Spin-flop multicritical points in systems with random fields and in spin glasses*

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Mean-field theory and renormalization-group arguments are used to study the phase diagram of an anisotropic n-component d-dimensional magnetic system with a uniaxially random magnetic field. The resulting phase diagram is shown to be very similar to that of anisotropic antiferromagnets in a uniform field: For small random fields, the system orders along the direction of uniaxial anisotropy, with exponents which are related to those of nonrandom Ising systems in $d-2$ dimensions. For larger random fields, parallel to the direction of uniaxial anisotropy, the transverse $n-1$ spin components order, with exponents which are unaffected by the random field. The two regions are separated by a spin-flop first-order line, by an intermediate "mixed" phase, and by a tetracritical (or bicritical) point. The exponents at this multicritical point are shown to coincide, near $d=6$, with those of the random-field Ising model. This phase diagram is shown to describe the behavior of random-site spin glasses in a uniform magnetic field. Other types of anisotropic random fields, related experimental realizations and other generalizations are also mentioned. Although some of the quantitative results are found only near $d=6$, qualitative results are believed to apply at $d=3$ as well.

I. INTRODUCTION

The Hamiltonian of a magnetic system with a random quenched magnetic field may be written in the form

$$\mathcal{H}=-\frac{1}{2}\sum_{i\neq j}\sum_{\alpha=1}^{n}J_{ij}^{\alpha}S_i^{\alpha}S_j^{\alpha}-\sum_i\vec{H}_i\cdot\vec{S}_i \quad . \tag{1.1}$$

Here, $\vec{S}_i\equiv\{S_i^1,\ldots,S_i^n\}$ is a classical n-component spin vector, located at the site i of a d-dimensional lattice, J_{ij}^{α} is the exchange-coupling coefficient between the αth spin components, and $\vec{H}_i$ is a local random field, with distribution $P\{\vec{H}_i\}$. We shall consider mainly symmetric distributions $P\{\vec{H}_i\}=P\{-\vec{H}_i\}$.

In a previous paper,[1] we showed that an *isotropic*-field distribution may change, under some conditions, the order of the magnetic transition from second to first order, at a *tricritical* point. In the present paper we shall be concerned with *anisotropic-field distributions*. In particular, we shall consider the case in which the random field is always along one spatial direction, i.e., $H_i^{\alpha}=H_i\delta_{\alpha 1}$. The actual local value of H_i will be random, with distribution $p(H_i)$, and will be assumed to be independent of the fields at any other lattice point. To be more explicit, we shall first assume that $H_i=\pm H_0$ at random, i.e.,

$$p_\delta(H_i)=\tfrac{1}{2}[\delta(H_i+H_0)+\delta(H_i-H_0)] \quad . \tag{1.2}$$

Many other distributions will turn out to be universally equivalent (in the renormalization-group sense) to the simple ones which we explicitly mention.[1]

The Hamiltonian (1.1) can thus be written

$$\mathcal{H}=-\frac{1}{2}\sum_{i\neq j}\sum_{\alpha}J_{ij}^{\alpha}S_i^{\alpha}S_j^{\alpha}-\sum_i H_iS_i^1 \quad . \tag{1.3}$$

If we now define a new order parameter

$$\vec{\tau}_i=\vec{S}_iH_i/H_0 \quad , \tag{1.4}$$

then (1.3) becomes

$$\mathcal{H}=-\frac{1}{2}\sum_{i\neq j}\sum_{\alpha}\tilde{J}_{ij}^{\alpha}\tau_i^{\alpha}\tau_j^{\alpha}-H_0\sum_i\tau_i^1 \quad , \tag{1.5}$$

where

$$\tilde{J}_{ij}^{\alpha}=J_{ij}^{\alpha}H_iH_j/H_0^2=\pm J_{ij}^{\alpha} \quad . \tag{1.6}$$

Equation (1.5) represents a magnetic system, with spins $\vec{\tau}_i$ and with random-exchange coefficients which have zero average, in a uniform magnetic field H_0 along the 1 axis. This is a model for some *random-site spin glasses.*[2-5] The spin-glass order parameter $\vec{S}_i$ is related to the magnetic order parameter $\vec{\tau}_i$ via Eq. (1.4).[6]

The transformation from (1.3) to (1.5) is also useful in the study of *antiferromagnets.* In simple cases, the antiferromagnetic order amounts to dividing the system into two interpenetrating sublattices, so that in the ground state the spins on one sublattice are parallel to each other, and those in different sublattices are antiparallel. Such a situation, for small uniform fields

H_0, will result from (1.5), if $\tilde{J}_{ij} < 0$ for i and j on different sublattices, and $\tilde{J}_{ij} > 0$ for i and j on the same sublattice. If we change the signs of the spins on one sublattice, via Eq. (1.4), we recover Eq. (1.3), with $\vec{S}_i$ now representing the *staggered magnetization*. The uniform field H_0 now looks like a *staggered field*. Equation (1.3) can thus represent a ferromagnet in a random field, a spin glass, or an antiferromagnet. The only difference between the antiferromagnet and the other cases is that for the former, the fields H_i's are *not random*.

The effects of a uniform magnetic field on the antiferromagnetic phase transition have been studied for a long time.[7] If the system is uniaxially anisotropic, i.e.,

$$J_{ij}^1 > J_{ij}^2 = \cdots = J_{ij}^n > 0 \quad , \tag{1.7}$$

then for sufficiently weak fields one expects antiferromagnetic ordering in the 1 direction ($\langle S_i^1 \rangle \neq 0$). However, Néel[8] already pointed out that for sufficiently strong fields, parallel to this direction, the spins will "*flop*" over, via a first-order transition, into an alignment that is predominantly perpendicular to the 1 axis. This first-order line ends at a *bicritical point*.[9–12] Under different conditions, the uniaxial "longitudinal" phase does not flop directly into the "transverse" one, but the two form an intermediate "mixed" phase, into which the transitions are second order.[12,13] In this case, the four second-order lines meet at a *tetracritical point*.

