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Quasiequilibrium statistical mechanics of two-dimensional superfluids and the two-dimensional Coulomb gas

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The virtually rigorous renormalization-group technique of Kosterlitz is applied to the vortex nucleation theory of superfluidity. The rate of decay of superfluid velocity and the superfluid density for moving helium films are obtained. The results are applicable to all temperatures on and below the critical point. Velocity exponents are directly related to the superfluid density of a film at rest. Finite-velocity effects are shown to account for pronounced changes in the dissipation and in the superfluid density near the critical point. This paper also confirms the prediction that, for a film at rest, the critical point value of ρ_s/T is a universal constant. At any nonzero velocity, however, finite-velocity effects will mask this result. The subcritical static polarizability of a two-dimensional Coulomb gas with hard-core interactions is obtained, in direct analogy to the ρ_s calculation.

I. INTRODUCTION

The phenomenon of superfluidity in extremely thin liquid ^{4}He films is one of fundamental physical interest. The usual explanation for superfluidity in the bulk involves invoking macroscopic occupation of a single quantum state. In two dimensions, however, it has been proven that there is no Bose-Einstein condensation.[1] Therefore, at least in two dimensions, it would appear that the explanation for superfluidity cannot rely on a condensate.

It should be noted that there exists, in the literature, a series of papers which attempt to circumvent the absence of Bose-Einstein condensation in the true $d=2$ thermodynamic limit.[2–4] The basic idea in this work is that the divergences which prohibit a condensate for a helium film of infinite area are marginal logarithmic divergences. For example, the author has examined the dynamics of the approach to thermal equilibrium and has shown that low-temperature ordering of the type found in the bulk can, in two-dimensional systems at sufficiently low temperatures, develop and persist for effectively infinite times before full thermal equilibrium sets in. However, given that the characteristic time for fluctuations on the microscopic scale to reach thermal equilibrium is in the range of 10^{-10}– 10^{-13} sec, then the virtual condensate predicted by this theory will occur at temperatures an order of magnitude less that the observed critical temperature. Films display superfluidity at temperatures at which even a transitory virtual condensate is ruled out.

The correct explanation for the observed superfluidity in two dimensions was developed intuitively by Kosterlitz and Thouless.[5] The Kosterlitz and Thouless paper showed that a number of systems, including Bose fluids, of order-parameter dimensionality 2 can, in two spatial dimensions, have a phase transition without the symmetry breaking that characterizes phase transitions in the bulk. As applied to helium, the basic physics in the Kosterlitz-Thouless argument is that entropy and energy are such that it is free energetically favorable to create an isolated quantum of circulation only above a nonzero temperature. In the low-temperature regime vortices of opposing vorticity will remain in bound pairs. The relevance of this description to two-dimensional superfluidity may be seen by considering a thin incompressible film of fluid initially moving over a stationary substrate. The substrate will dissipate flow by creating "eddies"—pairs of vortices which drift apart. In the low-temperature regime there is a free-energy barrier to be overcome before a pair of vortices can be pulled arbitrarily far apart. It is this that impedes low-temperature dissipation of flow and makes metastable currents possible. It should be noted that the discreteness of circulation in the quantum fluid is crucial to the Kosterlitz-Thouless picture. No superflow is possible in a classical two-dimensional fluid. This is because at nonzero temperature there will always be a small enough circulation below which it is free energetically favorable to create isolated vortices. These ideas were somewhat anticipated in earlier papers by Fisher and Langer,[6] Langer and Reppy,[7] and in still earlier work by others.[8,9]

In the present paper the decay with time of the superfluid velocity and the dependence of the superfluid density on velocity will be calculated. The technique used here is the renormalization-group formulation developed by Kosterlitz[10] to solve the critical properties of the two-dimensional X-Y model. It was derived from a technique of Anderson and Yuval.[11] The Kosterlitz work is one of the rare instances in which a nontrivial critical fixed point has been found exactly. Critical exponents and some other properties found by the Kosterlitz technique are as exact as any renormalization-group prediction can be.

