Phase diagrams and multicritical points in randomly mixed magnets. I. Mixed anisotropies*

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The critical behavior of a quenched random alloy of two materials with competing anisotropies is studied by renormalization-group techniques. Averaging over the random variables yields a translationally invariant effective Hamiltonian, in which the *m*-component spin is replaced by a *nm*-component spin vector and the limit $n \rightarrow 0$ is taken at the end of the calculation. The only physical stable fixed point is "decoupled," leading to a Hamiltonian which separates into two parts, each depending on m_1 or $m_2 = m - m_1$ spin components only, asymptotically close to criticality. This leads to breakdown of standard scaling in the vicinity of the multicritical point. The resulting phase diagram exhibits two critical lines, corresponding to ordering of only m_1 (or m_2) spin components, intersecting at an angle at a tetracritical point. The phases where only m_1 or m_2 components order are separated by an intermediate phase, where all components order. The effect of corrections to scaling on the shape of the critical lines in the vicinity of the tetracritical point is examined. Experiments on rare-earth alloys and layered materials are discussed.

I. INTRODUCTION

Many physical systems exhibit critical behavior which depends on the interplay of more than one order parameter (or the different components of a multicomponent order parameter). A well-studied example is that of a uniaxially anisotropic antiferromagnet in a uniform magnetic field, in which the spins may order antiferromagnetically parallel to the field (at low fields) or transverse to the field (at high fields).¹ A second example is that of the structural displacive phase transitions in stressed perovskite crystals, where the direction of the rotations of the atomic octahedra depends on the external uniaxial stress.^{2,3} Recent renormalization-group calculations, following earlier meanfield theory studies, show that these, and many other examples of competing order parameters,³ exhibit phase diagrams like those shown in Fig. 1. The parameter g in this figure denotes the additional external variable, e.g., the magnetic field in the antiferromagnet or the uniaxial stress in the perovskites. Depending on details of the system, the phase diagram exhibits a *bicritical point* and a first-order "flop" line [Fig 1(a)] or a tetracritical point and a mixed intermediate phase [Fig. 1(b)]. Note that calculations of these phase diagrams predict that the second-order lines approach the multicritical point tangential to each other.3

In the present series of papers we consider a special type of such phase diagrams, i.e., that of random quenched alloys of materials, in which each component has a different order parameter. The variable g thus represents the relative concentration of the components in the alloy p. Clearly, there exists a large variety of such alloys. One can mix components which have different types of

magnetic ordering (e.g., ferromagnetic and antiferromagnetic), like $Fe(Pd_pPt_{1-p})_{3}$, $^4(Mn_{1-p}Fe_p)WO_4$, 5 or UAs_pSe_{1-p} , 6 different types of structural ordering, like $NH_4Cl_{1-p}Br_p$, 7 different easy axes, like $K_2Mn_pFe_{1-p}F_4$, 8 different helical orderings, 9 etc. One can also consider introducing completely different order parameters, such as magnetic ions embedded in superconductors, 10 etc. Unlike the examples mentioned earlier, most alloys seem to exhibit a *tetracritical* (and not a bicritical) point. 4,5,8,9 The shapes of these observed phase diagrams are quite different from those predicted for nonrandom systems. In particular, they do not exhibit the tangential approach of the secondorder lines to the tetracritical point. It is our



FIG. 1. Schematic possible phase diagrams for systems with two competing order parameters. One of these order parameters orders in phase I (or II), while both order simultaneously in the "mixed" phase. (a) Phases I and II are separated by a first-order "flop" line, ending at a bicritical point. (b) There exists a "mixed" phase, and the four secondorder lines meet at a tetracritical point.



aim to understand these differences, and to classify the possible phase diagrams for such alloys.

In the present paper we consider a mixture of ions with competing spin anisotropies. If the spin space is m dimensional, then one component of the mixture tends to align the first m_1 spin components, while the other tends to align the remaining (perpendicular) $m_2 (= m - m_1)$ components.¹¹ The same theory is expected to apply to many other mixtures, in which the interdependence of the two order parameters is not too strong, e.g., a mixture of materials undergoing ferromagnetic and ferroelectric transitions. Mixtures of ions which have antiferromagnetic orderings with different periodicities (or ferromagnetic and antiferromagnetic orderings) will be treated in the second paper in this series. It turns out, that the fact that both order parameters in that case derive from the same basic spins affects the analysis, and usually does not allow a stable multicritical fixed point (see also Sec. VI).

The nature of phase transitions in quenched disordered systems with a single order parameter has been the subject of several recent renormalization-group studies.¹²⁻¹⁷ The lack of translational invariance means that one should either construct recursion relations for the distribution function of the random variables,¹² or derive an effective translationally invariant Hamiltonian which will yield the same free-energy density.¹³⁻¹⁵ In the present paper we follow the latter approach, i.e., the " $n \rightarrow 0$ replica trick." The main conclusion of these studies was that the random system still exhibits a sharp transition. The critical exponents at this transition are the same as those of the pure system if its specific-heat exponent α_{pure} is negative¹⁸ (i.e., for XY or Heisenberg systems at d=3), and have new, "random" values if α_{pure} is positive.¹⁷ These new exponents deviate from meanfield values at order $\epsilon = 4 - d$ (d is the dimensionality of the system) for m-component order parameters with m > 1,^{12,13} and at order $\epsilon^{1/2}$ if m =1.¹⁶ The exponents do not depend on the concentration of impurities. In the present papers we generalize these results to competing order parameters.¹⁹

As we reported earlier,¹¹ the main result of our analysis of the phase diagram of the alloy with competing anisotropies is shown in Fig. 2. The tetracritical point is "decoupled," i.e., each order parameter undergoes its own transition even when the two transitions occur simultaneously. Such decoupled behavior was previously predicted only for unphysically high values of the number of spin components, m.^{1b,17} In our earlier report we concentrated on discussing the stability of this "decoupled" fixed point. In the present paper we give



FIG. 2. Schematic phase diagram for an alloy with competing anisotropies. The tetracritical point is described by a "decoupled" fixed point. Phase I (II) denotes ordering of the m_1 - $(m_2$ -) component vector \vec{S}_1 (\vec{S}_2). $\hat{D} = \hat{D}$ ($\vec{q} = 0$) is a measure of the average anisotropy [see discussion following Eq. (2.10)].

all the details of the calculation and describe the complete renormalization-group analysis. The effective translationally invariant Hamiltonian is derived in Sec. II for bond randomness. Site randomness is considered in Appendix A. The recursion relations, fixed points, and stability exponents are obtained in Secs. III and IV. Some of the numerical calculations involved are described in Appendix B, and and a specific stability exponent, which can be calculated generally, is discussed in Appendix C. The resulting shape of the phase diagram is described in Sec V, with details of the corrections to scaling given in Appendix D. In Sec. VI we discuss experimental and further theoretical consequences of the results.

II. HAMILTONIAN

The Hamiltonian of anisotropic spin systems may be written in the form

$$\mathfrak{C}\left(\left\{J_{ij}, D_{ij}, \overline{\mathbf{S}}\left(i\right)\right\}\right)$$
$$= -\sum_{\langle ij \rangle} \left\{J_{ij} \overline{\mathbf{S}}(i) \cdot \overline{\mathbf{S}}(j) + D_{ij} \left[m_1^{-1} \overline{\mathbf{S}}_1(i) \cdot \overline{\mathbf{S}}_1(j) - m_2^{-1} \overline{\mathbf{S}}_2(i) \cdot \overline{\mathbf{S}}_2(j)\right]\right\},$$
$$(2.1)$$

where $\mathbf{\bar{S}}(i) \equiv (\mathbf{\bar{S}}_1(i); \mathbf{\bar{S}}_2(i))$ is an *m*-component spin vector at the site *i* of a *d*-dimensional lattice, $\mathbf{\bar{S}}_1$ and $\mathbf{\bar{S}}_2$ are m_1 - and m_2 -component parts of $\mathbf{\bar{S}}$ $(m_1 + m_2 = m)$, J_{ij} is an isotropic exchange coupling, while D_{ij} introduces an anisotropy aligning the spins in the $\mathbf{\bar{S}}_1$ $(D_{ij}>0)$ or the $\mathbf{\bar{S}}_2$ $(D_{ij}<0)$ subspaces. If J_{ij} and D_{ij} are uniform in space, then one expects a crossover from the isotropic *m*-component critical behavior to that of m_1 components or m_2 components as function of D^{20} . This was the basis of the calculations leading to the phase diagrams in Fig. 1.¹⁻³ In the case of interest here, J_{ij} and D_{ij} are random variables, characterized by a distribution $P(\{J_{ij}, D_{ij}\})$. Denoting their averages by \overline{J}_{ij} and \overline{D}_{ij} , we shall assume that $|\overline{D}_{ij}/\overline{J}_{ij}| < 1$, so that each component orders ferromagnetically (or antiferromagnetically) for all concentrations.