The analogy mentioned above suggests, that a similar situation may occur for the random-field model. In this paper we show that, indeed, *a spin-flop transition and a multicritical point do occur in this model*. Section II is devoted to a mean-field analysis of the model Hamiltonian (1.3), with the distribution (1.2). Indeed, a bicritical point is found, when the random field H_0 is sufficiently strong to overcome the uniaxial anisotropy [Eq. (1.7)]. The resulting phase diagram is shown in Fig. 1. Renormalization-group recursion relations, for a continuous spin version of the model, are set up in Sec. III, and the resulting multicritical and anisotropic fixed points are analyzed and discussed in Secs. IV and V. Generalizations of the model are considered in Sec. VI, and the results and their experimental implications are summarized in Sec. VII.

II. MEAN-FIELD THEORY

As in Ref. 1,[14] we now replace (1.3) by its mean-field approximation

$$\mathcal{H}_0 = \tfrac{1}{2} Nc \sum_\alpha J^\alpha (M^\alpha)^2 - \sum_i \sum_\alpha (cJ^\alpha M^\alpha + H_i^\alpha) S_i^\alpha \quad , \tag{2.1}$$

where c is the coordination number and N is the

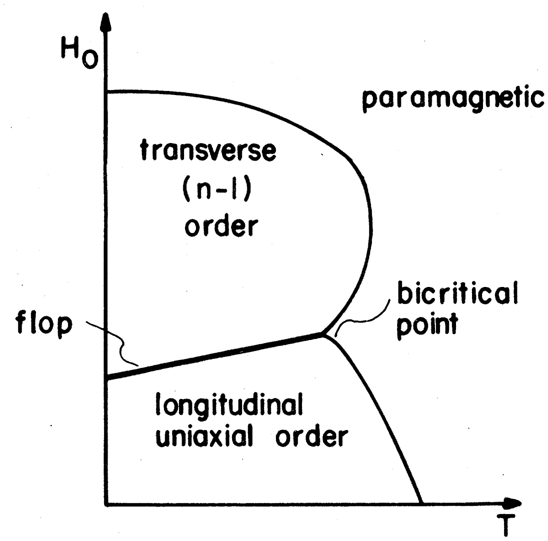

FIG. 1. Schematic mean-field phase diagram for the uniaxial random-field model. The local field is $\pm H_0$, and it is directed along the axis of uniaxial anisotropy.

number of lattice sites. For convenience, we concentrate here on the Heisenberg case $n = 3$. The free energy per spin thus becomes

$$\bar{F} = \tfrac{1}{2} c \sum_\alpha J^\alpha (M^\alpha)^2 - \frac{1}{\beta}\left\{ \ln g\left[\beta\left(\sum_\alpha (cJ^\alpha M^\alpha + H_i^\alpha)^2\right)^{1/2}\right]\right\}_{\text{av}} \quad , \tag{2.2}$$

where $\{\ \}_{\text{av}}$ denotes averaging over the magnetic field distribution, $\beta = 1/k_B T$, and

$$g(|\vec{x}|) = \int d\hat{S}_i \exp(\vec{x} \cdot \hat{S}_i) = \frac{4\pi \sinh|\vec{x}|}{|\vec{x}|} \quad . \tag{2.3}$$

The magnetization $\vec{M}$ is the solution of the self-consistency equation

$$M^\alpha = \left[\left(\frac{\coth x_i}{x_i} - \frac{1}{x_i^2}\right) \beta (cJ^\alpha M^\alpha + H_i^\alpha)\right]_{\text{av.}} \quad , \tag{2.4}$$

where

$$x_i^2 = \beta^2 \left(H_i^2 + 2c \sum_\alpha J^\alpha M^\alpha H_i^\alpha + c^2 \sum_\alpha (J^\alpha)^2 (M^\alpha)^2 \right) \quad . \tag{2.5}$$

In the particular case of interest [Eq. (1.2)],

$H_i^\alpha = \pm H_0 \delta_{\alpha 1}$, Eq. (2.4) simply becomes

$$M^\alpha = \tfrac{1}{2}(f_+ + f_-)\beta c J^\alpha M^\alpha + \tfrac{1}{2}(f_+ - f_-)\beta H_0 \delta_{\alpha 1} \ , \tag{2.6}$$

with

$$f_\pm = \coth x_\pm / x_\pm - 1/x_\pm^2 \ ,$$

$$x_\pm^2 = \beta^2 \left[H_0^2 + c^2 \sum_\alpha (J^\alpha)^2 (M^\alpha)^2 \pm 2cH_0 J^1 M^1 \right] . \tag{2.7}$$

One can now solve Eq. (2.6), and choose the solution which minimizes the free energy

$$(F)_{\text{av.}} = \tfrac{1}{2} c \sum_\alpha J^\alpha (M^\alpha)^2 - \frac{1}{2\beta} [\ln g(x_+) + \ln g(x_-)] \ . \tag{2.8}$$

