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Tricritical points in systems with random fields*

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Mean-field theory and renormalization-group arguments are used to show that the phase transition in a system with a random ordering field becomes first order at sufficiently low transition temperature, provided the (symmetric) random-field distribution function has a minimum at zero field. The first-order region is separated from the second-order region by a tricritical point. Both the critical and the tricritical exponents at $d>4$ dimensions are shown to be the same as for the pure system at $d-2$ dimensions. The relevance to spin glasses and other systems is discussed. The new tricritical point is very different from all previously studied tricritical points, as it deviates from mean-field theory at $d=5$, and not at $d=3$. Although quantitative results are calculated only at $d=5-\epsilon$ dimensions, the qualitative results are expected to apply at $d=3$.

I. INTRODUCTION

The effects of a random quenched magnetic field (or, in general, field conjugate to the order parameter) on the critical behavior near a ferromagnetic (general) phase transition has recently attracted some attention.[1–5] To discuss these effects, one considers a lattice system with the Hamiltonian

$$\mathcal{H}=-\frac{1}{2}\sum_{i\neq j}J_{ij}\vec{S}_i\cdot\vec{S}_j-\sum_i\vec{H}_i\cdot\vec{S}_i \quad , \tag{1.1}$$

where $\vec{S}_i$ is an n-component classical spin vector of unit length, located at the lattice site i, J_{ij} is the exchange coupling between sites i and j, and $\vec{H}_i$ is a local field. One is especially concerned with fields $\vec{H}_i$ which have a random distribution, characterized by a distribution function $P\{\vec{H}_i\}$, in the limit when the fields are *quenched*, i.e., they vary on a much slower time scale compared to the spin degrees of freedom.

Although it is not easy to imagine an experimental realization of such random fields in the magnetic case (local magnetic moments?), it is quite possible to have them in other cases. Examples are random electric fields in ferroelectric or in charge-density-wave transitions, strain fields due to impurities which couple linearly to the order parameter in displacive transitions,[6,7] or random sources and sinks of superfluid particles in the ideal Bose-gas case.[1] For convenience, we shall use the "magnetic" language through this paper.

A very important realization of the model of a random quenched field is related to *random-site spin glasses*.[8–12] In these, the order parameter is the projection of the local spin on the direction in which it freezes at zero temperature, and the uniform magnetic field plays the role of a random field, coupled linearly to this order parameter. It has recently been shown by Binder,[13] that many spin glasses can be described by this model, if one considers them as assemblies of magnetic clusters.

In order to obtain the thermodynamic properties of the model, one must first calculate all thermodynamic quantities for a given random distribution of fields $\{\vec{H}_i\}$ and then average the results with the distribution $P\{\vec{H}_i\}$. For example, the magnetization per spin will be given by[1–5]

$$\begin{aligned}\vec{M}&=(\langle M\{H_i\}\rangle)_{av}\\&\equiv\int d^{nN}H_iP\{\vec{H}_i\}\langle\vec{M}\{\vec{H}_i\}\rangle\\&\equiv\frac{1}{N}\int d^{nN}H_iP\{H_i\}\\&\quad\times\left[\mathrm{Tr}\left(\sum_i\vec{S}_ie^{-\beta\mathcal{H}}\right)\Big/\mathrm{Tr}e^{-\beta\mathcal{H}}\right] \quad .\end{aligned} \tag{1.2}$$

Here, the angular brackets denote thermal averaging, with all $\vec{H}_i$'s given, and $(\ )_{av}$ denotes averaging over the random-field distribution. As usual, $\beta=1/k_BT$, and N is the number of lattice sites.

As in previous work,[1–5] we shall concentrate here on distributions with no correlations among the random fields, i.e.,

$$P\{\vec{H}_i\}=\prod_i p(\vec{H}_i) \quad . \tag{1.3}$$

(Short-range correlations turn out to be irrelevant.) However, we wish to draw attention to the importance of the properties of the *local-field distribution function* $p(\vec{H})$. Previous work[1–5] concentrated only on Gaussian distributions, i.e.,

$$p_G(\vec{H})=(2\pi\lambda)^{-n/2}\exp(-|\vec{H}|^2/2\lambda) \quad . \tag{1.4}$$

Renormalization-group studies[2] for small λ showed that λ is a highly relevant variable near the nonrandom critical point $\lambda=0$. Its crossover exponent is equal to the nonrandom susceptibility exponent, γ.

For any finite λ, one expects a crossover to a new type of critical behavior. At dimensions $d>4$, this new behavior is directly related to that of the pure system in $d-2$ dimensions.[4] For $n \geqslant 2$, there is no ferromagnetic long-range order at $d<4$. For $n=1$, the behavior at $2<d<4$ is expected to be very different from that of the pure system,[4,11] and there is no long-range order at $d<2$.[2]

Hamiltonian (1.1) with Gaussian distribution (1.4) was recently studied, using mean-field theory for $n=1$ by Schneider and Pytte.[5] As might be expected intuitively, they find that as λ increases the critical temperature for ferromagnetic ordering T_c decreases, until it reaches $T_c=0$ for $\lambda^{1/2}/cJ=(2/\pi)^{1/2}$ (c is the coordination number). The transition for *all* values of λ remains second order. Schneider and Pytte also emphasize the role of the spin glass order parameter in this decrease in T_c: Above T_c, one has an "independent spin phase," in which each spin independently orders parallel to its local random field. The ferromagnetic ordering at T_c then has to compete with this "spin-glass" ordering.

One should note the similarity between the phase diagram (in the $T-\lambda^{1/2}$ plane) found by Schneider and Pytte[5] and that of an antiferromagnet in a uniform magnetic field (a "metamagnet"). In the antiferromagnet, above the transition point, the system behaves paramagnetically, with the spins ordering parallel to the field. This leads to a decrease in the antiferromagnetic transition temperature with increasing field: The antiferromagnetic order parameter (staggered magnetization) has to compete with this paramagnetic order. The analogies paramagnetic $\leftrightarrow$ independent (spin glass), antiferromagnetic $\leftrightarrow$ ferromagnetic, uniform field $\leftrightarrow$ random field are thus quite appealing. The analogy becomes even more obvious when we replace the ferromagnetic phase by the analogous spin-glass phase,[11] and the "independent" phase by the corresponding paramagnetic phase. In both the spin-glass and the antiferromagnetic ordered phases the spins point both parallel and antiparallel to the uniform magnetic field. The only difference is, that the antiferromagnet has a finite repeating unit cell, which becomes infinite for the spin glass.

