k,\omega) = 2\Lambda_0 k^2/|-i\omega+\Lambda_0 k^2|^2\ . \tag{B10}$$

3. Irreversible vertices (Fig. 3)

$$\text{(a)}:\ \Gamma_0(1+ib_0)u_0\ .$$
$$\text{(b)}:\ \Gamma_0(1-ib_0)u_0\ . \tag{B11}$$

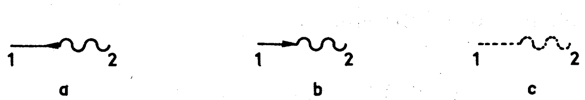

FIG. 1. Propagator lines.

FIG. 2. Correlation lines.

FIG. 3. Irreversible vertices.

4. Reversible vertices (Fig. 4)

$$\begin{aligned}&(\mathrm{a}):\ ig_0\ ,\\ &(\mathrm{b}):\ -ig_0\ .\end{aligned} \tag{B12}$$

$$(\mathrm{c}):\ ig_0(-\nabla_1^2+\nabla_2^2)\ \text{(in configuration space)}\ , \tag{B13}$$

$$ig_0(k_1^2-k_2^2)\ \text{(in momentum space)}\ . \tag{B14}$$

5. Sources (Fig. 5)

$$\begin{aligned}&(\mathrm{a}):\ -\Gamma_0(1+ib_0)h\ .\\ &(\mathrm{b}):\ -\Gamma_0(1-ib_0)h^*\ .\end{aligned} \tag{B15}$$

$$\begin{aligned}&(\mathrm{c}):\ -ig_0h\ .\\ &(\mathrm{d}):\ ig_0h^*\ .\end{aligned} \tag{B16}$$

$$(\mathrm{e}):\ \Lambda_0\nabla^2h_m\ \text{or}\ -\Lambda_0k^2h_m\ . \tag{B17}$$

$$\begin{aligned}&(\mathrm{f}):\ -ig_0h_m\ .\\ &(\mathrm{g}):\ ig_0h_m\ .\end{aligned} \tag{B18}$$

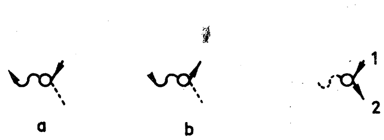

FIG. 4. Reversible vertices.

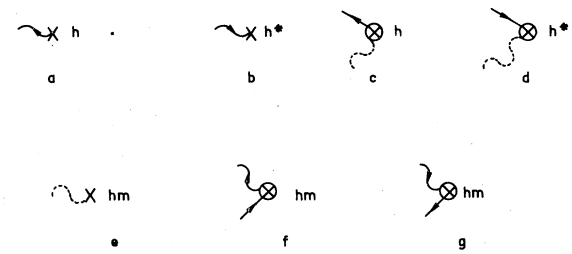

FIG. 5. Sources.

B. Diagram rules after integration over internal frequencies

(i) Arrange all vertices and sources from right to left in the order of their time labels, taking care of the retarded nature of the propagator lines. (ii) Associate with each vertex and source the above contributions. (iii) Associate with each correlation line the corresponding static contribution: $(k^2+r_0)^{-1}$ to $G^0_{\psi\psi^*}$ and 1 to G^0_{mm}. (iv) Associate with each time interval between successive vertices the factor

$$\left[-i\Omega+\sum_i\Gamma_0(1+ib_0)(k_i^2+r_0)+\sum_j\Gamma_0(1-ib_0)(k_j^2+r_0)+\sum_l\Lambda_0k_l^2\right]^{-1}\ , \tag{B19}$$

where the summation runs over all lines present in the interval, i runs over all lines with arrow pointing to the left, j over those with arrow pointing to the right, and l over all broken lines. The quantity Ω is the sum of all external frequencies present in the interval. (v) Multiply by $(-1)^l$ where l is the number of propagator lines and by the multiplicity factor of the diagram computed as if it were a static diagram.