The distribution $P(\{J_{ij}, D_{ij}\})$ may arise from many sources. In the case of bond randomness, it is usually assumed that each bond $\langle ij \rangle$ has its own distribution function,

$$P(\{J_{ij}, D_{ij}\}) = \prod_{\langle ij \rangle} p(J_{ij}, D_{ij}) , \qquad (2.2)$$

and that each bond is independent of all other bonds. This may occur for superexchange interactions, mediated by different impurities, or for amorphous systems, where the distances r_{ij} are random. In the case of site randomness, J_{ij} and D_{ij} depend on the identity of the ions at the sites *i* and *j*, and $P(\{J_{ij}, D_{ij}\})$ results from the distribution of site occupancy. We consider this case in more detail in Appendix A.

As usual in renormalization-group studies, 17,21,22 we next consider a continuous spin model, in which the spin $\overline{S}(i)$ has a weight function $e^{-w(\overline{S}(i))}$, with

$$w(\bar{S}(i)) = \frac{1}{2} |\bar{S}(i)|^2 + v_0 |\bar{S}(i)|^4 + \cdots$$

The partition function is

$$Z = \operatorname{Tr} \exp(-\Im (kT))$$
$$= \int \prod_{i} \left[d^{m} S(i) e^{-\Im (i)} \right] e^{-\Im (kT)}, \qquad (2.3)$$

where k is the Boltzmann constant and T is the temperature. Instead of calculating the free energy $F = -kT \ln Z$, and then averaging over the distribution (2.2),²³ we now replace $\ln Z$ by $(Z^n - 1)/n$, average, perform calculations for finite n, and then take the limit $n \to 0$.¹³⁻¹⁵ Considering Z^n amounts to replacing each m-component spin vector $\vec{S}(i)$ by an nm-component vector $\vec{\sigma}(i) \equiv (\vec{S}^1(i), \ldots, \vec{S}^n(i))$, and

$$Z^{n} = \int \prod_{i} d^{nm} \sigma(i) e^{\mathcal{K}_{n}}, \qquad (2.4)$$

$$\Im C_{n} = -\sum_{\alpha=1}^{n} \left(\frac{\Im C(\{J_{ij}, D_{ij}, \tilde{\mathbf{S}}^{\alpha}(i)\})}{kT} + \sum_{i} w(\tilde{\mathbf{S}}^{\alpha}(i)) \right) . \qquad (2.5)$$

We next write

$$\mathfrak{C}(\{J_{ij}, D_{ij}, \widetilde{\mathbf{S}}^{\alpha}(i)\}) = \mathfrak{K}(\{\overline{J}_{ij}, \overline{D}_{ij}, \widetilde{\mathbf{S}}^{\alpha}(i)\}) + \mathfrak{K}(\{\Delta J_{ij}, \Delta D_{ij}, \widetilde{\mathbf{S}}^{\alpha}(i)\}) , (2.6)$$

where $\Delta J_{ij} = J_{ij} - \overline{J}_{ij}$, $\Delta D_{ij} = D_{ij} - \overline{D}_{ij}$, expand the exponential in (2.4) in powers of ΔJ_{ij} and ΔD_{ij} , average over the distribution (2.2), and then exponentiate again. Finally, the configurational average $[Z^n]$ becomes

$$[Z^n]_{av} = \int \prod_i d^{nm} \sigma(i) e^{\mathcal{K}_{eff}} , \qquad (2.7)$$

with

$$\mathcal{W}_{eff} = -\sum_{\alpha=1}^{n} \left(\frac{\mathcal{U}\left\{ \left\{ \overline{J}_{ij}, \overline{D}_{ij}, \overline{S}^{\alpha}(i) \right\} \right\} \right\}}{kT} + \sum_{i} w(\overline{S}^{\alpha}(i)) \right)$$

+
$$\sum_{\langle ij \rangle} \sum_{\langle kl \rangle} \sum_{\alpha,\beta=1}^{n} \left[A^{11}_{ijkl} \overline{S}^{\alpha}_{1}(i) \cdot \overline{S}^{\alpha}_{1}(j) \overline{S}^{\beta}_{1}(k) \cdot \overline{S}^{\beta}_{1}(l) + 2A^{12}_{ijkl} \overline{S}^{\alpha}_{1}(i) \cdot \overline{S}^{\alpha}_{1}(j) \overline{S}^{\beta}_{2}(k) \cdot \overline{S}^{\beta}_{2}(l) + A^{22}_{ijkl} \overline{S}^{\alpha}_{2}(i) \cdot \overline{S}^{\alpha}_{2}(j) \overline{S}^{\beta}_{2}(k) \cdot \overline{S}^{\beta}_{2}(l) + \cdots,$$

(2.8)

where the dots represent higher-order terms and

$$A_{ijkl}^{11} = [(\Delta J_{ij} + \Delta D_{ij}/m_1) \times (\Delta J_{kl} + \Delta D_{kl}/m_1)]_{av}/2(kT)^2 ,$$

$$A_{ijkl}^{12} = [(\Delta J_{ij} + \Delta D_{ij}/m_1) \times (\Delta J_{kl} - \Delta D_{kl}/m_2)]_{av}/2(kT)^2 ,$$

$$A_{ijkl}^{22} = [(\Delta J_{ij} - \Delta D_{ij}/m_2) \times (\Delta J_{kl} - \Delta D_{kl}/m_2)]_{av}/2(kT)^2 .$$
(2.9)

By the assumption (2.2), $[\Delta J_{ij}\Delta J_{kl}]_{av} \neq 0$ only if $\langle ij \rangle = \langle kl \rangle$. Our results also remain valid if these correlations have a finite range. With (2.2), A_{ijij}^{11} and A_{ijij}^{22} are positive.

We now follow the usual renormalization group routine.^{17,21,22} We Fourier transform $\overline{\sigma}(i)$ into $\overline{\sigma}(\mathbf{q})$, rewrite Eq. (2.8) as an expansion in powers of the components of $\overline{\sigma}(\mathbf{q})$, expand the coefficients in powers of the momenta \mathbf{q} , and rescale the spins $\overline{\sigma}_1(\mathbf{q})$ and $\overline{\sigma}_2(\mathbf{q})$ so that the final form of \mathcal{H}_{eff} becomes

$$\mathcal{W}_{eff} = -\frac{1}{2} \int_{\vec{q}} \left[(r_1 + q^2 + \cdots) \vec{\sigma}_1(\vec{q}) \cdot \vec{\sigma}_1(-\vec{q}) + (r_2 + q^2 + \cdots) \vec{\sigma}_2(\vec{q}) \cdot \vec{\sigma}_2(-\vec{q}) \right] \\ - \int_{\vec{q}_1} \int_{\vec{q}_2} \int_{\vec{q}_3} \sum_{\alpha,\beta=1}^n \sum_{\mu,\nu=1}^2 (u_{\mu\nu} + v_{\mu\nu} \delta_{\alpha\beta}) \vec{S}^{\alpha}_{\mu}(\vec{q}_1) \cdot \vec{S}^{\alpha}_{\mu}(\vec{q}_2) \vec{S}^{\beta}_{\nu}(\vec{q}_3) \cdot \vec{S}^{\beta}_{\nu}(-\vec{q}_1 - \vec{q}_2 - \vec{q}_3) - \cdots$$
(2.10)

The integrals are over the range $|\vec{\mathbf{q}}| < \Lambda$. The initial values of $v_{\mu\nu}$ are all proportional to v_0 , those of $u_{\mu\nu}$ are proportional to $\sum_{j,k} A_{ijik}^{\mu\nu}$, while the coefficients r_1 and r_2 are proportional to $kT - \hat{J}(0) - \hat{D}(0)/m_1$ and $kT - \hat{J}(0) + \hat{D}(0)/m_2$, where $\hat{J}(\vec{\mathbf{q}})$ is the Fourier transform of \hat{J}_{ij} , etc. Variation of the temperature and of the relative concentration of the two components (i.e., of $\hat{D}(0)$) is equivalent to variation of r_1 and r_2 . When $\hat{D}(0) > 0$, one has $r_1 < r_2$ and \hat{S}_1 orders first. The tetracritical point should occur near the point $\hat{D}(0) = 0$ (see Fig. 2). In the following analysis we thus concentrate on the region $r_1 \simeq r_2$.