The basic physics underlying the calculation of this paper is an elaboration of the intuition mentioned above. In a moving helium film, an isolated pair of vortices of opposite vorticity will remain in a metastable bound state, so long as their separation is less than the inverse of the fluid's velocity (with appropriate units). Thus the energy of a vortex-antivortex pair separated a distance y normal to the direction of flow is[7]

$$E(y) \propto \ln(y) - \text{const}\, vy \quad ,$$

which has a local maximum at $y \sim 1/v$. When thermal fluctuations move the pair apart to a separation exceeding $\sim 1/v$, it becomes energetically favorable for the pair to rapidly unbind, decreasing the fluid velocity by $2\pi/L$, where L is the macroscopic length of the system. One may obtain two bits of information from this physical intuition. As Langer and Reppy have observed, the rate at which the superfluid velocity decreases should be proportional to the number of vortex pairs on the threshold of instability $y \sim 1/v$. At low temperatures this means the rate of decrease of the superfluid velocity v is

$$\frac{dv}{dt} \propto -\exp\left[\frac{-E_0(v)}{k_B T}\right] \quad ,$$

where $E_0(v) \sim \ln(1/v)$ is the energy of a vortex pair on the instability threshold. The low-temperature theory of Langer and Reppy is an application to two dimensions of the three-dimensional work of Fisher and Langer. Note that, at sufficiently low temperatures for small enough v, the velocity is virtually constant with time. The Fisher-Langer-Reppy theory identifies the critical velocity as that value of v for which observable changes in v occur within laboratory times.

The second bit of information that may be obtained is the velocity dependence of the superfluid density. The total mass flow of the film relative to its substrate is diminished by a net backflow due to the metastable bound vortices. These will be assumed to be distributed in quasiequilibrium according to their energy as seen in the rest frame of the substrate. Since it is energetically favorable to orient a vortex pair in such a way as to minimize the local flow, the bound vortices will produce an average current—a vortex backflow—in the direction opposite to $\vec{v}$. (Remember that the vortex pairs separated by distances exceeding $\sim 1/v$ contribute to the reduction of current, but in a different way: they fly apart and cause the deterioration of v with time.) The superfluid density for a film is simply the total metastable mass flow divided by the velocity of the film relative to its substrate. Since both the distribution of bound vortices and the maximum stable separation of vortex-antivortex pairs are dependent on the velocity, it should not be surprising that the eddy backflow, and therefore ρ_s, is also velocity dependent.

In this paper the Kosterlitz technique will be used to go beyond the simple isolated pair approximation and take multiple vortex interactions into account. It is important that this be done because it permits making testable predictions at all subcritical temperatures, not simply at low temperatures. It turns out that the most pronounced effects are, in fact, near T_c. Because of the quantitative predictions that the Kosterlitz renormalization group can make, ^{4}He thin films provide a system for which the general vortex nucleation theory of superfludity can be tested quite strenuously. It is worth summarizing, in this introduction, the results of the "marriage" of renormalization-group and vortex nucleation theory that takes place in this paper. One of the interesting features is the appearance of factors of the velocity raised to a power which vanishes as the critical point is approached from below. These factors come from the number density of vortex-antivortex pairs on the instability threshold (at separation $\propto 1/v$). The intuitive explanation for the finite-velocity effects is discussed in a letter by Huberman, Myerson, and Doniach.[12] The underlying idea, however, is simply the Kosterlitz and Thouless intuition that the free energy of a vortex-antivortex pair at infinite separation vanishes as the critical point is approached from below. This means that the number density of vortex-antivortex pairs at separation $\propto 1/v$ is finite sufficiently close to T_c, no matter how small v is.