The temperature-uniform field phase diagrams of metamagnets exhibit several interesting multicritical points.[14] These involve a *tricritical point,* at which the paramagnetic to antiferromagnetic transition becomes first order,[15,16] and a *bicritical point,* at which a spin flopped phase, with ordering perpendicular to the uniform field, appears.[17–19] In this paper and in the subsequent one,[20] we show that *such multicritical points may also appear for the magnet in a random field,* depending on details of the distribution function $p(\vec{H})$.

This paper is devoted to the discussion of *tricritical* points. We show, that *whenever the distribution function $p(\vec{H})$ has a minimum at zero field one should expect a tricritical point and a first-order transition* for sufficiently low transition temperatures T_c. Schneider and Pytte[5] did not find a tricritical point, because the Gaussian distribution (1.4) is maximal around the origin. In fact, Binder[13] finds, that the effective random-field distribution in spin glasses is not Gaussian, but rather has a maximum at finite $|\vec{H}|$, and a minimum at $\vec{H}=0$. One should therefore expect to find a tricritical point for spin glass transitions. In the following paper[20] we show, that a *spatial anisotropy* in the distribution $p(\vec{H})$ may lead to spin flop transitions and to *bicritical* points.

To exhibit the existence of a tricritical point for non-Gaussian field distributions, we first replace the distribution (1.4), for the Ising case $n=1$, by

$$p_\delta(H)=\tfrac{1}{2}[\delta(H-H_0)+\delta(H+H_0)] \quad . \tag{1.5}$$

This distribution represents a special case of the random-site spin-glass model in a uniform magnetic field H_0, as discussed in Refs. 8–12. In particular, its mean-field solution can be directly read out of Ref. 9. Instead, we simply follow the analysis of Ref. 5, replacing (1.4) by (1.5), in Sec. II. The resulting phase diagram is shown in Fig. 1, exhibiting a tricritical point and a first-order transition for sufficiently large values of H_0. This discussion is then generalized to other distributions $p(H)$. The model is then transformed into a continuous spin model in Sec. III, in preparation for the renormalization-group analysis of its critical and tricritical properties in Secs. IV and V. We

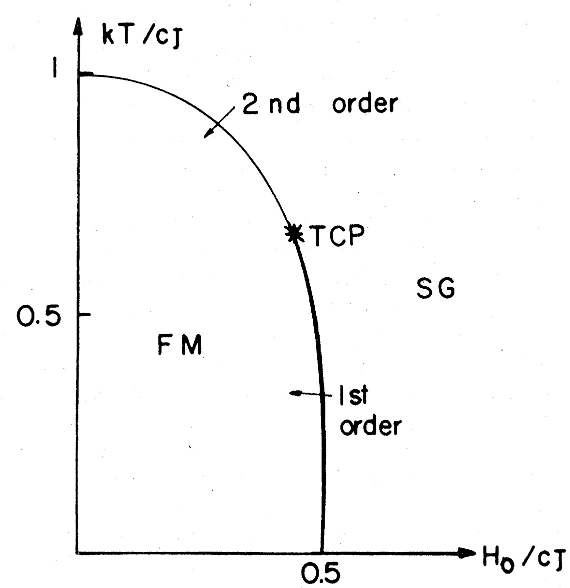

FIG. 1. Phase diagram for the Ising case, with the delta distribution, Eq. (1.5). FM is the ferromagnetic phase, and SG is the "spin glass," or "independent" phase. TCP is the tricritical point.

find that the critical behavior in $6-\epsilon$ dimensions is the same as that of the "pure" system in $4-\epsilon$ dimensions, and that the tricritical behavior in $5-\epsilon$ dimensions is the same as that of the "pure" system in $3-\epsilon$ dimensions. The case $n \geq 2$ is then separately discussed in Sec. VI, and the results are summarized in Sec. VII.

II. MEAN-FIELD THEORY

For convenience, we now concentrate on discussing the spin-$\frac{1}{2}$ Ising case $n=1$. We shall return to the general case in Sec. VI. Following Schneider and Pytte,[5] mean-field theory is obtained by replacing Hamiltonian (1.1) by

$$\mathcal{H}_0 = \tfrac{1}{2} NcJM^2 - \sum_i (cJM + H_i) S_i \quad , \tag{2.1}$$

where c is the coordination number (the number of spins which couple to a given spin with interaction J). In the notation of Ref. 5, $c=N$ and our J must be replaced by $2J/N$. The free energy per spin is thus ($S_i = \pm 1$)

$$\bar{F} = \tfrac{1}{2} cJM^2 - (1/\beta) \times \{\ln[2\cosh\beta(cJM + H_i)]\}_{\text{av}} \quad , \tag{2.2}$$

and the magnetization M is the solution of the equation

$$M = [\tanh\beta(cJM + H_i)]_{\text{av}} \quad , \tag{2.3}$$

which minimizes (2.2).

For any symmetric distribution of fields $p(H) = p(-H)$, it is clear that $M=0$ is always a solution of (2.3). If the distribution is nonsymmetric, $M=0$ is never a solution, unless we add a uniform field H_u such that

$$\{\tanh[\beta(H_u + H_i)]\}_{\text{av}} = 0 \quad . \tag{2.4}$$

From now on we shall ignore this possibility, and concentrate on symmetric distributions.

The solution $M=0$ will have the lowest free energy for sufficiently high temperatures and random fields. As the temperature is lowered, one might find an additional solution $M \neq 0$, with lower free energy. If the transition is second order, we can find the transition point by expanding (2.3) around $M=0$,

$$M \simeq AM - BM^3 - CM^5 - \cdots \quad , \tag{2.5}$$

with

$$A = \beta cJ[1 - (t_i^2)_{\text{av}}] \quad , \tag{2.6}$$

$$B = \tfrac{1}{3}(\beta cJ)^3[(1-t_i^2)(1-3t_i^2)]_{\text{av}} \quad , \tag{2.7}$$

etc., where $t_i = \tanh\beta H_i$.