III. RENORMALIZATION-GROUP RECURSION RELATIONS AND FIXED POINTS

We now perform the standard Wilson renormalization-group transformation^{17,21,22} on the Hamiltonian (2.9). We integrate over spins $\overline{\sigma}(\mathbf{q})$ with $\Lambda/b < |\mathbf{q}| < \Lambda$, rescale all momenta by $\mathbf{q} \rightarrow b\mathbf{q}$, and rescale all spins by $\overline{\sigma}_{\mu}(\mathbf{q}) \rightarrow \overline{\sigma}_{\mu}(b\mathbf{q}) = \xi_{\mu}\overline{\sigma}_{\mu}(\mathbf{q})$, with $\xi_{\mu}^{2} = b^{d+2-\eta_{\mu}}$ ($\mu = 1, 2$). The recursion relations are obtained under the assumptions of Bruce *et al.*,²⁴ from the diagrams depicted in Fig. 3. Figure 3(a) yields, to lowest order,

$$r_{1}' = b^{2 - \eta_{1}} \left\{ r_{1} + 4K_{d} I(r_{1}) \left[(nm_{1} + 2)u_{11} + (m_{1} + 2)v_{11} \right] + 4K_{d} I(r_{2})(nm_{2}u_{12} + m_{2}v_{12}) + \cdots \right\} ,$$

$$r_{2}' = b^{2 - \eta_{2}} \left\{ r_{2} + 4K_{d} I(r_{2}) \left[(nm_{2} + 2)u_{22} + (m_{2} + 2)v_{22} \right] + 4K_{d} I(r_{1})(nm_{1}u_{12} + m_{1}v_{12}) + \cdots \right\} ,$$

$$(3.1)$$

where

$$K_d^{-1} = 2^{d-1} \pi^{d/2} \Gamma(d/2)$$

$$I(r) = \int_{\Lambda/b}^{\Lambda} \frac{q^{d-1} dq}{r+q^2} = \frac{1}{2}\Lambda^2(1-b^{-2}) - r \ln b + O(r^2) ,$$

and the η_{μ} 's are calculated from Fig. 3(b),

$$\eta_1 = 8K_4^2 [(nm_1 + 2)u_{11}^2 + (2m_1 + 4)u_{11}v_{11} + (m_1 + 2)v_{11}^2 + nm_2u_{12}^2 + 2m_2u_{12}v_{12} + m_2v_{12}^2] + \cdots ,$$

with η_2 being obtained by interchanging the indices 1 and 2. From Fig. 3(c),

$$u_{11}' = b^{\epsilon - 2\eta_1} \left\{ u_{11} - 4K_d \ln b \left[(8 + nm_1)u_{11}^2 + (4 + 2m_1)u_{11}v_{11} + nm_2u_{12}^2 + 2m_2u_{12}v_{12} \right] + 32K_4^2 \ln b \left[(5nm_1 + 22)u_{11}^3 + (12m_1 + 24)u_{11}^2v_{11} + (3m_1 + 6)u_{11}v_{11}^2 + 2nm_2u_{12}^3 + 6m_2u_{12}^2v_{12} + 2m_2u_{12}v_{12}^2 + 3nm_2u_{11}u_{12}^2 + 6m_2u_{11}u_{12}v_{12} + m_2u_{11}v_{12}^2 \right] + \cdots \right\},$$

$$v_{11}' = b^{\epsilon - 2\eta_1} \left\{ v_{11} - 4K_d \ln b \left[(8 + m_1)v_{11}^2 + 12u_{11}v_{11} + m_2v_{12}^2 \right] + 32K_4^2 \ln b \left[(5m_1 + 22)v_{11}^3 + (12m_1 + 60)u_{11}v_{11}^2 + (3nm_1 + 42)u_{11}^2v_{11} + 2m_2v_{12}^3 + 4m_2u_{12}v_{12}^2 + 3nm_2v_{11}u_{12}^2 + 6m_2v_{11}u_{12}v_{12} + (m_2 + 2)v_{11}v_{12}^2 + 2m_2u_{11}v_{12}^2 \right] + \cdots \right\},$$

$$u_{12}' = b^{\epsilon - \eta_1 - \eta_2} \left\{ u_{12} - 4K_d \ln b \left[4u_{12}^2 + (2 + nm_1)u_{11}u_{12} + (2 + nm_2)u_{22}u_{12} + (2 + m_2)v_{22}u_{12} + (2 + m_2$$

$$v_{12}' = b^{\epsilon - \eta_1 - \eta_2} \left\{ v_{12} - 4K_d \ln b \left[4v_{12} + 8u_{12} + 2(u_{11} + u_{22}) + (2 + m_1)v_{11} + (2 + m_2)v_{22} \right] v_{12} + \cdots \right\} ,$$

with $\epsilon = 4 - d$ and with the recursion relations for u_{22} and v_{22} again obtained by $1 \leftrightarrow 2$.

We now turn to search for all the possible fixed points of Eqs. (3.4a). In order to find solutions with $u_{\mu\nu}$ and $v_{\mu\nu}$ of order ϵ , one may ignore the η_{μ} 's in the prefactors, and the terms of third order. However, if m_1 or m_2 is equal to 1, then some of these solutions go to infinity as $n \to 0$. In these cases, one must add the higher-order terms



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and then solutions of order $\epsilon^{1/2}$ are found.¹⁶ In the case $m_1 = m_2 = 1$, one needs the explicit expressions for the cubic terms in the recursion relations for u_{12} and v_{12} . The dots in the last two Eqs. (3.4a) then represent

$$\delta u_{12} = 32K_4^2 \ln \left[(n+4)u_{12}^3 + (3n+6)(u_{11}+u_{22})u_{12}^2 + 9(v_{11}+v_{22})u_{12}^2 + 2u_{12}^2 v_{12} + (3n/2+3)(u_{11}^2+u_{22}^2)u_{12} + (3n/2+3)(u_{11}^2+u_{22}^2)u_{12}^2 + (3n/2+3)(u_{11}^2+u_{12}^2)u_{12}^2 + (3n/2+3)(u_{11}^2+u_{12}^2)u$$

$$+9(v_{11}^2+v_{22}^2)u_{12}/2+3u_{12}v_{12}^2+9(u_{11}v_{11}+u_{22}v_{22})u_{12}$$

$$+2(u_{11}^2+u_{22}^2)v_{12}+(u_{11}+u_{22})v_{12}^2+4(u_{11}+u_{22})u_{12}v_{12}]+\cdots$$

(3.4b)

 $\delta v_{12} = 32K_4^2 \ln b \left[5v_{12}^3 + 8(u_{11} + u_{22})v_{12}^2 + 9(v_{11} + v_{22})v_{12}^2 + 14v_{12}^2u_{12} + (n/2 + 3)(u_{11}^2 + u_{22}^2)v_{12} + 9(v_{11}^2 + v_{22}^2)v_{12} / 2 \right]$

 $+ 3nv_{12}u_{12}^2 + 9(u_{11}v_{11} + u_{22}v_{22})v_{12} + 18(v_{11} + v_{22})v_{12}u_{12} + (2n+12)(u_{11} + u_{22})u_{12}v_{12}] + \cdots$

We first note that $v_{12}^* = 0$ is a solution of (3.4). When $v_{12} = 0$, in the limit $n \to 0$, u_{12} does not appear in the recursion relations for $u_{\mu\mu}$ and $v_{\mu\mu}$. Thus, one can solve separately the recursion relations for u_{11} and v_{11} and those for u_{22} and v_{22} . The recursion relations for u_{11} and v_{11} , in the decoupled case, are the same as those considered in the case of a single order parameter.^{12,13,16} They yield four fixed points

Gaussian,

$$u_{11}^* = v_{11}^* = 0$$
,

 m_1 -component "pure,"

$$u_{11}^* = 0, v_{11}^* = \epsilon / [4K_4(m_1 + 8)] + \cdots,$$

isotropic "n = 0,"

$$u_{11}^* = \epsilon/32K_4 + \cdots , v_{11}^* = 0$$
,

"random,"

$$u_{11}^* = (m_1 - 4)\epsilon / [64K_4(m_1 - 1)] + \cdots,$$

$$v_{11}^* = \epsilon / [16K_4(m_1 - 1)] + \cdots, (m_1 > 1),$$

or "Khmel'nitzkii,"

$$u_{11}^* = \mp (3\epsilon/106)^{1/2}/4K_4 + \cdots ,$$

$$v_{11}^* = \pm (3\epsilon/106)^{1/2}/3K_4 + \cdots (m_1 = 1) . \qquad (3.5)$$

Similarly, there are four fixed points for u_{22}^* and v_{22}^* .

We now turn to the recursion relation for u_{12} (still with $v_{12} = 0$). For each of the above solutions, this has two fixed-point solutions,

$$u_{12}^* = 0$$

or

$$u_{12}^{*} = \frac{1}{4} \{ \epsilon / 4K_4 - [2(u_{11}^{*} + u_{22}^{*}) + (2 + m_1)v_{11}^{*} + (2 + m_2)v_{22}^{*}] \} + \cdots .$$
(3.6)

In total, we have thus found 32 fixed points with $v_{12}^* = 0$ {counting the double solution for the Khmel'nitzkii random fixed point, in the case $m_1 = 1$ [Eq. (3.5)], only once}. Of these, 16 fixed points also have $u_{12}^* = 0$. At these latter fixed points,

the Hamiltonian breaks into two separate parts, involving only \bar{S}_1 or \bar{S}_2 . We therefore call these fixed points *decoupled*.