Since we are mainly concerned with the phase transition from the paramagnetic ($\vec{M}=0$) to the ferromagnetic ($\vec{M} \neq 0$) phase, it is useful to expand (2.8) around $\vec{M}=0$. For small M,

$$(F)_{\text{av}} = -\frac{1}{\beta} \ln \left[\frac{4\pi \sinh \beta H_0}{\beta H_0} \right] + \frac{1}{2} \sum_\alpha a_\alpha (M^\alpha)^2 + \sum_{\alpha, \beta} b_{\alpha\beta} (M^\alpha)^2 (M^\beta)^2 + \cdots \ , \tag{2.9}$$

with

$$a_\alpha = \beta (cJ^\alpha) \{ k_B T - cJ^\alpha [f_0 - 2f_1 (\beta H_0)^2 \delta_{\alpha 1}] \} \ , \tag{2.10}$$

$$b_{\alpha\beta} = \tfrac{1}{4} \beta^3 c^4 (J^\alpha)^2 (J^\beta)^2 f_{\alpha\beta} \ , \tag{2.11}$$

$$f_{\alpha\beta} = f_1 + 2f_2 (\delta_{\alpha 1} + \delta_{\beta 1}) + \tfrac{4}{3} f_3 \delta_{\alpha 1} \delta_{\beta 1} \ , \tag{2.12}$$

and

$$\begin{aligned} f_0 &= \coth x / x - 1/x^2 \ , \\ f_1 &= \coth x / (2x) + 1/(2\sinh^2 x) - 1/x^2 \ , \\ f_2 &= 2/x^2 - x \coth x / (2\sinh^2 x) - 3/(4\sinh^2 x) - 3\coth x/(4x) \ , \\ f_3 &= 15\coth x/(8x) + 15/(8\sinh^2 x) + x^2 \coth^2 x/(2\sinh^2 x) + x^2/(4\sinh^2 x) + 3x\coth x/(2\sinh^2 x) - 6/x^2 \ , \end{aligned} \tag{2.13}$$

where $x = \beta H_0$.

An explicit calculation shows, that all $f_{\alpha\beta}$ are positive for $\beta H_0 \lesssim 1.37$, and the transition is second order for these values of βH_0. As the temperature is lowered at fixed H_0, one of the coefficients a_α will become negative, and the appropriate spin component will order. For isotropic exchange, $J^\alpha \equiv J$, the transverse spin components $\alpha = 2, 3$ will order first for *any* finite H_0, since one always has $f_1(x) > 0$. For the uniaxially anisotropic case, Eq. (1.7), the longitudinal ($\alpha = 1$) spin component will order first only if

$$(J^1 - J^2)/J^1 > 2f_1 (\beta H_0)^2 / f_0 \ . \tag{2.14}$$

For higher values of H_0 (or lower anisotropies, the transverse components again order first. The point at which the two sides of Eq. (2.14) are equal is therefore a *bicritical point,* and the phase diagram in temperature—random-field plane is schematically shown in Fig. 1.

Another explicit calculation shows that $b_{12}^2 > b_{11} b_{22}$ for all the values of βH_0. Hence, the transition from the uniaxially ordered phase to the flopped phase is indeed first order, and there is no intermediate mixed phase.[12,13] The first-order nature of the flop line has also been checked directly, from Eqs. (2.6) and (2.8).

If the anisotropy is strong, condition (2.14) may still hold even at $\beta H_0 \simeq 1.37$. For $\beta H_0 > 1.37$ one finds that $b_{11} < 0$, and the transition into the uniaxially ordered phase becomes *first order,* as discussed in detail in Ref. 1. We shall not consider this first-order regime any further here except for noting that at $\beta H_0 \simeq 1.37$, the bicritical point will have new properties, as it becomes a higher-order multicritical point.

The phase diagram shown in Fig. 1 has exactly the same form as found for antiferromagnets in a uniform field,[9–11] in agreement with the conjectured analogy between the two cases.

One can easily generalize the above discussion. For example, if the magnitude of the field H_0 is also random, Eqs. (2.6), (2.8), (2.10), and (2.11) must be averaged over all possible magnitudes. This will change the numerical details, but not the general features of the phase diagram. Similarly, if the random field is not exactly aligned in the direction of the uniaxial anisotropy (the 1 axis), this will introduce terms like $M^1 M^2$ into Eq. (2.9), and the analysis will become somewhat more complicated, as the magnetization will not be along an axis. If there is a *random*-field component in the transverse plane, with average zero, then the analysis again becomes similar to the one mentioned above, but with modified coefficients in Eq. (2.9). We shall return to some of these cases below, in Sec. VI.

III. RENORMALIZATION GROUP

A. Continuous-spin Hamiltonian

In order to analyze Hamiltonian (1.1), or (1.3), with standard renormalization-group techniques,[15–17] it is useful to transform it into a continuous spin form. For simplicity, we follow here the standard approach,[15–17] i.e., we associate with each spin a weight factor

$$\exp[-W(\vec{S})]=\exp(-\tfrac{1}{2}|\vec{S}|^2-f_4|\vec{S}|^4-\cdots) , \tag{3.1}$$

and write the partition function as

$$Z=\int d^{nN}S\,\exp(\bar{\mathcal{H}}) , \tag{3.2}$$

with

$$\bar{\mathcal{H}}=-\beta\mathcal{H}-\sum_i W(\vec{S}_i) . \tag{3.3}$$

For simplicity, we also follow previous renormalization-group studies of the random-field problem,[18–21] and replace the δ distribution (1.2) by a Gaussian distribution, i.e.,

$$(H_i^\alpha H_j^\beta)_{\rm av}=\tilde{\lambda}\delta_{ij}\delta_{\alpha 1}\delta_{\beta 1} . \tag{3.4}$$

We shall return to this assumption later, to argue that many other distributions will be universally equivalent to the Gaussian one[1] (see Sec. VI A).