C. Role of the Jacobian

The Jacobian associated with the δ functions is

$$J=\left\{\det\left[\left(\delta_{jj'}\frac{\partial}{\partial t}+\frac{\delta K_j(t)}{\delta\varphi_{j'}(t)}\right)\delta(t-t')\right]\right\}\ , \tag{B20}$$

i.e., up to a constant multiplicative factor [that drops out in the ratio (1.7)], in symbolic form,

$$J=\exp\left\{\mathrm{Tr}\ln\left[\left(\frac{\partial}{\partial t}+\frac{\delta K(t)}{\delta\varphi(t)}\right)\frac{\delta(t-t')}{\partial\delta(t-t')/\partial t}\right]\right\} \tag{B21}$$

$$=\exp\left\{\mathrm{Tr}\ln\left[1+\left(\frac{\partial}{\partial t}\right)^{-1}\frac{\delta K(t)}{\delta\varphi(t')}\right]\right\}\ . \tag{B22}$$

Since the operator $(\partial/\partial t)^{-1}$ is retarded only the lowest-order term survives by taking the trace, therefore,

$$J = \exp\left[-\frac{1}{2}\int dt \,\frac{\delta K_j\{\varphi(t)\}}{\delta\varphi_j(t)}\right] , \tag{B23}$$

where the factor $\frac{1}{2}$ comes from the value of the step function at the zero value of its argument [(B23) may also be proved from (B22) and from the theory of Fredholm determinants of Volterra equations].

Consider now the contributions arising from the action in (1.8), where $\hat{\varphi}_j$ and one φ_j from the *same* coupling term in $\hat{\varphi}_j K_j\{\varphi\}$ close onto a loop. Since $G_{\hat{\varphi}\varphi}$ is retarded, all such contributions vanish except the one with a single propagator line (this is also known as the Deker-Haake theorem[23]). One may easily see that the corresponding contribution exactly cancels (B23) order by order in perturbation theory. One may therefore eliminate all such loops and forget about the $\ln J$ term.

APPENDIX C: VERTEX RENORMALIZATION OF MSR FIELD THEORIES

For ordinary field theories as in statics or purely relaxational dynamics, renormalization of the coupling constants introduces Z functions (Z_u, Z_γ) that absorb primitive divergences of the corresponding vertices ($\Gamma_{\psi^*\psi\psi^*\psi}$ for Z_u, $\Gamma_{m\psi\psi^*}$ for Z_γ). In MSR field theories each coupling constant appears several times in the Lagrangian density. For example, g_0 as such, appears for *model E* as

$$ig_0(\hat{\psi}\psi m - \hat{\psi}^*\psi^* m) - g_0\hat{m}(\psi^*\nabla^2\psi - \hat{m}\psi\nabla^2\psi^*) , \tag{C1}$$

where one reads, as usual, the zero-loop contribution to $\Gamma_{\hat{\psi}\psi m}$, $\Gamma_{\hat{\psi}^*\psi^* m}$, and $\Gamma_{\hat{m}\psi\psi^*}$. These Γ vertices possess independent primitive divergences that *a priori* cannot be taken care of by just renormalizing the coupling constant g_0. One should thus write (C1) in its renormalized form as

$$ig\frac{\mu^{\epsilon/2}}{K_d^{1/2}} Z_g Z_\psi Z_m^{1/2}[\hat{Z}_\psi^{1/2}(\hat{\psi}_R\psi_R m_R - \hat{\psi}_R{}^*\psi_R{}^* m_R)$$
$$- \hat{Z}_m^{-1}\zeta_g \hat{m}_R(\psi_R{}^*\nabla^2\psi_R - \psi_R\nabla^2\psi^*)] , \tag{C2}$$

where ζ_g (the tenth renormalization function of Sec. III B) should take care of the extra independent primitive divergences.

We show here that $\zeta_g = 1$, together with the results of Secs. III B–III D $[Z_g = Z_m = \hat{Z}_m = 1,\ \hat{Z}_\psi/Z_\psi$ determined by the finiteness of $\partial/\partial(-i\omega)\Gamma_{\hat{\psi}\psi}]$, is enough to take care of primitive divergences of both $\Gamma_{\hat{\psi}\psi m}$ and $\Gamma_{\hat{m}\psi\psi^*}$.