It now remains to find the fixed points with $v_{12}^* \neq 0$. This was done numerically; the procedure and the results are described in Appendix B. It turns out that none of these additional fixed points has much physical relevance. For each of the fixed points found above, we can now solve Eqs. (3.1) for r_1^* and r_2^* . Both are of order ϵ (or $\epsilon^{1/2}$) at the multicritical point. For the decoupled cases, the solutions are the same as for each order parameter separately.^{12,13}

IV. FIXED-POINT STABILITY

In this section we shall examine the stability of the fixed points found in the previous section and in Appendix B. For this purpose, we linearize Eqs. (3.4) about the fixed points, and calculate the stability exponents. If $v_{12}^* = 0$, then the matrix of the linearized renormalization-group transformation can easily by diagonalized, because it decouples into submatrices operating in the (u_{11}, v_{11}) , (u_{22}, v_{22}) , u_{12} , and v_{12} subspaces separately. In the (u_{11}, v_{11}) subspace, the stability exponents about the fixed points are¹³⁻¹⁷

Gaussian,

 $\lambda_1 = \lambda_2 = \epsilon$;

 m_1 component,

$$\lambda_1 = -\epsilon + \ldots, \quad \lambda_2 = (4 - m_1)\epsilon/(m_1 + 8) + \ldots;$$

isotropic "n = 0,"

$$\lambda_1 = -\epsilon + \ldots, \quad \lambda_2 = -\epsilon/2 + \ldots;$$

"random,"

$$\lambda_1 = -\epsilon + \dots, \quad \lambda_2 = (m_1 - 4)\epsilon/[4(m_1 - 1)] + \dots$$

"Khmel'nitzkii,"

 $\lambda_1 = -2\epsilon + \ldots, \quad \lambda_2 = \pm 4(3\epsilon/106)^{1/2} + \ldots \quad (4.1)$

.;

The calculation of λ_1 for the Khmel'nitzkii fixed point is described in Appendix C. The exponent λ_2

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for the m_1 -component fixed point is exactly equal to α_{m_1}/ν_{m_1} , where α_{m_1} and ν_{m_1} are the specific heat and the correlation length exponents of this fixed point.^{17,25} The m_1 -component "pure" behavior is thus stable when $\alpha_{m_1} < 0$, i.e., for $m_1 \ge 2$ at d = 3. The "random" fixed point thus has little relevance at d = 3.

As we mentioned above, we expect u_{11}^* to be negative $(A_{ijij}^{11}$ is positive) in the physical region. Therefore, one must choose the upper sign for the "Khmel'nitzkii" fixed point in Eqs. (3.5) and (4.1), and this fixed point is stable (at least for small ϵ).

The "isotropic n = 0" fixed point is also stable, but is unphysical, since $u_{11}^* > 0$. It will never be reached under renormalization-group iterations.¹⁵

In summary, the stable physical fixed point for u_{11} and v_{11} is the m_1 -component "pure" one for $m_1 \ge 2$, and the "Khmel'nitzkii" one for $m_1 = 1$. In both cases, the specific-heat exponent is negative. The same applies to the fixed point in the subspace of u_{22} and v_{22} .

We next turn to the eigenvalues for u_{12} and for v_{12} . At $v_{12}^* = 0$, these turn out to be equal to each other, having the value

$$\lambda_{12} = \epsilon - 4K_4 [8u_{12}^* + 2(u_{11}^* + u_{22}^*) + (2 + m_1)v_{11}^* + (2 + m_2)v_{22}^*] + \cdots \qquad (4.2)$$

The values of u_{12}^* are given in Eq. (3.6), and one easily checks that at this order,

$$\Lambda(u_{12}^* \neq 0) = -\lambda(u_{12}^* = 0) , \qquad (4.3)$$

and only one of each pair of fixed points is stable. An explicit calculation shows, that the "decoupled" fixed point, with $u_{12}^* = v_{12}^* = 0$ and with u_{11}^* , v_{11}^* , and u_{22}^* , v_{22}^* at their respective stable fixed point values ("pure" or "Khmel'nitzkii") is always stable, and that it is the only physical stable fixed point.

This conclusion may also be derived from general scaling arguments.¹¹ The parameter u_{12} is the coefficient of the operator $(\bar{\sigma}_1 \cdot \bar{\sigma}_1)(\bar{\sigma}_2 \cdot \bar{\sigma}_2)$. Since at the decoupled fixed point $\bar{\sigma}_1$ and $\bar{\sigma}_2$ have separate scaling behavior, the two factor operators scale as the energy densities of \bar{S}_1^{α} and of \bar{S}_2^{β} , i.e., as $\xi^{-(1-\alpha_1)/\nu_1}$ and $\xi^{-(1-\alpha_2)/\nu_2}$. Remembering the factor ξ^d from the volume, one finds¹⁷

$$\lambda_{12} = d - (1 - \alpha_1)/\nu_1 - (1 - \alpha_2)/\nu_2$$
$$= \frac{1}{2}(\alpha_1/\nu_1 + \alpha_2/\nu_2) . \qquad (4.4)$$

Since α_1 and α_2 are negative at the stable one component fixed points, one always has $\lambda_{12}(u_{12}^*=0)<0$ at the corresponding decoupled fixed point. A similar argument works for v_{12} .

The stability exponents of the fixed points with $v_{12}^* \neq 0$ were calculated numerically for $m_1 = 1$, $m_2 = 2$, and for $m_1 = m_2 = 1$ (see Appendix B). None of these fixed points is physical and stable.

V. PHASE DIAGRAM

In Sec. IV we concluded that the multicritical point is decoupled, i.e., $u_{12}^* = v_{12}^* = 0$ at the corresponding fixed point. We therefore start our discussion of the phase diagram with the simple case u_{12} $=v_{12}=0$. In this case, the Hamiltonian (2.10) simply breaks into two separate parts, involving only $\bar{\sigma}_1$ or $\bar{\sigma}_2$. Each of these has the critical behavior of a single order parameter random material, as discussed in Refs. 12-17. The only effect of one order parameter on the other is through the dependence of r_1 and r_2 on $\overline{D}(0)$, or on the relative concentration p of the two components in the alloy. The critical temperature T_1 (or T_2) for \vec{S}_1 ordering (or for \overline{S}_2 ordering) will decrease with the addition of the \overline{S}_2 -ordering (or \overline{S}_1 -ordering) component. This results with the phase diagram shown in Fig. 2.

The average value of $[\langle \vec{S}_1 \rangle]_{av}$ is nonzero for all $T < T_1(p)$, and it approaches zero as $|T_1(p) - T|^{\beta_1}$, irrespective of \vec{S}_2 (and vice versa). The exponents are those of the pure system for $m_1 = 2$ (d = 3), and those of the "Khmel'nitzkii" fixed point for $m_1 = 1$. The intersection of the lines $T_1(p)$ and $T_2(p)$ is thus a tetracritical point, at which both \vec{S}_1 and \vec{S}_2 undergo a phase transition simultaneously. The lines cross each other *at an angle*, and not tangential - $ly \cdot 3^{\cdot 4,9}$ In the "mixed" phase, both $[\langle \vec{S}_1 \rangle]_{av}$ and $[\langle \vec{S}_2 \rangle]_{av}$ are nonzero.

In the usual analysis of bicritical and tetracritical points,¹⁻³ the two relevant scaling parameters are the temperature T and the anisotropy $\hat{D}(0)$, corresponding to $m_1r_1 + m_2r_2$ and to $g = r_1 - r_2$. The recursion relation for the former yields the correlation length exponent ν , and the one for the latter yields the crossover exponent²⁰ ϕ , which also determines the shape of the phase diagram. The situation at the decoupled fixed point is quite different: the correct scaling fields are r_1 and r_2 , each having its own correlation length exponent ν_1 and ν_2 . There is no crossover exponent, as the free energy is simply the sum of the free energies of the two order parameters.

We now turn to the general case, where $u_{12} \neq 0$ and $v_{12} \neq 0$. Asymptotically close to the decoupled fixed point, these are irrelevant fields. However, irrelevant fields may some time affect the shapes of phase diagrams,³ and therefore, one should consider them with care. From Eq. (3.1) it is clear that in the limit $n \rightarrow 0$ u_{12} does not affect the recursion relations for r_1 and r_2 . This is expected to remain true to all orders.¹⁹ The effects of v_{12} on these recursion relations are carefully studied in Appendix D. The main result is shown schematically in Fig. 4. The shift in T_2 , due to v_{12} (i.e., relative to Fig. 2), is of the form



FIG. 4. Effects of corrections to scaling, due to v_{12} , on the phase diagram of Fig. 2 (see text).

$$\Delta T_2 = -A_1 v_{12} + A_2 v_{12} (t_1 - t_1^{\psi}) , \qquad (5.1)$$

where $t_1 = T - T_1$, where the coefficients A_1 and A_2 are positive and where

$$\psi = \nu_1 (-\lambda_{12} + 1/\nu_2) . \qquad (5.2)$$

Asymptotically, the shape of the phase diagram near the tetracritical point is thus unaffected by $v_{\rm 12}.$

VI. DISCUSSION

Our results for the alloy with competing anisotropies are summarized in Fig. 2. In this section we wish to make clear the limits in which these results apply, discuss other cases to which they may be relevant, and mention existing and wanted experiments.