As usual,[15–17] we next Fourier transform the spin variables $\vec{S}_i$ into $\vec{\sigma}_{\vec{q}}$, and rescale the spins so that the final form of $\bar{\mathcal{H}}$ becomes

$$\bar{\mathcal{H}}=-\frac{1}{2}\int_{\vec{q}}\sum_\alpha (r_\alpha+q^2)\sigma^\alpha_{\vec{q}}\sigma^\alpha_{-\vec{q}}-\sum_{\alpha\beta}u_{\alpha\beta}\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma^\alpha_{\vec{q}}\sigma^\alpha_{\vec{q}'}\sigma^\beta_{\vec{q}''}\sigma^\beta_{-\vec{q}-\vec{q}'-\vec{q}''}-\int_{\vec{q}}h^1_{\vec{q}}\sigma^1_{-\vec{q}} , \tag{3.5}$$

where r_α is now linear in (k_BT-cJ^α), $\int_{\vec{q}}$ stands for $(2\pi)^{-d}\int d^dq$ with $|\vec{q}|<\Lambda$, and

$$(h^1_{\vec{q}}h^1_{\vec{p}})_{\rm av}=\lambda\delta(\vec{q}+\vec{p}) , \tag{3.6}$$

with $\lambda\propto\tilde{\lambda}$ and $\delta(\vec{q})\equiv(2\pi)^d\delta^{(d)}(\vec{q})$.

An alternative approach, following Hubbard,[22] will lead to similar results. This approach was used explicitly in Ref. 1. In particular, it was shown there that using this approach one finds that the initial values of the parameters r_α, $u_{\alpha\beta}$, etc., are proportional to the appropriate coefficients a_α and $b_{\alpha\beta}$ of Eqs. (2.9)–(2.11).

B. Recursion relations

We now follow the usual routine of renormalization group in random systems[23]: We integrate out the spins in the partition function with $\Lambda/b<|\vec{q}|<\Lambda$, and rescale $\vec{q}$ into $b\vec{q}$ and $\sigma^\alpha_{\vec{q}}$ into $\zeta_\alpha\sigma^\alpha_{b\vec{q}}$, where ζ_α is chosen so that some coefficient in the first (quadratic) term in the new Hamiltonian (3.5) remains unchanged. If the spin component S^α is critical at a given fixed point, then we shall choose

$$\zeta_\alpha^2=b^{d+2-\eta_\alpha} , \tag{3.7a}$$

where η_α is fixed so that the coefficient of $q^2\sigma^\alpha_{\vec{q}}\sigma^\alpha_{-\vec{q}}$ remains unchanged. If there are no critical fluctuations in S^α, then we shall choose[24]

$$\zeta_\alpha^2=b^d[1+O(u_{\alpha\beta},u_{\alpha 1}\lambda)] , \tag{3.7b}$$

so that r_α remains unchanged. At the end we average over the field variables $h^1_{\vec{q}}$ which have $\Lambda/b<|\vec{q}|<\Lambda$, and also calculate the new value of λ, defined in (3.6). The explicit calculation involves a diagrammatic expansion in the last two terms in Eq. (3.5). The final diagrams will involve[1,18–21] four-line vertices representing $u_{\alpha\beta}$, two-line vertices (represented by empty circles) representing λ, and internal lines representing the propagators $(r_\alpha+q^2)^{-1}$. At low order, some diagrams which contribute to the new inverse propagators and to the new vertices are shown in Fig. 2. The appropriate recursion relations are [for $r_1\neq r_2=\cdots=r_n$, $u_{\alpha\beta}\equiv u_{22}$, $u_{1\alpha}\equiv u_{12}$, for $\alpha,\beta\neq 1$, see (1.7)]

$$r_1'=\zeta_1^2b^{-d}\{r_1+4K_d[3u_{11}(A_1(r_1)+\lambda A_2(r_1))+(n-1)u_{12}A_1(r_2)]+\cdots\} , \tag{3.8}$$

$$r_2'=\zeta_2^2b^{-d}\{r_2+4K_d[u_{12}(A_1(r_1)+\lambda A_2(r_1))+(n+1)u_{22}A_1(r_2)]+\cdots\} , \tag{3.9}$$

$$u_{11}'=\zeta_1^4b^{-3d}\{u_{11}-4K_d[9u_{11}^2(A_2(r_1)+2\lambda A_3(r_1))+(n-1)u_{12}^2A_2(r_2)]+\cdots\} , \tag{3.10}$$

$$u_{22}'=\zeta_2^4b^{-3d}\{u_{22}-4K_d[u_{12}^2(A_2(r_1)+2\lambda A_3(r_1))+(n+7)u_{22}^2A_2(r_2)]+\cdots\} , \tag{3.11}$$

$$u_{12}'=\zeta_1^2\zeta_2^2b^{-3d}\{u_{12}-4K_du_{12}[4u_{12}(B_{11}(r_1,r_2)+\lambda B_{21}(r_1,r_2))+3u_{11}(A_2(r_1)+2\lambda A_3(r_1))+(n+1)u_{22}A_2(r_2)]+\cdots\} , \tag{3.12}$$

$$\eta_\alpha = O(u_{\alpha\beta}^2, u_{\alpha 1}^2 \lambda^2) \ , \tag{3.13}$$

and

$$\lambda' = \zeta_1^2 b^{-d} \lambda [1 + O(u_{11}^2 \lambda^2)] \ , \tag{3.14}$$

where

$$A_m(r) = \int_{\Lambda/b}^{\Lambda} q^{d-1}\, dq\, (r+q^2)^{-m}$$
$$\simeq A_m(0) - mrA_{m+1}(0) \ , \tag{3.15}$$

$$B_{mn}(r_1,r_2) = \int_{\Lambda/b}^{\Lambda} q^{d-1}\, dq\, (r_1+q^2)^{-m} \times (r_2+q^2)^{-n} \ , \tag{3.16}$$

and

$$K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(\tfrac{1}{2}d) \ . \tag{3.17}$$