Indeed, a second-order transition is found at $A=1$, *provided* that $B>0$. Sufficiently close to the "pure" transition, the "spin-glass" order parameters

$$q = (\langle S_i \rangle^2)_{\text{av}} = [\tanh^2\beta(cJM + H_i)]_{\text{av}} \tag{2.8a}$$

or

$$\tau = (|\langle S_i \rangle|)_{\text{av}} = [|\tanh\beta(cJM + H_i)|]_{\text{av}} \tag{2.8b}$$

will not be very large. Similarly, $(t_i^4)_{\text{av}}$ will also be small compared to unity. Thus, the transition will occur close to the "pure" transition point $\beta cJ = 1$, and will be second order. For the Gaussian distribution (1.4), one always has $B>0$ when $A=1$. Therefore, the transition is always second order.

The δ distribution (1.5) yields different results: the condition $A=1$ now simply reduces to

$$A = \beta cJ(1 - \tanh^2\beta H_0) = 1 \quad , \tag{2.9}$$

while Eq. (2.7) becomes

$$B = \tfrac{1}{3} A(\beta cJ)^3(1 - 3\tanh^2\beta H_0) \quad . \tag{2.10}$$

Thus, B becomes negative when $\tanh^2\beta H_0 > \frac{1}{3}$. For higher values of βH_0, one can no longer use the expansion (2.5) to find the nonzero solution for M. A direct numerical solution of Eq. (2.3), however, shows that in this range the transition becomes first order. The resulting phase diagram is shown in Fig. 1. The point

$$(\beta cJ)_t = \tfrac{3}{2}, \quad \tanh^2(\beta H_0)_t = \tfrac{1}{3} \quad (n=1) \tag{2.11}$$

is thus a *tricritical point.*

For more-general distributions, one can no longer separate the product in (2.7), to obtain (2.10). Instead, one must calculate explicitly the averages in Eqs. (2.6) and (2.7). It is instructive to consider these averages for large β (low temperature), in the case of a symmetric distribution $p(H)$ which is an analytic function of H. For any even function $Q(\beta H)$, one has

$$[Q(\beta H)]_{\text{av}} = \int_{-\infty}^{\infty} dH\, p(H) Q(\beta H) = \frac{2}{\beta}\int_0^\infty dx\, p\left(\frac{x}{\beta}\right) Q(x) \quad . \tag{2.12}$$

Expanding $p(x/\beta)$ near $(x/\beta) \to 0$,

$$p(x/\beta) = p(0) + \tfrac{1}{2} p''(0)(x/\beta)^2 + \cdots \quad , \tag{2.13}$$

Eq. (2.12) becomes an expansion in inverse powers of β,

$$[Q(\beta H)]_{\text{av}} = \frac{2}{\beta} p(0) \int_0^\infty dx\, Q(x) + \frac{1}{\beta^3} p''(0) \int_0^\infty dx\, x^2 Q(x) + \cdots \quad . \tag{2.14}$$

In particular, Eqs. (2.6) and (2.7) yield

$$A = 2cJp(0) + O(\beta^{-2}) \ ,$$
$$B = -\tfrac{1}{3}(cJ)^3 p''(0) + O(\beta^{-2}) \ . \qquad (2.15)$$

For the Gaussian case, Eq. (1.4), $p_G(0) = (2\pi\lambda)^{-1/2}$, leading to the zero-temperature Schneider-Pytte transition condition $\lambda^{1/2}/cJ = (2/\pi)^{1/2}$, and $p_G''(0) = -(2/\pi\lambda)^{1/2}$, i.e., $B > 0$, even at the zero-temperature transition. The transition at zero temperature will become first order once $p''(0) > 0$. Note that the δ distribution (1.5) is the limit

$$p_\delta(H) = \lim_{\sigma \to 0} (2\pi\sigma)^{-1/2} \exp\left[\frac{-(H^2 - H_0^2)^2}{2\sigma}\right] \ , \qquad (2.16)$$

and indeed $p''(0) > 0$ at any finite σ. If the transition at zero temperature is first order, it is clear that B must change sign as a function of temperature, and a tricritical point results.

It is interesting to note, that the random-field distribution found by Binder[13] for the Ising spin glass indeed seems to have a minimum at zero field, rather than the Gaussian maximum. Thus, one should expect a tricritical point for these spin glasses.

III. CONTINUOUS-SPIN MODEL

In order to perform a renormalization-group, or a diagrammatic analysis of the Hamiltonian (1.1), it is convenient first to transform it into a continuous spin model.[21] Again, we concentrate first on the spin-$\frac{1}{2}$ Ising case. A convenient way to do this transformation is to use the identity[22]

$$\exp\left[\tfrac{1}{2}\beta \sum_{ij} J_{ij} S_i S_j\right] = \pi^{-N/2} D^{-1/2} \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\sigma_i \exp\left[-\frac{1}{2} \sum_{ij} (K^{-1})_{ij} \sigma_i \sigma_j + \sum_i \sigma_i S_i\right] \ , \qquad (3.1)$$

where D is the determinant and K^{-1} is the inverse of the $N \times N$ matrix $K_{ij} \equiv \beta J_{ij}$. Substituting (3.1) into the partition function $Z = \mathrm{Tr}\exp(-\beta\mathcal{H})$, and performing the simple traces over $S_i = \pm 1$, we thus find

$$Z = \pi^{-N/2} D^{-1/2} \int_{-\infty}^{\infty} \prod_i d\sigma_i \exp(\bar{\mathcal{H}}) \ , \qquad (3.2)$$

with

$$\bar{\mathcal{H}} = -\frac{1}{2} \sum_{ij} (K^{-1})_{ij} \sigma_i \sigma_j + \sum_i \ln[2\cosh(\beta H_i + \sigma_i)] \ . \qquad (3.3)$$

The second sum in (3.3) can now be expanded in powers of σ_i,

$$\ln[2\cosh(\beta H_i + \sigma_i)] = \ln(2\cosh\beta H_i) - \sum_{k=1}^{\infty} \tilde{a}_i^{(k)} \sigma_i^k \ , \qquad (3.4)$$

with

$$\tilde{a}_i^{(1)} = -t_i, \quad \tilde{a}_i^{(2)} = -\tfrac{1}{2}(1 - t_i^2), \quad \tilde{a}_i^{(3)} = \tfrac{1}{3} t_i (1 - t_i^2), \quad \tilde{a}_i^{(4)} = \tfrac{1}{12}(1 - t_i^2)(1 - 3t_i^2) \ ,$$
$$\tilde{a}_i^{(5)} = -\tfrac{1}{15} t_i (1 - t_i^2)(2 - 3t_i^2), \quad \tilde{a}_i^{(6)} = -\tfrac{1}{90}(1 - t_i^2)(2 - 15t_i^2 + 15t_i^4) \ , \ldots, \qquad (3.5)$$

where $t_i = \tanh(\beta H_i)$.