First, the theoretical limitations. One repeated question has to do with the approachability of the stable fixed point. If the initial Hamiltonian is not in the range of attraction of the decoupled fixed point, it will not be relevant to the tetracritical point even though it is stable. Returning to Eq. (2.9), one sees that by an appropriate choice of the distributions of D_{ij} and J_{ij} one can make u_{12} as small as needed. It is less easy to affect the initial value of v_{12} . However, explicit numerical calculations show that even large values of v_{12} , e.g., of the order $10v_{11}$, flow to zero. Thus, there is clearly a finite physical range of parameters for which the decoupled fixed point may be approached. If it cannot be approached, then there is no other physical stable fixed point, and the flow "runs away." The resulting behavior in this case is not clear. It may lead to a smeared transition, to a nonuniform ordering, to a spin-glass phase, etc.26

Another basic assumption is the use of the "n - 0

replica trick." Unlike the case of spin glasses, where this trick raises some questions,²⁶ we feel that it is justified here just as it was in the case of a single order parameter, due to its equivalence to recursion relations for the cumulants of the random distribution.¹²

Of more practical importance is the question of the width of the asymptotic region, in which the decoupled critical behavior is to be observed. As in the case of a single order parameter,¹²⁻¹⁸ the crossover from "pure" to "random" behavior is rather slow, being governed by the small specificheat exponent α . The width of the critical region is of order $p^{1/\alpha}$, where p is the concentration. However, in our case we are in the most favorable situation, since the relative concentration of the two components is of order $\frac{1}{2}$. Note also that other deviations from the decoupled behavior, e.g., Eq. (5.1), involve larger crossover exponents and a faster approach to asymptotic behavior. In any case, even if delicate details of the critical behavior are difficult to observe, the observation of the general structure of the phase diagram, and especially of the existence of the mixed phase, will be quite helpful.

Another word of caution refers to the uniformity of the random alloy. If the length scale of the impurity correlations is larger than the correlation length for magnetic ordering, then one will observe a *smeared* transition with parts of the system undergoing the transition at different temperatures. One, therefore, must be careful to use homogeneously distributed impurities without clustering.

The Hamiltonian (2.1) is quite general and should apply to many alloys with competing interactions, e.g., ferromagnetic and ferroelectric, etc. We should note, however, an important exception. If $m_1 = m_2$, then in some cases Eq. (2.1) may also involve couplings like $K_{ij} \tilde{S}_1(i) \cdot \tilde{S}_2(j)$, with $\overline{K}_{ij} = 0$. Such terms arise, e.g., in alloys of ferromagnets and antiferromagnets. Following the procedure outlined in Sec. II, these terms generate a new quartic coupling, of the form $[\tilde{\sigma}_1(i) \cdot \tilde{\sigma}_2(j)]^2$. The coefficient of this term is *relevant* near the decoupled fixed point and leads to new critical effects. We discuss such cases in the second paper in this series.

Finally, we mention experiments. In order to check experimentally the predictions of this work, one has to prepare a random alloy of two magnetic materials, such that the components of the magnetic moments that order in one (pure) material are perpendicular to the components that order in the other (pure) material. To avoid structural effects, one should study mixtures of materials having the same crystal structure. The heavyrare-earth metals Tb, Tm, Er, Ho, Dy, and Gd are natural candidates for this purpose, because (a) they all crystallize in the simple hexagonal close-packed structure with nearly the same c/a ratio and (b) different components of the magnetic moments order. For the following materials the component in the c direction orders at temperatures lower than given in the brackets²⁷: Tm (~60 °K), Er (84 °K), Ho (20 °K), Gd (293 °K, remains ordered in the c direction only down to 248 °K). The basal components (perpendicular to c) order for the following materials²⁷: Er (53.5 °K), Tb (229 °K), Ho (132 °K), Dy (178 °K). To study the effect of the competing anisotropies, we should mix randomly a material from the first group with one of the second group, for which different components order over an overlapping temperature range. Studies⁹ on random mixtures of Tm-Tb and Er-Tb confirm the existence of the mixed phase and of the tetracritical point, as predicted in the present work. Detailed measurements near the tetracritical point, as well as studies of additional such alloys, will be very helpful.

Another case of competing anisotropies has been observed in $K_2Mn_{1-x}Fe_xF_4$.⁸ The Mn ions in K_2MnF_4 order in the *c* direction, while the Fe ions in Rb₂FeF₄ order in the perpendicular direction. A mixed phase was observed in the alloy.⁸ However, the analysis of the phase diagram is complicated here due to the layered nature of the compound, which leads to crossover from two- to three-dimensional behavior. It would be interesting to do measurements on similar materials with stronger (or weaker) interlayer coupling, so that the observed phase diagrams correspond purely to d=3 (d=2).

Future experiments should concentrate on the following main features: (a) the existence of the mixed phase (coexistence of the two order parameters), (b) the shape of the phase diagram in the close vicinity of the tetracritical point, and (c) critical behavior near this point and near other points on the second-order lines. For example, the asymptotic behavior of the specific heat at the tetracritical point should be of the form

$$C = A_1 t_1^{-\alpha_1} + A_2 t_2^{-\alpha_2}, \tag{6.1}$$

where t_1 and t_2 measure the distances from the two lines T_1 and T_2 , and where α_1 and α_2 are two distinct (for $m_1 \neq m_2$) negative exponents.

APPENDIX A: SITE RANDOMNESS

In the discussion of Sec. II, J_{ij} and D_{ij} were considered random variables, with a separate distribution for each bond $\langle ij \rangle$. We now wish to

consider a random alloy of the type $A_{p}B_{1-p}$, with exchange and anisotropy couplings J_{ij}^{AA} , J_{ij}^{AB} , J_{ij}^{BB} and D_{ij}^{AA} , D_{ij}^{AB} , D_{ij}^{BB} depending only on the identity of the ions at the ends of a given bond. Defining a random site variable p_{i} , which is equal to 1(0) if site *i* is occupied by A(B) we have¹⁹

$$J_{ij} = J_{ij}^{AA} p_i p_j + J_{ij}^{AB} [p_i (1 - p_j) + p_j (1 - p_i)] + J_{ij}^{BB} (1 - p_i) (1 - p_j), \qquad (A1)$$

and a similar expression for D_{ij} . The distribution function (2.2) is now replaced (if there are no correlations among the p_i 's) by

$$P(\{J_{ij}, D_{ij}\}) = \prod \tilde{p}(p_i).$$
(A2)

If $\overline{p}_i = p$ is the concentration of A ions, then we have

$$\overline{J}_{ij} = p^2 J_{ij}^{AA} + 2p(1-p) J_{ij}^{AB} + (1-p)^2 J_{ij}^{BB}$$
(A3)

and

$$\Delta J_{ij} = A_{ij}(p_i - p)(p_j - p) + B_{ij}(p_i + p_j - 2p), \quad (A4)$$

where

$$A_{ij} = J_{ij}^{AA} - 2J_{ij}^{AB} + J_{ij}^{BB},$$

$$B_{ij} = J_{ij}^{AB} - J_{ij}^{BB} + pA_{ij}.$$
(A5)

A simple calculation then yields^{13,19}

$$A_{ijkl}^{11} = 2p(1-p)[p(1-p)(A_{ij}+A_{ij}/m_1)^2 \delta_{jk} + 2(B_{ij}+\tilde{B}_{ij}/m_1)(B_{ij}+\tilde{B}_{ik}/m_1)]\delta_{il}, \quad (A6)$$

where \tilde{A} and \tilde{B} are defined similarly to (A.5), with D_{ij}^{AA} replacing J_{ij}^{AA} , etc. We can now follow the same steps as in Sec. II, and finally arrive again at Eq. (2.9). One again checks that u_{11} and u_{22} must be negative. The rest of the analysis is exactly the same as for the random bond case.

Note that in the discussion based on the effective Hamiltonian presented in Sec. II, only the *average* magnetization $[\langle \mathbf{\tilde{S}} \rangle]_{av}$ can be calculated. One does not calculate the average magnetization of type A atoms, $[\langle P_1 \mathbf{\tilde{S}}_i \rangle]_{av}$, or of type B atoms, separately. For the random site case, one probably can perform a more detailed calculation, to find these separate average magnetizations.^{26,28}

APPENDIX B: FIXED POINTS FOR $v_{12} \neq 0$

In this Appendix we search for fixed points of Eqs. (3.4), with $v_{12} \neq 0$. The last equation in (3.4) then yields, at lowest order,

$$v_{12} = \frac{1}{4} \{ \tilde{\epsilon} - [8u_{12} + 2(u_{11} + u_{22}) + (2 + m_1)v_{11} + (2 + m_2)v_{22}] \},$$
(B1)

with $\tilde{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}/4K_4$. To the same order, we thus remain with five quadratic equations,

exponents.
stability
nd corresponding
$m_2 = 2, a$
$m_1 = 1$,
s (B2) for
equations
of the fixed-point
Solutions
TABLE I.