The recursion relations for r_1 and r_2 involve the variables $u_{11}\lambda$ and $u_{12}\lambda$ (but *not* $u_{22}\lambda$!). Therefore, we must also consider recursion relations for these variables

$$(u_{11}\lambda)' = \zeta_1^6 b^{-4d} \{ u_{11}\lambda - 72K_d(u_{11}\lambda)^2 A_3(r_1) + O[(u_{11}\lambda)^3, u_{\alpha\beta}^2\lambda, \ldots] \} \tag{3.18}$$

and

$$(u_{12}\lambda)' = \zeta_1^4 \zeta_2^2 b^{-4d} \{ u_{12}\lambda - 16K_d(u_{12}\lambda)^2 B_{21}(r_1,r_2) - 24K_d(u_{11}\lambda)(u_{12}\lambda)A_3(r_1) + O[(u_{11}\lambda)^3, u_{\alpha\beta}^2\lambda, \ldots] \} \ . \tag{3.19}$$

Having set up all the necessary recursion relations, we now turn to a study of the possible fixed points.

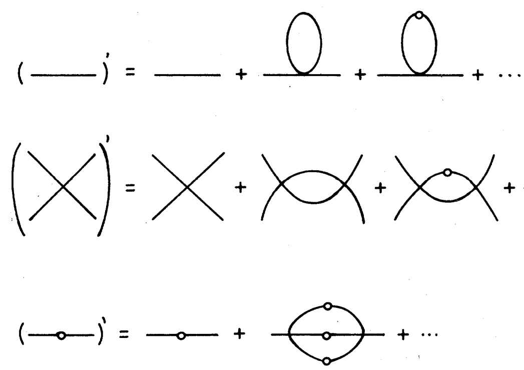

FIG. 2. Diagrams for the recursion relations for the temperature variables r_α [Eqs. (3.8) and (3.9)], the four spin coefficients $u_{\alpha\beta}$ [Eqs. (3.10)–(3.12)] and the random-field second cumulant λ [Eq. (3.14)].

IV. MULTICRITICAL POINT

A. Fixed point and exponents

At this stage, we must choose ζ_α. We start with the *multicritical point,* at which we expect all spin components to become critical simultaneously. Using the choice (3.7a), the prefactor in Eqs. (3.8), (3.9), and (3.14) becomes $b^{2-\eta_\alpha}$, that in Eqs. (3.18) and (3.19) becomes $b^{6-d}[1+O(\eta_\alpha)]$, and that in Eq. (3.11) becomes $b^{4-d-2\eta_2}$. Thus, both $u_{11}\lambda$ and $u_{12}\lambda$ decay to zero for $d > 6$, where we expect the mean-field results of Sec. II to apply. For $d \lesssim 6$, we can find nontrivial fixed points with $u_{11}\lambda$ and $u_{12}\lambda$ of order $\epsilon = 6-d$. There are three such fixed points, i.e.,

$$\lambda u_{11}^{(a)} = 0 \ , \quad \lambda u_{12}^{(a)} = \epsilon/16K_d + O(\epsilon^2) \ , \tag{4.1}$$

$$\lambda u_{11}^{(b)} = \epsilon/72K_d + O(\epsilon^2) \ , \quad \lambda u_{12}^{(b)} = 0 \ , \tag{4.2}$$

and

$$\lambda u_{11}^{(c)} = \epsilon/72K_d + O(\epsilon^2) \ , \quad \lambda u_{12}^{(c)} = \epsilon/24K_d + O(\epsilon^2) \ . \tag{4.3}$$

Linearizing Eqs. (3.18) and (3.19) near each of these fixed points, we find that only the third one is stable, with order $-\epsilon$ eigenvalue exponents $-\epsilon$ and $-\frac{2}{3}\epsilon$.

One should note that at this fixed point, $\lambda \rightarrow \infty$ and $u_{\alpha\beta} \rightarrow 0$; in particular, $u_{22} \rightarrow 0$. Therefore, we can ignore terms of order $u_{\alpha\beta}$ compared to those of order $u_{\alpha\beta}\lambda$. Returning now to Eqs. (3.8) and (3.9), we find a fixed point of order ϵ,

$$r_1^{(c)} = r_2^{(c)} = -\epsilon/12K_d + O(\epsilon^2) \ . \tag{4.4}$$

Linearizing about this fixed point, we find the eigenvector $\Delta r_1 \equiv \Delta r_2$, with eigenvalue exponent

$$1/\nu = 2 - \tfrac{1}{3}\epsilon + O(\epsilon^2) \ , \tag{4.5}$$

and the eigenvector $\Delta r_1 \equiv 0$, with eigenvalue exponent $\lambda_2 = 2 + O(\epsilon^2)$.

The exponent ν, given in (4.5), is the *correlation-length critical exponent* at the multicritical point. Other exponents can now be derived from scaling relations. It is interesting to note that at order ϵ, these are given by exactly the same expressions as found for the Ising model ($n=1$) in a random field.[18] This is simply a result from the fact that the recursion relations for r_1 and for $u_{11}\lambda$ in our case coincide with those of the Ising case. The interesting new result is, that the *transverse* spin components, $S^2, \ldots, S^n$, also have *Ising-like exponents* at the multicritical point. T is may, of course, change at higher order in ϵ.