We now concentrate on the quadratic terms in (3.3), and define

$$\bar{\mathcal{H}}_0 = -\frac{1}{2} \sum_{ij} \{(K^{-1})_{ij} - \delta_{ij}[1 - (t_i^2)_{\mathrm{av}}]\} \sigma_i \sigma_j \ . \qquad (3.6)$$

As usual in such calculations,[21] we next Fourier transform the spin variables σ_i into $\tilde{\sigma}_{\vec{q}}$, replace the sum in the first Brillouin zone by an integral over $|\vec{q}| < \Lambda$, expand the Fourier transform of $\{(K^{-1})_{ij} - \delta_{ij}[1 - (t_i^2)_{\mathrm{av}}]\}$ in powers of $\vec{q}$ near $\vec{q} = 0$, and rescale all spin variables by a constant ζ, $\tilde{\sigma}_{\vec{q}} \to \zeta \sigma_{\vec{q}}$ so that the coefficient of $\frac{1}{2} q^2 \sigma_{\vec{q}} \sigma_{-\vec{q}}$ in $\mathcal{H}_0$ is equal to unity

$$\bar{\mathcal{H}}_0 = -\frac{1}{2} \int_{\vec{q}} (r + q^2) \sigma_{\vec{q}} \sigma_{-\vec{q}} \ . \qquad (3.7)$$

Here, $r \propto T - T_0$, where T_0 is related to the mean-field transition temperature [given, e.g., by Eq. (2.6) and $A = 1$], $\int_{\vec{q}}$ denotes $(2\pi)^{-d} \int d^d q$, $|q| < \Lambda$, and higher powers of $\vec{q}$ are ignored since they are irrelevant.[21]

Finally, our Hamiltonian becomes $\bar{\mathcal{H}} = \bar{\mathcal{H}}_0 + \bar{\mathcal{H}}_1$, where

$$\begin{aligned}\bar{\mathcal{H}}_1 &= -\sum_i \sum_{k=1}^{\infty} a_i^{(k)} \sigma_i^k \\ &= -\int_{\vec q} a_{\vec q}^{(1)} \sigma_{-\vec q} \\ &\quad -\int_{\vec q_1}\int_{\vec q_2} a_{\vec q_1}^{(2)} \sigma_{\vec q_2} \sigma_{-\vec q_1 - \vec q_2} \\ &\quad -\int_{\vec q_1}\int_{\vec q_2}\int_{\vec q_3} a_{\vec q_1}^{(3)} \\ &\qquad\qquad \times \sigma_{\vec q_2}\sigma_{\vec q_3}\sigma_{-\vec q_1-\vec q_2-\vec q_3} \\ &\quad - \cdots \;, \end{aligned} \tag{3.8}$$

with

$$a_i^{(2)} = \zeta^2 [t_i^2 - (t_i^2)_{\rm av}] \;, \tag{3.9}$$

$$a_i^{(k)} = \zeta^k \tilde{a}_i^{(k)}, \quad k \neq 2 \;, \tag{3.10}$$

and

$$a_{\vec q}^{(k)} = \sum_i a_i^{(k)} e^{i \vec q \cdot \vec r_i} \;. \tag{3.11}$$

In the following discussions, we shall always calculate thermodynamic quantities in terms of the variables $a_{\vec q}^{(k)}$, and then, at the end, average these over the distribution of random fields. Since we shall be expanding everything in powers of $\bar{\mathcal{H}}_1$, or of $a_{\vec q}^{(k)}$, the final results can always be expressed in terms of the cumulants of the variables $a_{\vec q}^{(k)}$. For symmetric distributions $p(H) = p(-H)$, we have

$$(a_{\vec q}^{(k)})_{\rm av} = \begin{cases} 0 & k = \text{odd} \;, \\ u_k \delta(\vec q) & k = \text{even} \;, \end{cases} \tag{3.12}$$

$$(a_{\vec q}^{(k)} a_{\vec p}^{(m)})_{\rm av} = \begin{cases} \lambda_{km} \delta(\vec q + \vec p) \;, \\ \quad k,m = \text{odd} \;, \\ u_k u_m \delta(\vec q)\delta(\vec p) \\ \quad + v_{km}\delta(\vec q + \vec p) \;, \\ \quad k,m = \text{even} \;, \\ 0, \;\; \text{otherwise} \;, \end{cases} \tag{3.13}$$

etc. Here,

$$\begin{aligned} u_k &= (a_i^{(k)})_{\rm av}, \quad \lambda_{km} = (a_i^{(k)} a_i^{(m)})_{\rm av} \;, \\ v_{km} &= (a_i^{(k)} a_i^{(m)})_{\rm av} - u_k u_m \;. \end{aligned} \tag{3.14}$$

One easily extends these to higher-order cumulants. Note that $u_2 \equiv 0$, from (3.9).

IV. RENORMALIZATION GROUP AT THE CRITICAL POINT

We are now in a position to study the Hamiltonian $\bar{\mathcal{H}} = \bar{\mathcal{H}}_0 + \bar{\mathcal{H}}_1$ using the renormalization-group recursion relations or diagrammatic expansions. Our natural variables for these treatments are r, or the temperature [Eq. (3.7)], and all the cumulants of the coefficients $a_{\vec q}^{(k)}$, as defined in Sec. III, i.e., λ_{km}, u_k, v_{mn}, etc. In principle, one should construct recursion relations for the full distribution function $P\{\vec H_i\}$, or $P\{a_{\vec q}^{(k)}\}$, but this distribution is fully characterized by all its cumulants.[23]

One should note that if all $a_i^{(k)}$'s are zero except $a_i^{(1)}$, the model reduces to the one considered previously in Refs. 2–4 (although the distribution is now more general). If all $a_i^{(k)}$'s are zero except $a_i^{(2)}$, the model reduces to that of a random "transition temperature" T_0.[23,24] One can also follow the same analysis for models with random "quadrupolar" coupling ($a_i^{(k)} = 0$ except for $k = 4$), etc., but it is difficult to realize a situation in which both $a_i^{(1)}$ and $a_i^{(2)}$ are zero for random systems.