For the superfluid density ρ_s, it will be shown that

$$\rho_s = \frac{2m^2 k_B T}{h^2 2\pi}\left[2 + \frac{f(T)(1+v^{2f(T)})}{1-v^{2f(T)}} - 2A\frac{[f(T)]^2 v^{2f(T)}}{(1-v^{2f(T)})^2}\right] \quad ,$$

$$f(T) \equiv \beta^*(T) - 2 \quad , \tag{1.1}$$

where ρ_s is in units of mass over area and $A \approx 5$. In (1.1) the standard constants are m: the mass of the helium atom; k_B: Boltzman's constant; and T: the temperature. $\beta^*(T)$ is the temperature-dependent fixed-point value of the renormalized vortex-vortex coupling constant. As the critical point is approached from below

$$\beta^*(T) = 2 + C[(T_c - T)/T_c]^{1/2} \quad . \tag{1.2}$$

At low temperatures $\beta^*(T)$ reduces to the bare vortex-vortex coupling constant [the quantity β_1 of

Eq. (1.3)]. Note that (1.1) and (1.2) imply that, in two dimensions at zero velocity, the superfluid density drops discontinuously to zero at the critical point. The coefficient C in (1.2) is substrate dependent and has no critical point singularities. It must be determined experimentally. A rough estimate[10] for near the critical point (which is where C is important) is $C \approx 1$. The velocity v of the superfluid relative to its substrate is in units of the inverse of the vortex core size ($\sim\text{Å}^{-1}$):

$$v \approx v_0(m/\hbar)\times 10^{-8} \approx 10^{-4}v_0 \ , \tag{1.3}$$

where v_0 is the velocity in cm/sec. Since the vortex core size is substrate dependent, the precise coefficient multiplying velocity must be determined from a data fit. In recent work, Nelson and Kosterlitz[13] have derived an expression for ρ_s of stationary ($v=0$) films. Their definition of superfluid density is appropriate to third sound[14] experiments and is rather different from the mass flow definition, appropriate to Andronikashvili-type[9] experiments, which is used in this paper. In addition, a different renormalization group, due to Jose *et al.*,[15] was used by them. It is gratifying, then, that, in the limit of zero velocity, (1.1) agrees with the Nelson and Kosterlitz result.

An experimental consequence of (1.1)–(1.3) is that, for a superfluid flowing at any finite velocity relative to its substrate, one can always come sufficiently close to the critical point for the velocity dependent term to become important. This is because the velocity exponent $2[\beta^*(T)-2]$ goes to zero as $T\rightarrow T_c$. If the superfluid's velocity relative to its substrate is within several orders of magnitude of 1 cm/sec and if the estimated value of C is roughly correct then velocity dependent effects should become important within $\sim$0.25% of T_c. The larger v is, the broader the region of rounding off will be. Preliminary reports of recent Andronikashvili-type experiments by Bishop and Reppy[16] for helium on Mylar are quite consistent with this theory. The order of magnitude, temperature range of the broadening are encouraging. They also find the temperature range over which the drop occurs broadens when the drive strength (and hence v) is increased. The Bishop and Reppy experiments are the most direct experiments for testing near critical theories of helium films. Prior work, deducing ρ_s from third sound velocities,[14] is not inconsistent with (1.1). However enhanced dissipation makes the third sound signal difficult to detect near T_c.

It is important to bear the velocity dependent effects in mind when testing data for consistency with the Nelson and Kosterlitz elegant observation that the critical point value of ρ_s/T for a film at rest is a universal constant. The $v^{2[\beta^*(T)-2]}$ rounding off near T_c must be subtracted when matching data to this prediction.

A more subtle implication of (1.1) is the prediction that the velocity exponent is directly related to the value of ρ_s for a film at rest:

$$2[\beta^*(T)-2]=\rho_s(v=0)(\hbar^2 2\pi/m^2k_BT)-4 \ .$$

Since the Kosterlitz renormalization group is good at all temperatures below T_c, this prediction is not limited to the critical region.