The second exponent, λ_2, represents the crossover from the "isotropic" behavior at the multicritical point to anisotropic lower-symmetry behavior. The appropriate crossover exponent is (to order ϵ)

$$\phi = \lambda_2 \nu \simeq 2\nu \simeq \gamma \quad . \tag{4.6}$$

We shall return to the resulting anisotropic behavior in Sec. V. We shall only mention here, that the fact that $\gamma > 1$ means that the two critical lines meet at the multicritical point *tangential* to each other,[9-12] so that the mean-field phase diagram in Fig. 1 must be slightly modified near this point (see Fig. 3).

B. Phase diagram

At the multicritical fixed point, all $u_{\alpha\beta}$'s are equal to zero. However, one cannot ignore the way in which they approach their zero value.[25] Near the fixed point (4.3) u_{12} and u_{11} decay to zero as $1/\lambda$, or exactly as[19,20] b^{-2}. However, u_{22} decays to zero more slowly, i.e., as $b^{-2+\epsilon}$ [see Eq. (3.11)]. Therefore, sufficiently close to the multicritical point one has

$$u_{12}^2 < u_{11}u_{22} \quad , \tag{4.7}$$

which is the condition for a *tetracritical* point![12,13] We therefore have here a very interesting situation: the initial values of $u_{\alpha\beta}$ as given by mean-field theory [Eq. (2.11)] or by the discussion which led to Eq. (3.5), have $u_{12}^{(0)2} \gtrsim u_{11}^{(0)}u_{22}^{(0)}$, or a *bicritical* point. This means, that if the fluctuations are negligible, i.e., we are far away from the multicritical temperature, we should observe a phase diagram of the form exhibited schematically in Fig. 1. However, as we approach closer to criticality, the effective renormalized values of $u_{\alpha\beta}$ change, and sufficiently close to the multicritical point the nature of the phase diagram seems to change. It is a little dangerous to draw conclusive statements about the nature of the phase diagram in the *ordered* phases from our recursion relations, which were set up only for the *disordered* phase. One should really follow the analysis of Ref. 12 or 26, for the ordered phases. However, experience in these references shows, that the recursion relations for $u_{\alpha\beta}$ are not altered in the ordered phases. This leads us to the conjecture, that *sufficiently close to the multicritical point the first-order spin-flop line will probably split up into two second-order lines, enclosing a region in which an intermediate "mixed" phase should be observed.* This situation is schematically drawn in Fig. 3. The lower end of this intermediate phase region, where its second order boundaries meet the flop line (at a bicritical point?), is highly speculative. At this end, the fluctuations related to the multicritical point are not very strong, and (4.7) is probably not yet fulfilled. Further details must await a full analysis in the ordered phases.

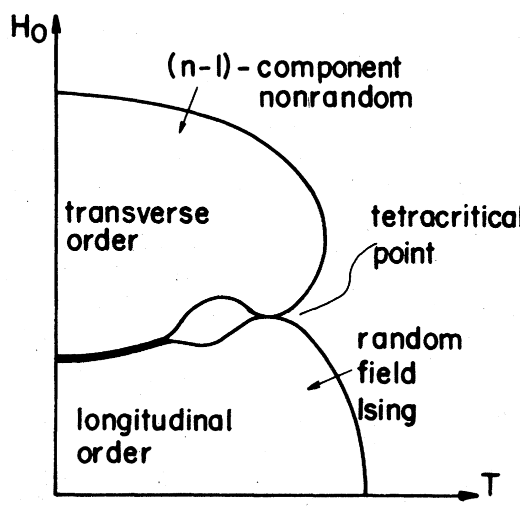

FIG. 3. Conjectured schematic phase diagram for the uniaxial random-field model. H_0 represents $\lambda^{1/2}$, with λ defined by Eq. (3.6). The area between the "transverse" and "longitudinal" phases represents the conjectured intermediate "mixed" phase.

The situation here is different from that discussed in Ref. 12, where only the case $u_{11} \equiv u_{22}$ was considered in any detail. For that case, the nature of the multicritical point (i.e., bicritical or tetracritical) is determined by the *initial* values of $u_{11}^{(0)} = u_{22}^{(0)}$ and $u_{12}^{(0)}$, and is unaffected by the renormalization-group iterations. It turns out, that this is no longer the case when $u_{11}^{(0)}$ and $u_{22}^{(0)}$ are sufficiently different from each other. Some changes in the nature of the phase diagram close to multicritical points were recently studied by Lyuksyutov *et al.*[27] However, the situation we have here is completely new, and was not anticipated by them.

V. ANISOTROPIC CRITICAL BEHAVIOR

We next wish to study the anisotropic behavior, to which one crosses over when r_1 and r_2 do not satisfy the conditions for bicriticality. If we try to keep the choice (3.7a), then clearly $r_1 \rightarrow \infty$ if $r_1^{(0)} > r_2^{(0)}$ and $r_2 \rightarrow \infty$ if $r_2^{(0)} > r_1^{(0)}$. Thus, fluctuations in the appropriate spin components become negligible ($q^2 << r_\alpha$). It is more convenient therefore to use the choice (3.7b).[24]