A. $\Gamma_{\hat{\psi}\psi_m}$ vertex

We analyze the bilinear response $\delta^2\langle\psi\rangle/\delta h\,\delta h_m$ as

$$\frac{\delta^2\langle\psi\rangle}{\delta h\,\delta m} = G_{\psi\hat{\psi}}(\Gamma_{\hat{\psi}\psi m}R_\psi R_m$$
$$+\Gamma_{\hat{\psi}\tilde{\psi}m}R_m + \Gamma_{\hat{\psi}\psi\tilde{m}}R_\psi + \Gamma_{\hat{\psi}\tilde{\psi}\tilde{m}}) , \tag{C3}$$

where we have introduced for short, the notation

$$\tilde{\psi} = ig_0\hat{m}\psi^* , \quad \tilde{m} = ig_0(\hat{\psi}\psi - \hat{\psi}^*\psi^*) ,$$

for the parts that couple to external fields h and h_m through mode coupling. For zero frequencies the left-hand side is the *static* response. It vanishes for model E $(\gamma_0 = 0)$ since the fields m and ψ are decoupled. Thus, we have (for *zero frequencies)*

$$\Gamma_{\hat{\psi}\psi m} = -\Gamma_{\hat{\psi}\tilde{\psi}m}R_\psi^{-1} - \Gamma_{\hat{\psi}\psi\tilde{m}}R_m^{-1} - \Gamma_{\hat{\psi}\tilde{\psi}\tilde{m}}R_\psi^{-1}R_m^{-1} , \quad \omega_i = 0 \tag{C4}$$

On the right-hand side the factors R_ψ, $\Gamma_{\hat{\psi}\psi\tilde{m}}$ contain primitive divergences but $\Gamma_{\hat{\psi}\tilde{\psi}m}$ and $\Gamma_{\hat{\psi}\tilde{\psi}\tilde{m}}$ do not; besides, $R_m = 1$ at zero frequency. To contribute to the primitive divergences[28] of $\Gamma_{\hat{\psi}\psi m}$, one is only allowed to keep zero-loop contributions of the factors that multiply an already primitively divergent factor. Now $\Gamma_{\hat{\psi}\tilde{\psi}m}$, $\Gamma_{\hat{\psi}\tilde{\psi}\tilde{m}}$ have no zero-loop contribution. Hence, when focusing only on primitive divergences ($\boldsymbol{\mathcal{P}}$) one has

$$(\Gamma_{\hat{\psi}\psi m})_{\mathcal{P}} = -(\Gamma_{\hat{\psi}\psi\tilde{m}})_{\mathcal{P}} = -ig_0\left[\frac{\partial}{\partial i\omega}\Gamma_{\hat{\psi}\psi}\right]_{\mathcal{P}} , \quad \omega_i = 0 \tag{C5}$$

where the last equation is a weak form of the Ward identity for model E. Equations (C5) show that renormalizing the vertex $\Gamma_{\hat{\psi}\psi m}$ (or its complex conjugate) as in (C2) that is, introducing for its zero-loop term $ig\mu^{\epsilon/2}K_d^{-1/2}Z_\psi\hat{Z}_\psi^{-1}$, is enough to absorb its primitive divergences.

Finally, going to nonzero frequencies introduces no new divergence since all the above primitive divergences are logarithmic.