The careful reader of this paper will note that there has been no mention of the backflow due to density fluctuations. In three dimensions phonons comprise a significant fraction of the normal component. In thin films, however, the phonon backflow is proportion to $(c_3)^{-4}$, where c_3 denotes the third sound velocity. Typically c_3 is on the order of 100 times the bulk sound velocity, so phonon contributions in two dimensions should be negligible. Furthermore, from the critical phenomenist's standpoint, density fluctuations in thin films are short-ranged irrelevant variables.[17,18] This means that, even if a substrate with small c_3 existed, the phonon backflow contribution to ρ_s would be a nonsingular function of temperature. It should be noted that density fluctuations are not irrelevant in three or more dimensions.

The second major result that will be developed in this paper is the time dependence of the superfluid velocity. This is found to be given by

$$\frac{dv}{dt}=-v_0 v\frac{[\beta^*(T)-2]\,v^{2\beta^*(T)}}{1-v^{2[\beta^*(T)-2]}} \ , \tag{1.4}$$

where $\beta^*(T)$ is related to ρ_s by (1.1). ν_0, the attempt frequency, is usually estimated to be 10^{10}–10^{13} sec^{-1}. The presence of the extra factor of v (which is not important at low temperature) is due to the fact that free vortices can be retrapped. The details are given in Sec. III. The velocity in (1.4) is in the units of (1.3). For a velocity v on the order of 1 mm/sec (1.4) and (1.2) imply that the decay will become substantial (time constant $\sim$ sec) within a few percent of the critical point. The enhanced dissipation near T_c is a standard feature of experiments on films.[14,16] Note that the Kosterlitz renormalization group effectively substitutes $(1-T/T_c)^{1/2}$ where one would have $(1-T/T_c)$ in an isolated vortex pair approximation. The square root is important—without it one would overestimate the regime of significant dissipation by an order of magnitude.

In very interesting work Telschow and Hallock[19] made direct observations of the time decay of superfluid velocity. Their film coverages were in the 10 Å range, so it is conceivable that non-two-dimensional effects come into play. In addition, their velocities were rather large $\gtrsim$10 cm/sec. Although the coverage dependence of their decays deviates from the Langer-Reppy low-temperature theory in a manner qualitatively consistent with (1.4), this experiment must presently be considered unexplained and worthy of future investigation.

The reader is reminded that the v in these formulas is the velocity of the superfluid relative to its substrate. Obviously this need not be the same as the laboratory velocity of the superfluid. Thus, in an Andronikashvili-type experiment, where the superfluid is essentially still and the substrate oscillates, v in this paper is proportional to the ratio of the amplitude to the period of the substrate oscillation.

In Sec. II of this paper, the details of the calculation of the basic results (1.1)–(1.3) for ρ_s will be discussed. In Sec. III the time development of v, Eq. (1.4), is derived. The calculation of Sec. II is a renormalization-group calculation. The discussion of Sec. III is of less esoteric matters and should be more accessible to nonspecialists.

The static properties of a two-dimensional Coulomb gas with a hard-core (nonzero minimum charge separation) interaction are equivalent to those of two-dimensional vortices in helium. In Sec. II this analogy is exploited to convert the expression (1.1) for ρ_s into an expression for the subcritical polarizability of the two-dimensional Coulomb gas. The analogy to (1.4) would be a relation for the electric current j due to a uniform electric displacement vector $\mathfrak{D}$. j would replace dv/dt and $\mathfrak{D}$ would replace the factors of v on the right side of (1.4). Unfortunately, there are dynamical differences between vortices and charges that make this second analogy questionable.