A. Longitudinal uniaxial ordering

We start with $r_2^{(0)} > r_1^{(0)}$. Choosing ζ_2 by (3.7b) and ζ_1 by (3.7a), we have $r_2' \equiv r_2$. For $4 < d < 6$, the coefficient of $q^2\sigma_{\vec{q}}^{\alpha}\sigma_{-\vec{q}}^{\alpha}$ with $\alpha > 1$, becomes irrelevant. Similarly, $u_{12}\lambda$ and u_{22} are irrelevant, and decay to zero. We are thus left with the variables $u_{11}\lambda$ and r_1, which obey the same recursion relations as they would for the Ising case, $n = 1$. We can now use the results of Refs. 18–21, and conclude that the longitudinal spin ordering has the same critical behavior as does the pure Ising model in $d-2$ dimensions.[20,21]

B. Transverse $(n-1)$ ordering

We now turn to the opposite anisotropy, $r_1^{(0)} > r_2^{(0)}$, or to "transverse" ordering. With the choices (3.7b) for ζ_1, and (3.7a) for ζ_2, the coefficient of $q^2\sigma^1_{\vec{q}}\sigma^1_{-\vec{q}}$ becomes irrelevant, and $r_1' \equiv r_1$. Similarly, $u_{11}\lambda$ and $u_{12}\lambda$ decay to zero for all $d>2$, and $\lambda' \simeq \lambda$. The recursion relation for u_{22} approaches, after a few iterations, the form

$$u_{22}' = b^{4-d}[u_{22} - 4K_d(n+7)u_{22}^2 A_2(r_2) + \cdots] \ , \qquad (5.1)$$

which is just the same as one would find for the usual $(n-1)$-component spin problem *without* a random field. This means mean field results for $d>4$, and the usual ϵ-expansion results[15–17] for $d \lesssim 4$. We thus see, that a *random-ordering field does not affect the critical behavior of the spin components which are perpendicular to its direction.*

C. Intermediate summary

We are now at a point where an overview summary of the results may help. Until this point, we have concentrated on the model (1.3), or (3.5), i.e., on the case in which the random-ordering field is uniaxial, and is directed along the axis of uniaxial anisotropy. The resulting phase diagram is shown in Fig. 1 (or in Fig. 3). For random fields which are weak (compared to the anisotropy), the ordering remains uniaxial, but the critical exponents change very significantly, as they would for the Ising model in a random field. For strong random fields (or weak anisotropy), the ordering is perpendicular to the field, and the exponents are the same as for a nonrandom $(n-1)$-component problem. Note that for sufficiently strong fields, e.g., $H_0 > cJ^2$ in the model discussed in Sec. II, there is no ferromagnetic ordering at all. The transition temperature into the transverse $(n-1)$-components phase decreases to zero at some finite value of H_0. Near this temperature, quantum effects may become important.[28]

One should especially note the case of isotropic exchange, when initially $r_\alpha \equiv r$ and $u_{\alpha\beta} \equiv u$. Once any small uniaxial random field is introduced, e.g., in the 1 direction, then the first iteration will yield $r_1' > r_2'$ [see Eqs. (3.8) and (3.9)], and the flow will go to the $(n-1)$-components nonrandom fixed point. The multicritical point is then on the temperature axis, and is described by the nonrandom n-component fixed point. Since $T_c(\lambda) - T_c(0) \sim \lambda^{1/\phi}$, with $\phi \equiv \gamma \gtrsim 1$, the critical line in the $T-\lambda^{1/2}$ plane meets the temperature axis at a right angle.[10]

VI. GENERALIZATIONS

The results described here are easily generalized.

A. Non-Gaussian distributions

First, we mention the Gaussian distribution of the field, assumed in Eqs. (3.4) or (3.6). For a general distribution, we shall have to consider also higher-order cumulants of the field distribution, e.g., $\{[(H_i^1)^4]_{\rm av} - 3[(H_i^1)^2]^2_{\rm av}\}$, etc. Similarly, one can consider higher-order random coefficients in Eq. (3.5), e.g., terms like

$$\int_{\vec{q}_1} \cdots \int_{\vec{q}_k} \sum_{\alpha_i} a^{\alpha_1\alpha_2\cdots\alpha_k}_{\vec{q}_1+\cdots+\vec{q}_k} \sigma^{\alpha_1}_{-\vec{q}_1} \cdots \sigma^{\alpha_k}_{-\vec{q}_k} \ , \qquad (6.1)$$

where $a^{\alpha_1\cdots\alpha_k}_{\vec{q}}$ is random, and one has all the cumulants of all these coefficients as variables. Such terms will arise if we start with the Hamiltonian (1.3), and transform it into a continuous spin model as done in Ref. 22. The analysis of this general case follows exactly the same lines as for the isotropic n-component case, treated in Ref. 1. The final conclusion is that none of these additional variables is relevant for $4<d<6$, and therefore all our results are applicable to the general case. Of course, we cannot give quantitative predictions in this context about $d=3$.

B. Nonuniaxial random field

The results will be significantly altered if the random field is not completely uniaxial, i.e., if Eq. (3.4) is generalized into

$$(H_i^\alpha H_j^\beta)_{\rm av} = \tilde{\lambda}_\alpha \delta_{ij}\delta_{\alpha\beta} \ , \qquad (6.2)$$

allowing for random-field components both along and perpendicular to the uniaxial anisotropy axis. Such terms will introduce new two line vertices, i.e., $\lambda_2 \cdots \lambda_n$, which will introduce terms of order $u_{\alpha 2}\lambda_2 A_2(r_2)$ into Eqs. (3.8) and (3.9), terms of order $u_{\alpha 2}^2\lambda_2 A_3(r_2)$ into Eqs. (3.10) and (3.11), etc. An immediate result of this is, that the appropriate variables near $d=6$ now become $u_{\alpha\beta}(\lambda_\alpha\lambda_\beta)^{1/2}$, with an "isotropic" bicritical point at

$$4K_d u_{\alpha\beta}(\lambda_\alpha\lambda_\beta)^{1/2} = \epsilon/(n+8) + O(\epsilon^2) \ . \qquad (6.3)$$