B. $\Gamma_{\hat{m}\psi\psi^*}$ vertex

The bilinear response considered is now

$$\frac{\delta^2\langle m\rangle}{\delta h\,\delta h^*} = G_{m\hat{m}}(\Gamma_{\hat{m}\psi\psi^*}R_\psi R_{\psi^*} + \Gamma_{\hat{m}\tilde{\psi}\psi^*}R_{\psi^*}$$
$$+\Gamma_{\hat{m}\psi\tilde{\psi}^*}R_\psi + \Gamma_{m\tilde{\psi}\tilde{\psi}^*}) . \tag{C6}$$

Again for *zero* external *frequencies,* one has

$$\Gamma_{\hat{m}\psi\psi^*} = -\Gamma_{\hat{m}\tilde{\psi}\psi^*}R_\psi^{-1}$$
$$-\Gamma_{\hat{m}\psi\tilde{\psi}^*}R_{\psi^*}^{-1} - \Gamma_{\hat{m}\tilde{\psi}\tilde{\psi}^*}R_\psi^{-1}R_{\psi^*}^{-1} , \quad \omega_i = 0 \tag{C7}$$

Here R_ψ, R_{ψ^*} are primitively divergent ; $\Gamma_{\hat{m}\hat{\psi}\psi^*}$, $\Gamma_{\hat{m}\psi\hat{\psi}^*}$, and $\Gamma_{\hat{m}\hat{\psi}\hat{\psi}^*}$ are not. Besides $\Gamma_{\hat{m}\hat{\psi}\psi^*}$ has a zero-loop contribution (ig_0), but $\Gamma_{\hat{m}\hat{\psi}\hat{\psi}^*}$ has none. Hence, one may write, for the primitively divergent parts

$$(\Gamma_{\hat{m}\psi\psi^*})_{\mathcal{P}} = -ig_0(R_\psi^{-1} - R_{\psi^*}^{-1})_{\mathcal{P}} \ , \quad \omega_i = 0 \ . \tag{C8}$$

In (C2) we have renormalized the vertex $\Gamma_{\hat{m}\psi\psi^*}$ with a factor $\simeq gZ_\psi\zeta_g$. Equation (C8) shows that the factor Z_ψ is able to take care of all primitive divergences of $R_\psi^{-1} - R_{\psi^*}^{-1}$ and hence we have

$$\zeta_g = 1 \ , \tag{C9}$$

the extension to nonzero frequencies bringing nothing novel as above.

When the Landau-Ginzburg functional contains a coupling $\gamma_0 m\psi\psi^*$ as in *model F*, the corresponding MSR Lagrangian density contains, besides the terms of (C1), the couplings

$$\gamma_0\Gamma_0(\hat{\psi}\psi m + \hat{\psi}^*\psi^* m) + \gamma_0\Lambda_0 k^2 \hat{m}\psi\psi^* \ , \tag{C10}$$

whose coefficients are supplementary zero-loop contributions to $\Gamma_{\hat{\psi}\psi m}$, $\Gamma_{\hat{\psi}^*\psi^* m}$, $\Gamma_{\hat{m}\psi\psi^*}$, respectively. With the γ_0 coupling it is no longer true that the left-hand side of (C3),(C6) vanish for zero external frequencies. Instead, e.g., (C6) yields

$$\Gamma_{\hat{m}\psi\psi^*} = \Lambda_0 k^2(1+P)\Gamma^{\rm st}_{m\psi\psi^*} - \Gamma_{\hat{m}\hat{\psi}\psi^*}R^{-1} - \Gamma_{\hat{m}\psi\hat{\psi}^*}R_{\psi^*}^{-1} - \Gamma_{\hat{m}\hat{\psi}\hat{\psi}^*}R_\psi^{-1}R_{\psi^*}^{-1} \ , \quad \omega_i = 0 \ , \tag{C11}$$

where we have used (2.20) and $\Gamma^{\rm st}_{m\psi\psi^*}$ is the one-irreducible vertex function of the static theory (as developed in Ref. 12). Here both P and $\Gamma^{\rm st}_{m\psi\psi^*}$ are (logarithmically) primitively divergent, but $\Gamma^{\rm st}_{m\psi\psi^*}$ only has a zero-loop contribution (γ_0). Hence, instead of (C8) one has

$$(\Gamma_{\hat{m}\psi\psi^*})_{\mathcal{P}} = \Lambda_0 k^2(\Gamma^{\rm st}_{m\psi\psi^*})_{\mathcal{P}} + \Lambda_0 k^2\gamma_0(P)_{\mathcal{P}} - ig_0(R_\psi^{-1} - R_{\psi^*}^{-1})_{\mathcal{P}} \ , \quad \omega_i = 0 \tag{C12}$$