We now follow Ref. 23, and construct recursion relations for all our variables: We first integrate out all variables $\sigma_{\vec q}$ with $\Lambda/b < |\vec q| < \Lambda$ in the partition function, then rescale momenta $\vec q \to b\vec q$ and spins $\sigma_{\vec q} \to \zeta \sigma_{b\vec q}$, obtain recursion relations for the new variables r' and $a_{\vec q}^{(k)\prime}$, and finally find the new cumulants of the $a_{\vec q}^{(k)}$'s by averaging over their "old" distribution. It is convenient to describe the various terms in Eq. (3.8) by diagrams, as exhibited in Fig. 2. An external line denotes a variable $\sigma_{\vec q}$, and an open circle at a vertex of k lines denotes a variable $a_{\vec q}^{(k)}$, with the sum of all momenta (including this $\vec q$) equal to zero. For $a_{\vec q}^{(4)}$, it is convenient to separate out its average u_4, which we denote by a simple point vertex, with momentum zero.

At the first stage of the renormalization-group iteration, we expand $\exp(\bar{\mathcal{H}}_0 + \bar{\mathcal{H}}_1)$ in powers of $\bar{\mathcal{H}}_1$, and integrate over spins $\sigma_{\vec q}$ with $\Lambda/b < |\vec q| < \Lambda$. After

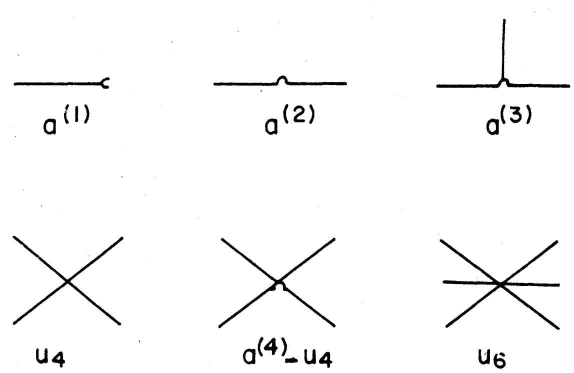

FIG. 2. Vertices describing the terms in the expansion (3.8).

rescaling momenta and spins, the new coefficients $a_{\vec{q}}^{(k)\prime}$ can be represented as sums of *connected* diagrams.[21] In these diagrams, all internal lines have momenta in the range $\Lambda/b < |\vec{q}| < \Lambda$, being integrated over. Some of the diagrams which contribute to the new coefficient of $\sigma_{\vec{q}}\sigma_{-\vec{q}}$ are exhibited in Fig. 3. For example, the fourth term here represents the contribution

$$12\zeta^2 u_4 \int_{\vec{q}_1}^{>} a_{\vec{q}_1}^{(1)}(r+q_1^2)^{-1} \times \int_{\vec{q}_2}^{>} a_{\vec{q}_2}^{(1)}(r+q_2^2)^{-1} \ , \tag{4.1}$$

where $\int_{\vec{q}}^{>}$ denotes integration over $\Lambda/b < |q| < \Lambda$. We now separate the new coefficient of $\sigma_{\vec{q}}\sigma_{-\vec{q}}$ into its average, which is obtained by averaging both sides over the field distribution, and its deviation from the average, which we denote by $a_{\vec{q}}^{(2)\prime}$. For example, the contribution of the term in Eq. (4.1) to the average will be [see (3.13)]

$$12\zeta^2 u_4 \lambda_{11} \int_{\vec{q}}^{>} (r+q^2)^{-2} \ , \tag{4.2}$$

which we represent by the first graph in Fig. 4. Here, the small empty circle represents the parameter λ_{11}, and the loop now has two propagators (the circle is simply a new kind of a two line vertex). Similarly, the fifth and sixth terms in Fig. 3 contribute the next two graphs in Fig. 4 to the average. Note that the fifth term yields v_{22}, which we denote by a full circle at a four-line vertex, while the sixth term yields λ_{13}, which we denote by an empty circle. One can now easily draw higher-order diagrams. These will yield many more kinds of new vertices. For example, the cumulant averages of $a_i^{(2)}(a_i^{(1)})^2$ and of $(a_i^{(1)})^4$ will lead to two more kinds of four line vertices, which will appear only in diagrams with more than one loop.

None of the diagrams in Fig. 4 contributes to the new coefficient of $\frac{1}{2}q^2\sigma_{\vec{q}}\sigma_{-\vec{q}}$ in the renormalized Hamiltonian. If we wish to keep this coefficient equal to unity, we thus choose[21]

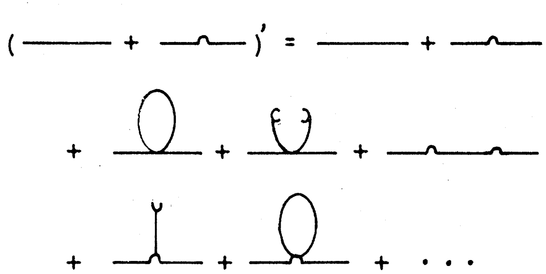

FIG. 3. Diagrams which contribute to the recursion relation for the nonaveraged coefficient of quadratic spin terms.