The behavior near this bicritical point, for $n \lesssim 3$,[17] will be the same as near the usual bicritical point at $d-2$ dimensions.[9–12] Moreover, if one is not exactly at this bicritical point then one crosses over to random-field Ising-like behavior (as discussed above) or to *random-field $(n-1)$-component behavior.* This latter behavior is very different from the one described in Sec. V. In particular, it leads to no ferromagnetic long-range order for $n-1 \geqslant 2$ and $d<4$.[18]

Hence, *any transverse random-field component, for the Heisenberg case $n=3$, will completely eliminate the transverse ordered phase (Fig. 1) at $d=3$.* There is therefore an extremely important difference between the *uniaxial* random field and the *general* one. Note that this difference also occurs for the isotropic exchange case: *A general random field, for $n \geq 2$, completely eliminates the ferromagnetic long-range order at $d=3$), whereas a uniaxial random field yields transverse ordering.* We emphasize again, that the spin-glass models[2–5] fall into the second category, and therefore are results apply to these models at $d=3$.

C. Skew random field

An intermediate situation seems to occur for a skew random field, i.e., one which is uniaxial, but not exactly parallel to the axis of uniaxial anisotropy. This would be relevant to the spin-glass case, if the magnetic field is not exactly aligned. If the field is in the 1-2 plane, at an angle θ to the 1 axis, then we have

$$(H_i^\alpha H_i^\beta)_{\rm av} = \begin{cases} \tilde{\lambda}\cos^2\theta\ , & \alpha=\beta=1\ , \\ \tilde{\lambda}\sin\theta\cos\theta\ , & \alpha=1,\ \beta=2 \text{ or } \alpha=2\ ,\ \beta=1\ , \\ \tilde{\lambda}\sin^2\theta\ , & \alpha=\beta=2\ . \end{cases} \tag{6.4}$$

or $\tilde{\lambda}_\alpha$ [Eq. (6.2)].

In the mean-field approximation, discussed in Sec. II, this will lead to additional terms, like M^1M^2, $M^1(M^2)^3$, etc., in Eq. (2.9). Diagonalizing the resulting bilinear form, we find that some linear combination of M^1 and M^2 will become critical. A similar diagonalization in the renormalization group analysis yields a random field Ising-like critical behavior for this rotated order parameter. The situation is thus quite similar to that of the antiferromagnet in a skew uniform field[10,11]: There is an Ising-like critical surface, in the H_{0_1}-H_{0_2}-T space, and its cut through the $H_{0_2}=0$ plane is shown in Fig. 1 (or Fig. 3).

We thus see, that most of the properties of an antiferromagnet in a uniform field can be also expected for the magnet in a random field. In particular, we can also introduce a nonrandom ordering field $\vec{H}_u$ (this is not realizable for the spin glass), to study the phase diagram in the $\vec{H}_u$-$\vec{H}_0$-T space.[10] Similarly, one can identify the scaling axes in the H_0-T plane,[29] etc.

VII. DISCUSSION

We first summarize our conclusions. The phase diagram of the uniaxial random-field model is summarized in Figs. 1 and 3, which speak for themselves. The phase diagram will become quite different if the random field has more than one component, and one should always be careful in determining which of the two pictures applies to a given situation.

As reviewed in Ref. 1 or 18, there are many experimental situations to which the random-field model applies, e.g., for ferroelectric or structural displacive transitions, etc. In most of these, one probably has more than one random field components, and therefore the phase diagram in anisotropic cases should be described in Sec. VI B. An experimental variation of the exchange anisotropy, e.g., by uniaxial stress[30,12] for frozen random fields, should yield interesting new effects at the multicritical point. The concentration of the impurities which are responsible for the random fields may serve as an experimental handle over H_0, or $\tilde{\lambda}_\alpha$ [Eq. (6.2)].

The uniaxial random-field model is directly applicable to the random-site spin glasses. For anisotropic spin glasses, one expects a phase diagram of the form shown in Fig. 3, with H_0 being the external magnetic field. The "longitudinal" phase represents an Ising-like spin glass, and the critical properties near the transition line (e.g. the magnetic susceptibility, related to derivatives of $(F)_{\rm av}$ with respect to H_0, which has a cusp) are as described in Ref. 3. Note that a finite magnetic field does not "smear" the spin-glass transition in the Ising case. Experiments on anisotropic spin glasses, to check our predicted phase diagram, will be quite helpful.

It should be emphasized again, that the whole discussion here is limited to "*random site,*" and not to "*random bond,*" spin glasses.[6] The effects of local anisotropy on the longitudinal ordering of the latter was recently studied using mean-field theory by Ghatak.[31] The effects of a magnetic field on the (isotropic) transition, within these models, seem to smear the cusp in the susceptibility.[6,32] There seems to exist serious differences between the effects of a field in the two models, which must be studied in the future. Some properties of the multicritical point in the "random-bond" spin glass, due to anisotropic exchange coefficients, will be reported elsewhere.[33] The general question, of determining which spin glass model should be used for a given system, also remains open.[34]

Finally, we note that much of our analysis was carried out for $d>4$. Other techniques, e.g., Monte Carlo calculations, series expansions, real-space renormalization group, etc., should be applied at $d=3$ for better quantitative results. We believe, however, that our qualitative results should apply for $d<4$ as well.

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