Correspondingly, the renormalized form of the $\Gamma_{\hat{m}\psi\psi^*}$ vertex as appearing in (C1),(C.10) should be written

$$-ig\frac{\mu^{\epsilon/2}}{K_d^{1/2}}Z_\psi\zeta_g\hat{m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) + (\gamma\mu^{\epsilon/2}/K_d^{1/2})(Z_\gamma\Lambda/\hat{Z}_m Z_\Lambda)\zeta_\gamma k^2\hat{m}\psi^*\psi \ , \tag{C13}$$

where we have used (3.72), the Ward identity (2.30), and relation (2.14). We have introduced along the same line as in the above discussion an extra renormalization function ζ_γ. This function together with Z_γ takes care of the fact that the coupling constant γ_0 appears in (C10) with (the real part of) both vertices $\Gamma_{\hat{\psi}\psi m}$ and $\Gamma_{\hat{m}\psi\psi^*}$. Comparing (C12) and (C13) one sees that Z_ψ absorbs the primitive divergences of $R_\psi^{-1} - R_{\psi^*}^{-1}$, $(\hat{Z}_m Z_\Lambda)^{-1}$ those of P, and Z_γ (as in static theory) those of $\Gamma^{\rm st}_{m\psi\psi^*}$. Thus, again, we have

$$\zeta_g = \zeta_\gamma = 1 \ . \tag{C14}$$

The same treatment applied to $\Gamma_{\hat{\psi}\psi m}$ confirms (C14). It is easily extended to $\Gamma_{\hat{\psi}\psi\psi^*\psi}$ by considering the trilinear responses $\delta^3\langle\psi\rangle/\delta h\,\delta h\,\delta h^*$ and complex conjugate to show eventually that the standard renormalization procedure as defined by (2.4)–(2.12) and (3.70)–(3.72) is enough to absorb all primitive divergences appearing in the theory.

Once primitive divergences are provided for, nonprimitive divergences can be shown to be also absorbed into the Z functions by the same order by order recurrent combinations that appears in standard field theories[39] either by working on the loop expansion of the one-irreducible Γ vertices as generated by (2.1),(2.2) or by working out or proof by introduction using Callan-Symanzik equations.[40,41]

APPENDIX D: RENORMALIZATION FUNCTIONS

We list here the graphs computed in the two-loop renormalization of model E. It is understood that the sum over all possible choices of propagator and correlation lines compatible with time ordering as well as all internal frequency summations have been performed. The graphs are expressed as a function of the parameters

$$f_0 \equiv K_d g_0^2/\Lambda_0\Gamma_0 \ , \tag{D1}$$

$$w_0 \equiv \Gamma_0/\Lambda_0 \ . \tag{D2}$$

1. Graphs for S (Fig. 6)

$$\text{(a)}: \quad -\frac{(ig_0)^2}{\Gamma_0}\int d^dq \frac{1}{-i\omega + \Gamma_0[(k+q)^2 + r_0] + \Lambda_0 q^2}\frac{1}{(k+q)^2 + r_0} = k^{-\epsilon}f_0\frac{1}{1+w_0}\frac{1}{\epsilon}[1 + O(\epsilon)] \ . \tag{D3}$$

(b):

$$-\frac{(ig_0)^4}{\Gamma_0}\left(\frac{n}{2}\right)\int d^dp\int d^dq\frac{1}{\{-i\omega+\Gamma_0[(k+p)^2+r_0]+\Lambda_0p^2\}^2}\frac{1}{(k+p)^2+r_0}$$

$$\times\frac{[(p+q)^2-q^2]^2}{-i\omega+\Gamma_0[(k+p)^2+q^2+(p+q)^2+3r_0]}\frac{1}{(q^2+r_0)[(p+q)^2+r_0]}$$

$$=k^{-2\epsilon}f_0^2\left(\frac{n}{2}\right)\frac{1}{(1+w_0)^2}\left(-\frac{1}{4\epsilon^2}\right)\left[1-\tfrac{1}{2}\epsilon\left(5-9\ln\frac{4}{3}\right)+\epsilon\mathfrak{a}+O(\epsilon^2)\right]. \quad \text{(D4)}$$