II. CALCULATION OF ρ_s

In this section the notation of the Kosterlitz X-Y model paper[10] will be followed. For a helium film moving at velocity v the likelihood of finding N vortices at locations $\vec{r}_1, \ldots, \vec{r}_N$ is proportional to $\exp(-H^N/k_BT)$. H^N is the energy as seen in the rest frame of the substrate[7]

$$\frac{H^N}{k_BT} = \sum_{\substack{i,j\le N\\ i\ne j}} \beta_1 p_i p_j \ln(|\vec{r}_i - \vec{r}_j|) + V_1 \sum_{i\le N} p_i y_i \ . \qquad (2.1)$$

In (2.1) p_i is proportional to the circulation of the ith vortex and

$$V_1 = 2\beta_1 v \ . \qquad (2.2)$$

The direction of flow v is taken to be in the x direction; y_i denotes the y component of the location $\vec{r}_i$ of the ith vortex. In a quantum fluid $p_i = \pm 1$ and

$$\beta_1 = \hbar^2 \rho_0 2\pi / 2m^2 k_B T \ , \qquad (2.3)$$

where ρ_0 is the average density of the helium film. For statistical mechanical purposes, non-neutral collections of vortices may be ignored—a $\ln(\Omega)$ energy must be supplied to obtain a film with a nonzero net vorticity. This means that

$$\sum_i p_i = 0 \ . \qquad (2.4)$$

There is an intrinsic minimum separation between vortices, corresponding to the vortex core size. There is no harm in making this a sharp cutoff, which, with an appropriate choice of units of distance, is unity. Thus

$$i \ne j \ , \quad |\vec{r}_i - \vec{r}_j| \ge 1 \ . \qquad (2.5)$$

The choice of units means that v is converted from velocity in cm/sec by the factor of $\sim 10^{-4}$ given in (1.3). Finally we note that the thermal average of some function f of the vortex parameters, p,r, is given by

$$\langle f \rangle = \frac{1}{Z} \sum_{n=0}^{\infty} (K_1)^{2n} \prod_{i=1}^{2n} \int d^2\vec{r}_i \sum_{p_i=\pm 1} f e^{-H^{2n}/k_BT} \ ,$$

where Z is the partition function and K_1, the vortex fugacity, depends on the core energy of a vortex. Changes in the substrate can change K_1. The integrals over the $\vec{r}_i$ and the sums over p_i must be consistent with (2.4) and (2.5).

In addition to the minimum separation of Eq. (2.5) there is a velocity dependent maximum scale of distance. This will be denoted $r_0(v)$. The maximum distance scale expresses the fact that, in a moving film, if a vortex-antivortex pair is pulled sufficiently far apart, it will then become energetically favorable for the pair to unbind. Such unstable pairs do not contribute to the quasiequilibrium statistics of the moving film but cause, instead, the deterioration of velocity with time. In determining $r_0(v)$ we will go a bit beyond the isolated vortex pair approximation and will take multiple vortex interactions into account. Let us consider an ensemble of vortices with some maximum scale of distance $\hat{r}$ and superimpose on it a vortex-antivortex pair separated a distance r which exceeds $\hat{r}$. After integrating over the ensemble of metastable bound vortices, we generate an effective interaction with parameters $\beta_{\hat{r}}, V_{\hat{r}}$ (other parameters are irrelevant at large r) between the superimposed pair. The minimum "dressed" energy of the superimposed pair occurs when the pair is oriented parallel to the y axis and is $2\beta_{\hat{r}} \ln(\hat{r}/r) - V_{\hat{r}} r$. $r_0(v)$ is defined by

$$r_0(v)\, V_{r_0(v)} = 2\beta_{r_0(v)} \ . \qquad (2.6)$$

With this definition, $r_0(v)$ is the smallest value of r having the property that any pair superimposed at separation $r > \hat{r}$ will be unstable.

Physically, one should picture a "gas" of vortex pairs with maximum separation $r_0(v)$ in near equilibrium. Occasionally a pair will leak past the threshold, rapidly unbinding. It is assumed that the leakage is rare enough for the gas of bound pairs to maintain itself in quasiequilibrium. Note that the quasiequilibrium statistical mechanics contains the time-dependent velocity v.