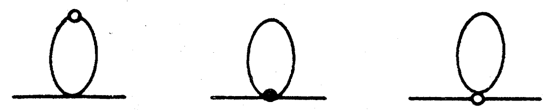

FIG. 4. Diagrams which contribute to the recursion relation for the temperature variable r. The open circle in the first diagram denotes λ_{11}, and in the last one$-\lambda_{13}$. The full circle denotes v_{22}.

$$\zeta^2 = b^{d+2-\eta} = b^{d+2}[1+O(u_4^2, u_4^2\lambda_{11}^2, u_4\lambda_{13}\lambda_{11}, \lambda_{33}\lambda_{11}, \ldots)] \ . \tag{4.3}$$

With this choice, we immediately have all the necessary recursion relations, e.g.,

$$r' = b^{2-\eta}[1+O(u_4, u_4\lambda_{11}, v_{22}, \lambda_{13})] \ , \tag{4.4}$$

$$u_4' = b^{4-d}[u_4 + O(u_4^2\lambda_{11}, u_4\lambda_{13}\, u_4 v_{22}, v_{24}, \lambda_{33}, \ldots)] \ , \tag{4.5}$$

$$\lambda_{11}' = b^{2-\eta}[\lambda_{11} + O(u_4^2\lambda_{11}^3, \ldots)] \ , \tag{4.6}$$

$$\lambda_{13}' = b^{4-d}[\lambda_{13} + O(u_4\lambda_{11}\lambda_{13}, \ldots)] \ , \tag{4.7}$$

$$v_{22}' = b^{4-d}[v_{22} + O(v_{22}^2, \ldots)] \ , \tag{4.8}$$

etc. Above four dimensions, all variables except r and λ_{11} are irrelevant, i.e., decay to zero. However, the recursion relations for these depend on products like $u_4\lambda_{11}$. Therefore, one must consider the recursion relation for this product, i.e.,

$$(u_4\lambda_{11})' = b^{6-d}[1+O(u_4^2\lambda_{11}^2, \ldots)] \ . \tag{4.9}$$

Thus, one recovers the Gaussian behavior for $d>6$, but one expects deviations from it for $d<6$. Indeed, if all the other variables are ignored, there exists a fixed point at $d=6-\epsilon$, with (for general n)

$$4K_d u_4 \lambda_{11} = \epsilon/(n+8) + O(\epsilon^2) \ , \tag{4.10}$$

where $K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(\frac{1}{2}d)$. This is the fixed point studied in earlier work,[2-4] and it clearly describes the behavior of the system when $u_4>0$, when only $a_{\vec{q}}^{(1)}$ appears in $\mathcal{H}_1$ and when only its second cumulant λ_{11} is nonzero. The new variables in our model are, for example, λ_{13}, v_{22}, etc. Clearly, these are irrelevant at $d=6-\epsilon$, since they decay like b^{-2}. All that remains to be checked is whether they are "*dangerously*" *irrelevant*,[25] as is the coefficient u_4. This coefficient is important, although "irrelevent," since it appears [e.g., in Eq. (4.4)] multiplied by the highly relevant variable λ_{11}. One can now explicitly check, that this will not happen to λ_{13} and v_{22}. If we want to replace the simple u_4 vertex in the first diagram of Fig. 4 by a λ_{13} or by a v_{22} vertex, to obtain contributions of order $\lambda_{13}\lambda_{11}$ or $v_{22}\lambda_{11}$ to r', we immediately see that such contributions must result from nonconnected graphs (at the nonaveraged stage, exhibited in Fig. 3), and therefore they must be absent. Thus, all the variables except r and $u_4\lambda_{11}$ can be completely ignored near $d=6$, and *the critical behavior of all models with random fields, with any kind of distribution, will lead to the same universal critical behavior, provided* $u_4>0$.

For any $u_4>0$, we can now consider all higher order diagrams. Keeping only those which involve powers of $u_4\lambda_{11}$ (like the first one in Fig. 4. and not the others in this figure or the third diagram in Fig. 3), these diagrams give the same results for critical exponents at $d=6-\epsilon$, to all orders in ϵ, as one would calculate for the nonrandom problem at $d=4-\epsilon$.[26] For small values of λ_{11}, Eq. (4.6) yields the crossover exponent for λ_{11}, which is equal to the nonrandom susceptibility exponent γ [only the first term in (4.6) contributes near $\lambda_{11}=0$]. Thus, any thermodynamic function will depend on λ_{11} through the scaled variable λ_{11}/t^γ, where $t=[T-T_c(\lambda_{11}=0)]/T_c$, and the line $T_c(\lambda_{11})$ will have the shape

$$T_c(\lambda_{11}=0)-T_c(\lambda_{11}) \propto \lambda_{11}^{1/\gamma} \ , \tag{4.11}$$

If one plots T_c vs $\lambda_{11}^{1/2}$, as done in Fig. 1, one thus should find that the critical line meets the line $\lambda_{11}=0$ at a right angle $(1<\gamma<2)$. For dimensions $4<d<6$, the pure system has mean field exponents, i.e., $\gamma=1$, and thus the line $T_c(\lambda_{11}^{1/2})$ starts as $(\lambda_{11}^{1/2})^2$.

One should note that the diagrammatic approach presented here works only for dimensions $4<d<6$. For $d<4$, many of the variables we ignored, e.g., λ_{13}, v_{22}, etc., become relevant, and other approaches are necessary. It is also possible that at $d=3$, the model with only $\lambda_{11}\neq 0$ and the more general model discussed here may have different types of critical behavior. One should note that if in addition to the random field one also has a *random exchange coefficient*, or random T_c, i.e., $v_{22}\neq 0$, these are *irrelevant* for $d>4$. Although our quantitative results apply only at $d=6-\epsilon$, we believe that the general qualitative features of the phase diagram are correct also at $d=3$.

V. RENORMALIZATION GROUP AT THE TRICRITICAL POINT

The discussion of Sec. IV applies only when $u_4>0$. Returning to Eqs. (3.14), (3.12), (3.10), and (3.5), we see that

$$u_4=\tfrac{1}{12}\zeta^4[(1-t_i^2)(1-3t_i^2)]_{\text{av}} \ , \tag{5.1}$$

which is very reminiscent of Eq. (2.7). Indeed, whenever mean-field theory predicts a first-order transition one should be very careful in the renormalization-group analysis.

Once $u_4\lesssim 0$, one must keep higher-order terms in the expansion (3.8). Returning to Eq. (3.5), one checks that for the examples considered above, $u_6>0$ at the point where u_4 becomes negative. We thus include in our analysis variables like u_6, v_{24}, λ_{33}, etc. The recursion relations for all these three variables have a form like

$$u_6'=b^{6-2d}[u_6+O(u_6^2\lambda_{11}^2,\dots)] \ . \tag{5.2}$$

In addition, one finds that u_6 contributes to the equation for r', Eq. (4.4), through terms like $u_6\lambda_{11}^2$, etc., whereas v_{24} and λ_{33} contribute through smaller powers of λ_{11}, e.g., $v_{24}\lambda_{11}$ and $\lambda_{33}\lambda_{11}$. Thus, we must consider the additional recursion relation for $u_6\lambda_{11}^2$, which reads

$$(u_6\lambda_{11}^2)'=b^{10-2d}[u_6\lambda_{11}^2+O(u_6^2\lambda_{11}^4,\dots)] \ , \tag{5.3}$$

and ignore all other variables.