(c):

$$-\frac{(ig_0)^4}{\Gamma_0}\int d^dp\int d^dq\frac{1}{\{-i\omega+\Gamma_0[(k+p)^2+r_0]+\Lambda_0p^2\}^2}\frac{1}{-i\omega+\Gamma_0[(k+p+q)^2+r_0]+\Lambda_0(p^2+q^2)}\frac{1}{(k+p+q)^2+r_0}$$

$$=k^{-2\epsilon}f_0^2\frac{w_0}{(1+w_0)^3}\left(-\frac{1}{2\epsilon^2}\right)\left[1-\tfrac{1}{2}\epsilon\left(1-\ln\frac{1+w_0}{2}+(1+2w_0)\ln\frac{2(1+w_0)}{1+2w_0}\right)+\epsilon B+O(\epsilon^2)\right] \quad \text{(D5)}$$

(d):

$$-\frac{(ig_0)^4}{\Gamma_0}\int d^dp\int d^dq\frac{1}{-i\omega+\Gamma_0[(k+p)^2+r_0]+\Lambda_0q^2}$$

$$\times\frac{1}{-i\omega+\Gamma_0[(k+p+q)^2+r_0]+\Lambda_0(p^2+q^2)}\frac{1}{-i\omega+\Gamma_0[(k+q)^2+r_0]+\Lambda_0q^2}\frac{1}{(k+p+q)^2+r_0}$$

$$=k^{-2\epsilon}f_0^2\frac{1}{(1+w_0)^2}\left(-\frac{1}{4\epsilon}\right)\left[(1+2w_0)\ln\frac{2(1+w_0)}{1+2w_0}+\ln\frac{1+w_0}{2}+O(\epsilon)\right], \quad \text{(D6)}$$

(e):

$$-\frac{(ig_0)^4}{\Gamma_0}\int d^dp\int d^dq\frac{1}{-i\omega+\Gamma_0[(k+p)^2+r_0]+\Lambda_0p^2}\frac{[(k+p)^2-(k+p+q)^2][(k+q)^2-(k+p+q)^2]}{-i\omega+\Gamma_0[(k+p)^2+(k+p+q)^2+(k+q)^2+3r_0]}$$

$$\times\frac{1}{-i\omega+\Gamma_0[(k+q)^2+r_0]+\Lambda_0q^2}\frac{1}{[(k+p)^2+r_0][(k+p)^2+r_0][(k+p+q)^2+r_0]}$$

$$=k^{-2\epsilon}f_0^2\frac{1}{(1+w_0)^2}\left(-\frac{1}{4\epsilon}\right)\left[9\ln\frac{4}{3}-2+O(\epsilon)\right]. \quad \text{(D7)}$$

The graph in Fig. 6(f) vanishes.

2. Graphs for P (Fig. 7)

(a):

$$-\frac{(ig_0)^2}{\Lambda_0k^2}\left(\frac{n}{2}\right)\int d^dq\frac{[(k+q)^2-q^2]^2}{-i\omega+\Gamma_0[(k+p)^2+q^2+2r_0]}\frac{1}{(k+q)^2+r_0}\frac{1}{q^2+r_0}=k^{-\epsilon}f_0\frac{n}{2}\frac{1}{2\epsilon}[1+O(\epsilon)] \quad \text{(D8)}$$