	$Exponents/\epsilon$	$-1, 0.8667, -0.3788 \pm 0.4222i, 0.4973 \pm 0.0352i$	$-1, 0.6984, -0.5151 \pm 0.2407i, 0.4416 \pm 0.0211i$	-1, 0.4132, 0.7742, -0.5402, -0.3063, -0.4130	-1, 0.4018, 0.2931, -0.8010, -0.5252, -0.2107	$-1, -1.708, 2.186, 2.793, 0.2016 \pm 1.564i$	-1, 0.8127, -0.4911, 0.4604, 0.1302, 0.1081	-1, -0.7658, 0.3850, -0.2045, -0.5515, 0.2032	-1, 0.6364, -0.7273, 0.3636, -0.0909, 0.0909	$-1, -0.5453, -0.0990, 0.2008, 1.034 \pm 0.0021i$	-1, -1, -0.5, -0.5, 0, 0	-1, 0.3325, -0.9986, 0.0593, -0.5007, -0.0592	$-1, 11.07, -4.469, 3.754, -0.8327 \pm 3.687i$	$-1, 1.575, 2.007 \pm 0.2132i, 1.429 \pm 0.0997i$	$-1, 1.406, -1.869 \pm 0.2906i, 1.283 \pm 0.1725i$	$-1, -0.9603, 0.5513 \pm 0.0943i, 1.129 \pm 0.0260i$	$-1, 1.908, 3.118 \pm 0.4681i, -2.818 \pm 0.4746i$	$-1, -0.8488, 1.474 \pm 0.5840i, 0.8125 \pm 0.1895i$	$-1, -1.627, 2.368, -2.874, -1.575 \pm 0.6799i$	$-1, -0.6113, 0.1324, 0.2797, 1.115 \pm 0.0107i$	$-1, -0.9891, -0.8742, -0.4461, 0.2206 \pm 0.1282i$	$-1, -1.049, -0.2940, 0.3923, 0.2035 \pm 0.0464i$	-1,1.149,0.4996,-0.4702,0.2089,-0.1591	$-1, -0.2519, 0.3718, -0.4607, -0.8371 \pm 0.1387i$	-1, 0.8750, 0.3887, -0.7637, -0.1250, -0.1250					Complex Solutions, which do not correspond	to physical lixed points.			
-	$v_{12}^*/\tilde{\epsilon}$	0.1196	0.0794	0.0449	0.0471	0.2723	0.1151	0.0553	6060.0	-0.0189	0	-0.0017	0.5763	-0.1908	-0.2393	-0.0808	-0.5192	-0.2069	0.2905	-0.0449	-0.0655	-0.0327	0.1628	0.0774	0.1250	0.1644	$\pm 0.0205i$	0.3745	$\pm 0.0598i$	0.2744	$\pm 0.0333i$	0.2080	$\pm 0.2682i$	
	$u_{12}^{*}/\tilde{\epsilon}$	0.0379	0.0454	0.0462	0.0377	0.2147	0	0.0260	0	0.0442	0	-0.0096	0.2892	0.2395	0.2043	0.1045	0.3648	0.1194	-0.2507	0.0717	0,0566	0.0417	-0.0373	-0.0308	-0.0313	0		0.2232	$\pm 0.1361i$	-0.0860	$\pm 0.0074i$	-0.0860	$\pm 0.2037i$	3ef. 17).
	v <u>*</u> 2∕€	0.0233	-0.0141	-0.0044	-0.0048	-0.2184	0.0843	0.0824	6060.0	0.2400	0.2500	0.2502	0.8356	0.0230	0.0375	0.0058	-0.1189	-0.0446	-0.0572	0.1946	0.1676	0.1169	0.1275	0.2308	0.1250	0.0337	$\pm 0.0207i$	0.0010	$\pm 0.0633i$	0.0871	$\pm 0.0281i$	0.4614	$\pm 0.1411i$	onent system (F
	u <u>*</u> 2∕€	0.0128	0.1323	0.1252	0.1262	0.2936	0	0.0116	0	-0.1168	-0.1250	-0.1252	-0.6461	-0.0674	-0.0751	-0.0157	0.3714	0.2004	0.2539	-0.0797	-0.0584	-0.0148	-0.0402	-0.1112	-0.0313	0		-0.0103	$\pm 0.1138i$	-0.0586	$\pm 0.0185i$	-0.2908	$\pm 0.1006i$	ure three-comp
	vħ/ẽ	-0.0832	0.0233	-0.0101	0.0793	-1.139	0.0675	-0.0141	6060.0	0.0007	0	0.1108	-3.909	0.0365	-0.0721	0.2258	0.1695	0.3491	0.9518	0.0036	-0.0132	0.1304	0.0957	-0.0201	0.1250	0.0692	$\pm 0.0549i$	-1.661	$\pm 1.201i$	0.1990	$\pm 0.0583i$	-0.3477	$\pm 0.6037i$	d point for the p
	u†1/⋶	0.1744	0.0207	0.1242	0.0192	0.9486	0	0.1300	0	-0.0033	0.1250	0.0002	3.029	-0.1100	0.2697	-0.0908	-0.3088	-0.1990	-0.6453	-0.0120	0.1475	-0.0158	-0.0346	0.1481	-0.0313	0		1.340	$\pm 0.8971i$	-0.1190	±0.0129	0.3175	$\pm 0.4454i$	he "cubic" fixe
Fixed	point	1	7	3	4	2	6 ^a	7	q 8,	6	10 °	11	12	13	14	15	16	17	18	19	20	21	22	23	24 ^d	25 (26)	27)	28 5	29 (30)	31 (32 5	^a This is t

^bThis is the pure isotropic three-component fixed point [Eq. (3.5)]. Its stability in the $v_{11}-v_{12}-v_{22}$ subspace is determined by the specific-heat exponent. ^oThis fixed point is degenerate with one of the decoupled fixed points. ^dThis is the "random" three-component fixed point [Eq. (3.5)].

Fixed							
point	u11/€	v∦1∕ẽ	$u_{22}^*/\tilde{\epsilon}$	$v_{22}^{*}/\tilde{\epsilon}$	$u_{12}^*/\tilde{\epsilon}$	v î2∕č	Exponents∕€
1 a	0		0	1	0		-1, 1, 0.6667, 0.3333, 0.3333, 0
2 b	0	° ⊷ ⊊	0	: ⊊	0	10	-1, -0.8, 0.6, 0.4, -0.2, 0.2
36		- II	k 	í -	~ 0 		-1, 1.5, 0.6514, -1.1514, -0.5000, -0.5000
9 4	0.1263	-0.0077	0.1263	-0.0077	0.0383	0.0587	-1, 0.4235, -0.8146, -0.3524, -0.4235, -0.5377
5	0.8257	-0.9730	0.8257	-0.9730	0.2500	0.3838	$-1, 1.1162 \pm 1.3244i, -2.3028, 2.7676, 2.7676$
9	-0.0757	0.0285	-0.0757	0.0285	0.2500	-0.2171	$-1, 1.7171 \pm 0.2269i, 1.3028, 1.5657, 1.5657$
7	-0.0183	0.1332	-0.0183	0.1332	0.0605	-0.0526	-1, -1.0657, -0.3790, 0.3153, 0.3790, 0.3810
8	0.1448	-0.0350	0.0094	0.0213	0.0308	0.1217	-1 $-0.4868 \pm 0.2890i$ 0.8584 0.5638 \pm 0.0293 i
6	0.0094	0.0213	0.1448	-0.0350	0.0308	0.1217)	
10	0.0094	0.0931	0.1292	-0.0100	0.0252	0.0679)	-1 0 440% 0 2965 -0.8152 -0.5297 -0.2662
11	0.1292	-0.0100	0.0094	0.0931	0.0252	0.0679)	
12	0.2347	-0.0573	-0.0876	0.0454	0.2292	-0.2730	-1 -1 $5629 \pm 0.1993i$ 1.0840 $1.6057 \pm 0.1341i$
13	-0.0876	0.0454	0.2347	-0.0573	0.2292	-0.2730)	
14	-0.0684	0.1962	-0.0224	0.0090	0.1224	-0.1032 }	-1 0 5521 ± 0.1238 <i>i</i> . $-0.8682.1.909 \pm 0.0274i$
15	-0.0224	0.0090	-0.0684	0.1962	0.1224	-0.1032)	
16	-0.3004	0.4655	-0.1382	0.1298	-0.2084	0.4397	-1 1 2164 \pm 0.4196 <i>i</i> . 2.6720. $-1.1145.$ -0.9964
17	-0.1382	0.1298	-0.3004	0.4655	-0.2084	0.4397)	
18	-0.0712	0.1957	0.1643	-0.0239	0.1045	-0.1345	-1 0.5252 ± 0.1466 <i>i</i> . -0.6099 . $-1.0862 \pm 0.2103i$
19	0.1643	-0.0239	-0.0712	0.1957	0.1045	-0.1345)	
20	-0.6712	0.9820	0.3283	-0.1083	-0.3476	0.4614	-1 -2 $1101 \pm 0.6986i$ 2.7055 -2.8655 -1.9873
21	0.3283	-0.1083	-0.6712	0.9820	-0.3476	0.4614)	
22)	3.3367	-4.3231	-0.0311	0.0075	0.5177	0.7985	
23	$\pm 1.4823i$	$\pm 1.9794i$	$\pm 0.2742i$	$\pm 0.1370i$	$\pm 0.2065i$	$\pm 0.0907i$	Complex solutions which do not correspond to
24)	-0.0311	0.0075	3.3367	-4.3231	0.5177	0.7985	physical fixed points.
25 }	$\pm 0.2742i$	$\pm 0.1370i$	$\pm 1.4823i$	$\pm 1.9794i$	$\pm 0.2065i$	±0.0907 <i>i</i>)	
å This is th	jųlu, "oulu, o	n' = 2 fixed no	oint				
dt of ofdrud	6 - w (onling, or	fived noint					
	, , , ,, ,, ,, ,, ,, ,, ,, ,, ,, ,, ,,						
"This is th	ie "random" m	= 2 11xed point.					

TABLE II. Solutions of the fixed-point equations (B2) for $m_1 = m_2 = 1$ and corresponding stability exponents.