At this stage we thus have three basic parameters, i.e., r, $u_4\lambda_{11}$, and $u_6\lambda_{11}^2$, and the situation is completely analogous to that of the pure system, where one considers r, u_4, and u_6.[27–29] One can now obtain the recusion relations for the present model from those discussed for the pure system by simply replacing d by $d+2$ everywhere. The rules for diagrams which contain u_6 are similar to those discussed in Sec. IV and in Ref. 4: In any given diagram, one must put a circle (i.e., a λ_{11} vertex) on one internal line for each loop.

From Eq. (5.3) it is clear, that $u_6\lambda_{11}^2$ is irrelevant for $d>5$. Thus, for $5<d<6$ one recovers the crossover from Gaussian tricritical behavior to the critical behavior described above, similarly to the pure case.[30] AT $d=5$ one expects logarithmic corrections to the mean-field-like tricritical behavior,[27] and at $d<5$ one can expand the tricritical exponents in powers of $\epsilon=5-d$, e.g., (for general n),[28,29]

$$\begin{aligned}\gamma&=1+\frac{5}{8}\frac{(n+2)(n+4)}{(3n+22)^2}\epsilon^2+O(\epsilon^3) \ ,\\ \eta&=\frac{1}{12}\frac{(n+2)(n+4)}{(3n+22)^2}\epsilon^2+O(\epsilon^3) \ ,\\ \alpha&=\tfrac{1}{2}+\tfrac{1}{2}\epsilon+O(\epsilon^2) \ .\end{aligned} \tag{5.4}$$

Again, it is probably unreasonable to extrapolate these results down to $d=3$. However, since for $n=1$ one expects a ferromagnetic transition at $d>2$,[2] one probably should also expect this transition to become first order whenever $u_4<0$. The exponents at this tricritical point are expected to be very different from those of the pure system discussed in Ref. 27.

One should note the difference between the tricritical point we find here and that of a metamagnet. At $d=3$, all "metamagnetic" tricritical points are universally equivalent, and exhibit mean field like behavior with logarithmic corrections at $d=3$.[16] The tricritical point generated by a random field does *not* belong to the same universality class, and its exponents deviate strongly from those predicted by mean-field theory at $d=3$.

VI. SPINS WITH MANY COMPONENTS

We now return to the general n-component spin Hamiltonian (1.1). The simple generalization of the δ distribution (1.5) for $n \geq 2$ is

$$p_\delta(\vec{H}) = \delta(|\vec{H}|^2 - H_0^2)\Gamma(\tfrac{1}{2}n)/\pi^{n/2}H_0^{n-2} \quad . \tag{6.1}$$

Here, all $\vec{H}_i$'s have one magnitude, $|\vec{H}_i| = H_0$, and all the spatial directions of $\vec{H}_i$ have the same probability. One can then also consider other generalizations of the Gaussian distribution (1.4), like the ones discussed towards the end of Sec. II.

In other cases, one may be interested in distribution of cubic symmetry, e.g., ones in which the random field may point only along one of the $2n$ cubic axes

$$p_C(\vec{H}) = \frac{1}{2n}\sum_{\alpha=1}^{n}[\delta(\vec{H} - H_0\hat{e}_\alpha) + \delta(\vec{H} + H_0\hat{e}_\alpha)] \quad , \tag{6.2}$$

where $\hat{e}_\alpha$ is a unit vector along the αth axis. Such distributions are probably relevent for systems like $SrTiO_3$, when the titanium ions are dislocated.[6,7]

For any of these distributions, we can now repeat the mean-field analysis of Sec. II. The free energy now becomes

$$\bar{F} = \tfrac{1}{2}cJ|\vec{M}|^2 - (1/\beta)[\ln g(\vec{H}_i)]_{av} \quad , \tag{6.3}$$

where

$$g(\vec{H}_i) = \int d\hat{S}_i \exp\beta(cJ\vec{M} + \vec{H}_i)\cdot\vec{S}_i = \frac{(2\pi)^{n/2}I_{n/2-1}(\beta|cJ\vec{M} + \vec{H}_i|)}{(\beta|cJ\vec{M} + \vec{H}_i|)^{n/2-1}} \quad , \tag{6.4}$$

where $I_\nu(x)$ is the Bessel function, and $\vec{M}$ is the solution of

$$\vec{M} = \left[\frac{I_{n/2}(\beta|cJ\vec{M} + \vec{H}_i|)(cJ\vec{M} + \vec{H}_i)}{I_{n/2-1}(\beta|cJ\vec{M} + \vec{H}_i|)|cj\vec{M} + \vec{H}_i|}\right]_{av} . \tag{6.5}$$

One can now solve (6.5) numerically to find the value of $\vec{M}$ which minimizes $\bar{F}$, or expand the right-hand side of (6.5) around $M = 0$ to find when a solution $\vec{M} \neq 0$ appears.

For $n = 3$, one has

$$g(\vec{H}_i) = \frac{4\pi\sinh(\beta|cJ\vec{M} + \vec{H}_i|)}{\beta|cJ\vec{M} + \vec{H}_i|} \tag{6.6a}$$

and

$$\vec{M} = \left[\left(\frac{\coth(\beta|cJ\vec{M} + \vec{H}_i|)}{\beta|cJ\vec{M} + \vec{H}_i|} - \beta^{-2}|cJ\vec{M} + \vec{H}_i|^{-2}\right) \times \beta(cJ\vec{M} + \vec{H}_i)\right]_{av} . \tag{6.6b}$$