(b):

$$-\frac{(ig_0)^4}{\Lambda_0k^2}n\int d^dp\int d^dq\,\frac{[(k+p)^2-p^2]^2}{\{-i\omega+\Gamma_0[(k+p)^2+p^2+2r_0]\}}\frac{1}{-i\omega+\Gamma_0[(k+p+q)^2+p^2+2r_0]+\Lambda_0q^2}\frac{1}{(k+p+q)^2+r_0}$$

$$=k^{-2\epsilon}f_0^2\frac{n}{1+w_0}\left(-\frac{1}{8\epsilon^2}\right)\left[1-\frac{3\epsilon}{4}-\frac{\epsilon}{2}w_0(2+w_0)\frac{\ln(1+w_0)^2}{w_0(2+w_0)}+\epsilon\mathfrak{C}+O(\epsilon^2)\right] \quad \text{(D9)}$$

(c):

$$-\frac{(ig_0)^4}{\Lambda_0 k^2}\, n \int d^dp \int d^dq \frac{(k+p)^2-p^2}{-i\omega+\Gamma_0[(k+p)^2+p^2+2r_0]}$$
$$\times \frac{q^2-(k+q)^2}{-i\omega+\Gamma_0[(k+q)^2+q^2+2r_0]}\,\frac{1}{-i\omega+\Gamma_0(p^2+q^2+2r_0)+\Lambda_0(k-p+q)^2}\,\frac{1}{(p^2+r_0)(q^2+r_0)}$$
$$=k^{-2\epsilon} f_0^2\, n\frac{1}{16\epsilon}\left[1-w_0(2+w_0)\ln\frac{(1+w_0)^2}{w_0(2+w_0)}+O(\epsilon)\right] \quad \text{(D10)}$$

3. Graph for $\partial/\partial(-i\omega)\Gamma_{\tilde\psi\psi}$ *(Fig. 8)*

$$\Gamma_0 u_0^2\frac{n+2}{18}\int d^dp\int d^dq \frac{\Gamma_0[p^2+q^2+(k+p-q)^2+3r_0]}{\{-i\omega+\Gamma_0[p^2+q^2+(k+p-q)^2+3r_0]\}^2}\,\frac{1}{(p^2+r_0)(q^2+r_0)[(k+p-q)^2+r_0]}$$
$$=k^{-2\epsilon}\Gamma_0 u_0^2\frac{n+2}{36}\frac{1}{\epsilon}\left[\frac{3}{2}\ln\frac{4}{3}+O(\epsilon)\right]. \quad \text{(D11)}$$

Graph multiplicities are expressed in terms of the number of real components of the order parameter ($n=2$ for He) for the generalized helium models. The quantities $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ represent frequency- and momentum-dependent functions which are canceled by opposite contributions arising from the development of one-loop graphs in terms of renormalized quantities. Taking into account the lowest-order expressions of Z_Γ, Z_Λ:

$$Z_\Gamma=1+(f/\epsilon)1/(1+w)+\dots, \quad \text{(D12)}$$

$$Z_\Lambda=1+(f/\epsilon)(n/4)+\dots, \quad \text{(D13)}$$

and the renormalization of f, w (4.6)–(4.7) one has in fact the following for Figs. 6(a) and 7(a). For Fig. 6(a):

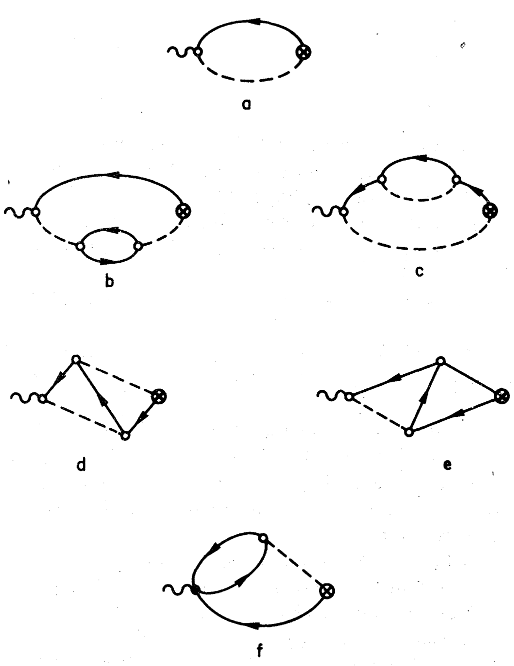

FIG. 6. Generic diagrams for S. The diagrams that refer to the computing rules of Appendix B are generated by drawing in all possible ways, a "tree" on a, b, c, d, e, f, and making each branch of the "tree" a propagator line (Fig. 1). This gives rise to eleven diagrams [a(1), b(2),c(2),d(2),e(2),f(2)].