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$$u_{11}\tilde{\epsilon} - [8u_{11}^{2} + (4 + 2m_{1})u_{11}v_{11} + 2gm_{2}u_{12}v_{12}] = 0,$$

$$v_{11}\tilde{\epsilon} - [(8 + m_{1})v_{11}^{2} + 12u_{11}v_{11} + gm_{2}v_{12}^{2}] = 0,$$

$$u_{22}\tilde{\epsilon} - [8u_{22}^{2} + (4 + 2m_{2})u_{22}v_{22} + 2gm_{1}u_{12}v_{12}] = 0,$$

$$v_{22}\tilde{\epsilon} - [(8 + m_{2})v_{22}^{2} + 12u_{22}v_{22} + gm_{1}v_{12}^{2}] = 0,$$

$$u_{12}\tilde{\epsilon} - [4u_{12} + 2(u_{11} + u_{22}) + (2 + m_{1})v_{11} + (2 + m_{2})v_{22}]u_{12} - gv_{12}(m_{1}u_{11} + m_{2}u_{22}) = 0.$$

(B2)

with g = 1 and with v_{12} given by (B1). Unfortunately, we are not aware of any algorithm that enables us to find all the solutions of (B2). A general system of five quadratic coupled equations has 32 solutions. However, specific systems may have fewer finite solutions, because some solutions may be multiple or may "run away" to infinity. Although we are interested in solutions of (B2) for g=1 and m_1 , m_2 integers, we can define generalized solutions for any continuous value of g, m_1 , and m_2 . These solutions can be followed along trajectories in the g, m_1 , m_2 , space. For g=0, $u_{\mu\mu}$ and $v_{\mu\mu}$ coincide with their values for $v_{12} = 0$, found in Sec. III, and $v_{\rm 12}$ is trivially found from (B1). We can now vary g by small steps, from g = 0 to g = 1. In each step, we solve the equations (B2) by Newton's method, using the solution of the previous step as an initial guess. In this way, we keep track of all the expected 32 solutions. If $m_1 = 1$ or $m_2 = 1$, then some of the solutions of (B2) are infinite even for g = 0, (corresponding to fixed points of order $\epsilon^{1/2}$).¹⁶ In these cases, we start with values $m_1 \neq 1$ and $m_2 \neq 1$, and vary m_1 and m_2 , as well as g, in the iteration procedure. Having the freedom to change m_1 and m_2 as well as g enables us to avoid singularities in $u_{\mu\nu}(g)$ and $v_{\mu\nu}(g)$.

We now define trajectories in the g, m_1 , m_2 space starting at g = 0, m_1 , and m_2 arbitrary and terminating at g=1 and at m_1 , m_2 , values for which the solution is required. Each solution can be classified by a parameter α specifying the trajectory and a "length" parameter β specifying the point on the trajectory. The solutions are defined as $u^{\alpha}_{\mu\nu}(\beta)$, $v^{\alpha}_{\mu\nu}(\beta)$, where $\beta = (g, m_1, m_2)$. We try to choose trajectories in a way that $u^{\alpha}_{\mu\nu}(\beta)$ and $v^{\alpha}_{\mu\nu}(\beta)$ are regular functions. If a solution of (B2) "runs away" to infinity, it will exhibit itself as a pole at the end point in one of the functions $u^{\alpha}_{\mu\nu}(\beta)$, $v^{\alpha}_{\mu\nu}(\beta)$ for all the trajectories. If the residues satisfy Eqs. (B2) when we substitute $\epsilon = 0$, the ratios of the residues will give the ratios of the values obtained (of order $\epsilon^{1/2}$) when the higher order terms in (3.4) are introduced. This enables us to ensure that we do not lose any solution, because the total number of the finite solutions (taking into account the multiplicities) and of the "run-away" solutions equals 32. The multiple solutions can easily be identified, because linearization of (3.4) around

these points (ignoring the higher-order terms) leads to a vanishing (order ϵ) eigenvalue due to the vanishing of the appropriate Jacobian.

The only physically interesting cases are m_1 = 1, m_2 =2, and $m_1 = m_2$ =1. The solutions of (B2) for m_1 =1, m_2 =2 are given in Table I. We do not have any physical interpretation for the complex values.²⁵ In this table we also present the stability exponents for the corresponding fixed points (real solutions). In Table II we present the solutions of (B2) and the stability exponents for $m_1 = m_2 = 1$. Solution 1 in Table II has multiplicity 3, as can be verified analytically. Solutions of the order $\epsilon^{1/2}$ are [see Eq. (B1)]

 $u_{11} = -u_{22}, \quad v_{11} = -v_{22}, \quad v_{12} = 0, \quad u_{12} = 0,$ (B3)

where $u_{\mu\mu}$ and $v_{\mu\mu}$ are given by (3.5), and

$$u_{11} = u_{22} = u_{12} = \pm (3\epsilon/68)^{1/2}/4K_4,$$

$$v_{11} = v_{22} = -\frac{2}{3}u_{12}, \quad v_{12} = -2u_{12}.$$
(B4)

The stability exponents of the last solution are

$$\lambda_1 = -2\delta, \quad \lambda_2 = 6\delta, \quad \lambda_{3,4} = (3 \pm i 1.732)\delta,$$

 $\lambda_5 = 2\delta, \quad \lambda_6 = 0,$
(B5)

where

 $\delta = 2(3\epsilon/68)^{1/2}$.

Study of the "run-away" solutions show that two solutions "run away" to $\epsilon^{1/2}$ solutions of the type (B3) and one to a solution of the type (B4). Two solutions run away in a way that can be extrapolated to poles with residues that do not satisfy (B2) when $\tilde{\epsilon} = 0$. Therefore, these "flows" do not correspond to a solution of the order $\epsilon^{1/2}$. The finite solutions and the distinct "run-away" solutions sum to 32; we conclude that we found all solutions.

All the fixed points found in this Appendix are unstable. The marginal fixed point 10 in Table I is unphysical (note it is degenerate with one of the decoupled fixed points discussed in Secs. III-IV).

APPENDIX C: STABILITY OF THE KHMEL'NITZKII POINT

Experience in ϵ expansions shows that if we linearize the recursion relations about a nontrivial fixed point of order ϵ , one eigenvector points, to order ϵ , towards the trivial (Gaussian) fixed point, and the corresponding eigenvalue is $-\epsilon$. We shall first give a general proof for this fact. Let the Hamiltonian depend on N quartic couplings X_i . The recursion relations to leading order are

$$X'_{i} = b^{6} \left(X_{i} - 4K_{4} \ln b \sum_{k, j=1}^{N} A_{kj}^{(i)} X_{k} X_{j} \right), \quad i = 1, \dots, N.$$
(C1)

The fixed points satisfy the equations

$$X_{i}^{*}\tilde{\epsilon} - \sum_{k,j}^{N} A_{kj}^{(i)} X_{k}^{*} X_{j}^{*} = 0 , \qquad (C2)$$

where $\tilde{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}/4K_4$. Linearizing (C1) about the fixed points, one obtains

$$\Delta X'_{i} = b^{\epsilon} \left(\Delta X_{i} - 4K_{4} \ln b \sum_{k,j=1}^{N} A^{(i)}_{kj} (X^{*}_{k} \Delta X_{j} + X^{*}_{j} \Delta X_{k}) \right). \quad (C3)$$

Consider now the vector $\Delta X_i^{(1)} \equiv a X_i^*$, pointing towards the Gaussian trivial fixed point $(X_i^* = 0)$. Substituting in (C3), and using (C2), we find for this vector (at order ϵ)

$$\Delta X_i^{(1)'} = b^{-\epsilon} \Delta X_i^{(1)} , \qquad (C4)$$

which completes the proof.

We now follow the same lines for a fixed point of order $\epsilon^{1/2}$, and show that one eigenvalue is always equal to -2ϵ . The recursion relations now read^{16,24}

$$X'_{i} = b^{6} \left(X_{i} - 4K_{d} \ln b \sum_{k, j=1}^{N} A^{(i)}_{kj} X_{k} X_{j} + 32K_{4}^{2} \ln b \sum_{k, j, l=1}^{N} A^{(i)}_{kjl} X_{k} X_{j} X_{l} \right), \quad (C5)$$

while the fixed point equations are

$$\tilde{\epsilon}X_{i}^{*} - \sum_{k,j=1}^{N} A_{kj}^{(i)}X_{k}^{*}X_{j}^{*} + 8K_{4} \sum_{k,j,l=1}^{N} A_{kjl}^{(i)}X_{k}^{*}X_{j}^{*}X_{l}^{*} = 0.$$
(C6)

For fixed points of order $\epsilon^{1/2}$, we write

$$X_i^* = Y_i \epsilon^{1/2} + Z_i \epsilon + \cdots, \qquad (C7)$$

and the coefficients must satisfy¹⁶

$$\sum_{k,j=1}^{N} A_{kj}^{(i)} Y_{k} Y_{j} \equiv 0 , \qquad (C8)$$

$$Y_{i} - 4K_{d} \sum_{k,j} A_{kj}^{(i)} (Y_{k}Z_{j} + Y_{j}Z_{k}) + 32K_{4}^{2} \sum_{k,j,l} A_{kjl}^{(i)} Y_{k}Y_{j}Y_{l} = 0.$$
 (C9)