Expanding this about $\vec{M} = 0$, for the isotropic δ distribution (6.1), yields, similarly to (2.5),

$$\vec{M} = A\,\vec{M} - BM^2\vec{M} - \cdots \quad , \tag{6.7}$$

with

$$A = \tfrac{1}{3}c\beta J[2\coth\beta H_0/\beta H_0 - (\beta H_0)^{-2} - (\sinh\beta H_0)^{-2}] \quad , \tag{6.8}$$

$$B = \tfrac{1}{15}(c\beta J)^3[5(\beta H_0\sinh\beta H_0)^{-2} + 5\coth\beta H_0(\beta H_0)^{-3} + 8\coth\beta H_0(\beta H_0\sinh^2\beta H_0)^{-1} - 17(\beta H_0)^{-4} - (\sinh\beta H_0)^{-4} - \tfrac{2}{3}(\sinh^2\beta H_0)^{-1}] \quad , \tag{6.9}$$

and a tricritical point is found at

$$(\beta cJ)_t \simeq 5.4, \quad (H_0/cJ)_t \simeq 5.5 \quad (n = 3) \quad . \tag{6.10}$$

The full phase diagram is very similar to the one exhibited in Fig. 1. If the magnitude of H_0 is also random, one can follow the same lines as done at the end of Sec. II, and find general conditions on the behavior of $p(\vec{H})$ near the origin. One should note, however, that the procedure here is somewhat more complicated than for the Ising case: Eq. (2.12) now has the form (for general n)

$$[Q(\beta H)]_{av} = 2\pi^{n/2}\Gamma\left(\frac{n}{2}\right)^{-1}\int_0^\infty H^{n-1}dH\,p(H)Q(\beta H) = 2\pi^{n/2}\Gamma\left(\frac{n}{2}\right)^{-1}\beta^{-n}\int_0^\infty x^{n-1}dx\,p\left(\frac{x}{\beta}\right)Q(x) \quad . \tag{6.11}$$

If we try to substitute for $Q(\beta H)$ the functions $A(\beta H)$ or $B(\beta H)$ of Eqs. (6.8) and (6.9), we find that $p(x/\beta)$ cannot be replaced by $p(0)$, or else the integral diverges. Thus, one must first integrate by parts. For $n = 3$, the resulting conditions will be on the properties of the function $\tilde{p}(H)$, where $p(H) = d^2\tilde{p}(H)/dH^2$.

We now turn to the renormalization-group analysis for $n \geq 2$. We can follow directly all the steps of Secs. III, IV, and V. First, we use the identity (3.1) for each spin component S_i^α, $\alpha = 1, \ldots, n$, to find the continuous spin Hamiltonian

$$\bar{\mathcal{H}} = -\frac{1}{2}\sum_{ij}(K^{-1})_{ij}\vec{\sigma}_i\cdot\vec{\sigma}_j + \sum_i \ln G_n(|\beta\vec{H}_i + \vec{\sigma}_i|) \quad , \tag{6.12}$$

where [see Eq. (6.4)]

$$G_n(x) = (2\pi)^{n/2}I_{n/2-1}(x)/x^{n/2-1} \quad . \tag{6.13}$$

We can now expand the last term in (6.12) in powers of σ_i^α,

$$\ln G_n(|\beta \vec{H}_i + \vec{\sigma}_i|) = \ln G_n(\beta|\vec{H}_i|) - \sum_{k=1}^{\infty} \tilde{a}_i^{\alpha_1 \cdots \alpha_k} \sigma_i^{\alpha_1} \cdots \sigma_i^{\alpha_k} , \quad (6.14)$$

with the coefficients $\tilde{a}_i^{\alpha_1 \cdots \alpha_k}$ now being tensors. The remaining discussion goes as for $n = 1$: Fourier transforms, cumulants of the random distribution of the coefficients $a_i^{\alpha_1 \cdots \alpha_k}$, etc. The resulting recursion relations near $d = 6$, for isotropic distributions, again lead to the fixed point (4.10), where now

$$(a_i^{\alpha} a_i^{\beta})_{\text{av}} = \lambda_{11} \delta_{\alpha\beta} \quad (6.15)$$

and

$$(a_i^{\alpha\beta\gamma\delta})_{\text{av}} = \tfrac{1}{3} u_4 (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \ . \quad (6.16)$$

The initial value of u_4 is now again proportional to the mean-field coefficient B [Eq. (6.9)], and thus a tricritical point arises when $u_4 \leqslant 0$. This fixed point will deviate from mean-field behavior at $d = 5$, with the exponents (5.4).

The situation for the cubic distribution, Eq. (6.3), is more complicated. Now, (6.16) will be replaced by

$$(a_i^{\alpha\beta\gamma\delta})_{\text{av}} = \tfrac{1}{3} u_4 (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) + v_4 \delta_{\alpha\beta}\delta_{\alpha\gamma}\delta_{\alpha\delta} \ . \quad (6.17)$$

Near $d = 6$, we now have two important parameters, i.e., $u_4\lambda_{11}$ and $v_4\lambda_{11}$. The recursion relations for these are exactly the same as those for the pure cubic system at $d - 2$ dimensions, and one expects new cubic critical exponents for $n \gtrsim 3$.[31] The tricritical exponents below $d = 5$ will also be modified.[32] More-complicated distribution functions, with lower symmetries, will be discussed in a separate paper.[20]

Before concluding this section, it is worth emphasizing again that the discussion of the critical behavior of the present model for $n \geqslant 2$, at any finite width of the random-field distribution, is somewhat academic, since there is no ferromagnetic long-range order for $n \geqslant 2$ at $d < 4$.[2] The results for $n = 1$ are believed to hold, qualitatively, also for $d < 4$.

VII. CONCLUSION

It has been demonstrated, using both mean-field theory and renormalization-group arguments, that a random-ordering field with an appropriate distribution function will lead to a tricritical point and to a first-order transition at sufficiently low temperatures. For $d > 4$, the critical and tricritical exponents are the same as those of the pure system in $d - 2$ dimensions. For $d < 4$, the system has no long-range order if $n \geqslant 2$, and therefore the experimental verification of our conclusions must be limited to the Ising case, $n = 1$.

The most promising system, for which the "random"-field distribution is of the form discussed here, is the spin glass. In this system, the magnitude of the "random"field H_0 is simply that of the external field. Thus, for an Ising-like spin glass, in a sufficiently high magnetic field, a tricritical point should be observed.

Since our explicit calculations were carried out only for $d > 4$, alternative calculation techniques should be applied for $n = 1$ at $d = 3$. Thus, Monte Carlo or high-temperature series studies of this problem would be very interesting, especially since the deviations from mean-field theory are expected to be so large. Very recent Monte Carlo calculations[33] indeed indicate the existence of a tricritical point.

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