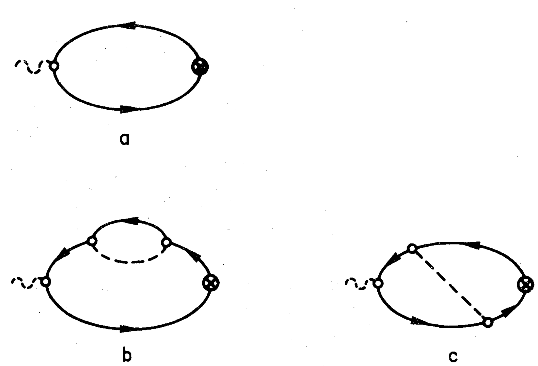

FIG. 7. Generic diagrams for P [giving rise to ten diagrams a(2), b(4), c(4)].

$$f\frac{1}{1+w}\frac{1}{\epsilon}+f^2\frac{1}{(1+w)^2}\frac{1}{2}n\frac{1}{2\epsilon^2}(1-\tfrac{3}{4}\epsilon+\tfrac{1}{2}\epsilon\mathfrak{A}+O(\epsilon^2))$$
$$+f^2\frac{w}{(1+w)^3}\frac{1}{\epsilon^2}[1-\tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon\mathfrak{B}+O(\epsilon^2)] \quad . \tag{D14}$$

For Fig. 7(a):

$$f\tfrac{1}{2}n\frac{1}{2\epsilon}+f^2\tfrac{1}{2}n\frac{1}{1+w}\frac{1}{2\epsilon^2}[1-\tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon E+O(\epsilon^2)] \quad .$$

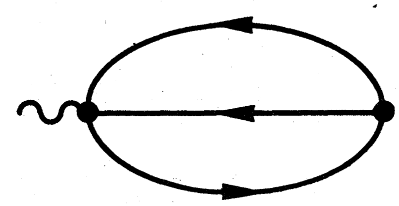

FIG. 8. Generic diagram for $\partial/\partial(-i\omega)\Gamma_{\hat{\psi}\psi}$ (three diagrams).

By requiring S^R and P^R to be finite we finally obtain

$$Z_\Gamma=1+\frac{f}{\epsilon}\frac{1}{1+w}+\frac{f^2}{\epsilon^2}\left[\frac{n}{8}\frac{1}{(1+w)^2}+\frac{1+\frac{3}{2}w}{(1+w)^3}\right]-\frac{f^2}{\epsilon}\frac{1}{16(1+w)^3}$$
$$\times\left[4(1+2w)\ln\frac{(1+w)^2}{1+2w}+9(4+n)(\ln\tfrac{4}{3})(1+w)-(4+n)w-(8+n)\right]$$
$$-\frac{u^2}{\epsilon}\frac{n+2}{36}\left[\frac{3}{2}\ln\frac{4}{3}-\frac{1}{4}\right]+\cdots \quad , \tag{D16}$$

$$Z_\Lambda=1+\frac{f}{\epsilon}\frac{n}{4}+\frac{f^2}{\epsilon^2}\left[\frac{n}{8}\frac{1}{1+w}+\frac{n^2}{16}\right]-\frac{f^2}{\epsilon}\frac{n}{16(1+w)}\left[w^2(2+w)\ln\frac{(1+w)^2}{w(2+w)}-w-\frac{1}{2}\right]+\cdots \quad , \tag{D17}$$

$$\hat{Z}_\psi=1+\frac{u^2}{\epsilon}\frac{n+2}{36}(\tfrac{3}{2}\ln\tfrac{4}{3}-\tfrac{1}{4})+\cdots \quad . \tag{D18}$$

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