Linearizing (C5) about X_i^* , and choosing

$$\Delta X_{i}^{(1)} = a(Y_{i}\epsilon^{1/2} + 2Z_{i}\epsilon + \cdots), \qquad (C10)$$

now immediately gives (at lowest order)

$$\Delta X_{i}^{(1)'} = b^{-2\epsilon} \, \Delta X_{i}^{(1)} \,. \tag{C11}$$

Note that the proof is independent of the values of Z_i . To calculate these, one must go beyond the diagrams of Fig. 3. Thus, the eigenvalue which vanishes at order $\epsilon^{1/2}$ is actually negative, and the "Khmel'nitzkii" fixed point is stable near d=4.²⁹

APPENDIX D: CORRECTIONS TO SCALING

In this appendix we study the effects of a nonzero value of v_{12} on the shape of the phase diagrams near the tetracritical point. We follow Wegner and Houghton,³⁰ and write Eq. (3.1) in the differential form

$$\frac{dr_1}{dl} = 2r_1 + \frac{A_1}{1+r_1} + 4K_4m_2\frac{v_{12}}{1+r_2},$$

$$\frac{dr_2}{dl} = 2r_2 + \frac{A_2}{1+r_2} + 4K_4m_1\frac{v_{12}}{1+r_1},$$
(D1)

where

$$A_{\mu} = -2\gamma_{\mu}^{*} = 4K_{4} [2u_{\mu\mu}^{*} + (m_{\mu} + 2)v_{\mu\mu}^{*}] \ [= O(\epsilon)]$$

and l is the continuous renormalization-group parameter. In the immediate neighborhood of the decoupled fixed point,

$$v_{12}(l) = v_{12}(0)e^{\lambda_{12}l}, \qquad (D2)$$

where λ_{12} is given by Eqs. (4.2) or (4.4). When $v_{12}(0) = 0$, the solutions to (D1) at lowest order are^{31}

$$r_{\mu}(l) = t_{\mu}(l) - \frac{1}{2}A_{\mu} + \frac{1}{2}A_{\mu}t_{\mu}(l)\ln[1 + t_{\mu}(l)]$$
$$= t_{\mu}(l) + O(\epsilon), \qquad (D3)$$

with

$$t_{\mu}(l) = t_{\mu}(0) \exp(2 - A_{\mu})l, \quad t_{\mu}(0) = r_{\mu}(0) - r_{\mu}^{*}.$$

The exponent $2 - A_{\mu}$ is equal to $1/\nu_{\mu}$.

We now assume $v_{12}(0) \ll u_{\mu\mu}^*$, $v_{\mu\mu}^*$, and consider for definiteness the shape of T_2 in the vicinity of the decoupled fixed point (Fig. 2). This means that \tilde{S}_2 is much more critical than \tilde{S}_1 , $r_2 \ll r_1$. We continue the renormalization-group iterations until $r_1(l^*)=1$, while $r_2(l^*)=O(r_2^*)$. At this point

$$t_1(l^*) = 1 + O(r_1^*, v_{12})$$

or

$$\exp[-(2 - A_1)l^*] = t_1(0)[1 + O(r_1^*, v_{12})].$$
 (D4)

During the iterations $r_2(l) = O(\epsilon)$ [or $O(\epsilon^{1/2})$], and therefore we may expand

$$\frac{dr_2}{dl} = (2 - A_2)r_2 + A_2 + 4K_4m_1\frac{v_{12}}{1 + r_1}.$$
 (D5)

The solution of this inhomogeneous linear equation is

$$r_{2}(l) = e^{(2-A_{2})l} \left[r_{2}(0) + \int_{0}^{l} \left(A_{2} + \frac{4K_{4}m_{1}v_{12}(l')}{1 + r_{1}(l')} \right) \times e^{-(2-A_{2})l'} dl' \right].$$
(D6)

At lowest order we replace r_1 by t_1 in the last term. Since $r_1(l) < 1$ for $l < l^*$, we also expand the denominator, use (D3) and integrate,

For $k \neq 1$, the denominator is $2(k-1) + O(\epsilon)$, and we ignore the order ϵ terms. Thus,

$$I \simeq -\frac{1}{2} \left[v_{12}(l) e^{-(2-A_2)t} - v_{12}(0) \right] - \left[v_{12}(l) t_1(l) e^{-(2-A_2)t} - v_{12}(0) t_1(0) \right] / (\lambda_{12} + A_2 - A_1) + \frac{1}{2} v_{12}(l) t_1(l) e^{-(2-A_2)t} \ln[1 + t_1(l)] - \frac{1}{2} v_{12}(0) t_1(0) \ln[1 + t_1(0)].$$
(D8)

Finally,

$$r_{2}(l) = \tilde{t}_{2}(0)e^{(2-A_{2})l} - A_{2}/2 - 2K_{4}m_{1}v_{12}(l) - 4K_{4}m_{1}v_{12}(l)t_{1}(l)/(\lambda_{12} + A_{2} - A_{1}) + 2K_{4}m_{1}v_{12}(l)t_{1}(l)\ln[1 + t_{1}(l)],$$
(D9)

with

$$\tilde{t}_{2}(0) = r_{2}(0) + A_{2}/2 + 2K_{4}m_{1}v_{12}(0) + 4K_{4}m_{1}v_{12}(0)t_{1}(0)/(\lambda_{12} + A_{2} - A_{1}) - 2K_{4}m_{1}v_{12}(0)t_{1}(0)\ln[1 + t_{1}(0)].$$
(D10)

At $l = l^*$, $r_1(l^*) = 1$, and we can integrate the variables \vec{S}_1 out of the partition function.³² As a result of this, we obtain an effective Hamiltonian in \vec{S}_2 , with

$$r_2^{\text{eff}} = r_2(l^*) + 4K_4 m_1 v_{12}(l^*) \int_0^1 \frac{q^3 dq}{r_1(l^*) + q^2} .$$
 (D11)

Therefore,

$$r_{2}^{\text{eff}} - r_{2}^{*} = \tilde{t}_{2}(0)e^{(2-A_{2})t^{*}} - \frac{4K_{4}m_{1}v_{12}(t^{*})t_{1}(t^{*})}{\lambda_{12} + A_{2} - A_{1}} .$$
(D12)

At the critical line $T = T_2$, $(r_2^{\text{eff}} - r_2^*)$ vanishes. Therefore,

$$\tilde{t}_{2}(0) = \frac{4K_{4}m_{1}v_{12}(l^{*})t_{1}(l^{*})e^{-(2-A_{2})l^{*}}}{\lambda_{12}+A_{2}-A_{1}} .$$
(D13)

By (D4), $t_1(l^*) \simeq 1$, and $\exp[-(2-A_1)l^*] \simeq t_1(0)$. Thus, using also (D2) and $(2-A_{\mu}) = 1/\nu_{\mu}$,

$$\tilde{t}_{2}(0) = \frac{4K_{4}m_{1}v_{12}(0)[t_{1}(0)]^{(\nu_{1}/\nu_{2})-\lambda_{12}\nu_{1}}}{\lambda_{12}+A_{2}-A_{1}} .$$
(D14)

The actual shift in the critical temperature T_2 , or in the critical value of $r_2(0)$, due to the existence of $v_{12} \neq 0$, is thus due to three sources: first, there is a constant shift to lower T_2 , proportional to $v_{12}(0)$ [the third term on the right-hand side, Eq. (D10)]. Second, there is a term proportional to $v_{12}(0)t_1(0)$, which changes the slope of the straight lines in Fig. 2, bringing them closer to each other. Last, there is the nonanalytic piece, Eq. (D14), which brings the lines in Fig. 2 further apart from each other at larger distances from the tetracritical point. These final lines approach the abovementioned straight lines tangentially. The modified phase diagram, near the tetracritical point, is shown in Fig. 4.

The result (D14) can also be derived by general scaling arguments, similar to the ones used in Ref. 3. To start with, the appropriate scaling fields at the decoupled fixed point are t_1 , t_2 , and v_{12} , with scaling exponents $1/\nu_1$, $1/\nu_2$, and λ_{12} . Therefore, the free energy has the scaling form

$$F(t_1, t_2, v_{12}) = b^{-d}F(b^{1/\nu_1}t_{1,} b^{1/\nu_2}t_{2,} b^{\lambda_{12}}v_{12}) .$$
(D15)

Scaling until $b^{1/\nu_1}t_1 = 1$, this becomes

$$F(t_1, t_2, v_{12}) = t_1^{d\nu_1} f(t_2 / t_1^{\nu_1 / \nu_2}, v_{12} t_1^{-\lambda_{12} \nu_1}).$$
(D16)

At the line $T = T_2$, the function f(X, Y) must be singular. In the X-Y plane, this singularity must have the form $X = AY^{\theta}$, with θ to be determined. Thus, the singular part of the shift in T_2 is given by

$$t_2^{\sin g} / t_1^{\nu_1 / \nu_2} = A \left(v_{12} t_1^{-\lambda_{12} \nu_1} \right)^{\theta} . \tag{D17}$$

Our explicit calculation, at low order, gives $\theta = 1$.

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