There is some ambiguity in how one might impose a "maximum scale of distance" on an ensemble of vortices. The most workable definition is to simply make $r_0(v)$ that scale of distance at which one stops the renormalization group. It is not fruitful to use less workable but perhaps more physically stringent definitions. These will only effect the coefficients A and v_0 in (1.1) and (1.4), and not the exponents. Since both A and v_0 will also depend on the nature of one's substrate, it is best to determine them experimentally.

For determining the superfluid density the quantity of interest is the vortex backflow, the average current due to the metastable bound pairs j_V:

$$j_V \equiv \rho_0 \left\langle \sum_i 2\pi p_i y_i \frac{1}{\Omega} \right\rangle . \tag{2.7}$$

This will be evaluated by the renormalization group. The idea is to successively integrate the contribution of vortex pairs separated by distances between 1 and τ, leaving a statistical mechanics problem of the same form as the original problem except that the minimum scale of distance in (2.5) is now τ instead of unity. After rescaling units of distance by τ one obtains a statistical mechanics problem with rescaled parameters β_τ, $\tau^2 K_\tau$, and τV_τ replaing β_1, K_1, V_1 of the original problem. In the process of thinning variables between 1 and τ, a contribution to j_V will be made. This will be denoted by $J(\tau)$. The full vortex backflow will be written

$$j_V = J(\tau) + \rho_0 \lambda_\tau \left\langle \sum_i 2\pi p_i y_i \frac{1}{\tau\Omega} \right\rangle_\tau . \tag{2.8}$$

In (2.8) $\langle\ \rangle_\tau$ denotes an expectation value taken with respect to the renormalized Hamiltonian. The quantities β_τ, $\tau^2 K_\tau$, V_τ, λ_τ, and $J(\tau)$ are given by the differential equations

$$\tau \frac{d}{d\tau} \beta_\tau = -2\pi (\beta_\tau)^2 (\tau^2 K_\tau)^2 I_0(\tau V_\tau) , \tag{2.9}$$

$$\tau \frac{d}{d\tau} (\tau V_\tau) = \tau V_\tau - (\tau^2 K_\tau)^2 4\pi \beta_\tau I_1(\tau V_\tau) , \tag{2.10}$$

$$\tau \frac{d\lambda_\tau}{d\tau} = -(K_\tau \tau^2)^2 4\pi \beta_\tau \lambda_\tau I_2(\tau V_\tau) , \tag{2.11}$$

$$\tau \frac{d}{d\tau} J(\tau) = -2\pi \rho_0 (K_\tau \tau^2)^2 \lambda_\tau \frac{I_1(\tau V_\tau)}{\tau} , \tag{2.12}$$

$$\tau \frac{d}{d\tau} (\tau^2 K_\tau) = (2-\beta_\tau)(\tau^2 K_\tau) + O((\tau^2 K_\tau)^3) , \tag{2.13}$$

where $I_m(x)$ is essentially a modified Bessel function:

$$I_m(x) \equiv \int_0^{2\pi} d\theta \cos^m\theta \exp(x\cos\theta) . \tag{2.14}$$

Initial values are $\lambda_1 = 1$ and $J(1) = 0$. β_1, K_1, and V_1 were given at the beginning of this section. Equations (2.9) and (2.13) were derived by Kosterlitz; the notation here is the same as his. The other renormalization-group equations (2.10)–(2.12) follow in a completely straightforward way from his paper. It is worth noting, however, that in obtaining [(2.10)–(2.12)] integrals of the type

$$\int d^2 r \left(\frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|^2} - \frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|^2} \right) = 2\pi(\vec{r}_2 - \vec{r}_1)$$

are not zero in two dimensions. The above example, with $\vec{r}$ denoting a two component vector, is related (components have been reversed) to the total current due to a vortex-antivortex pair at $\vec{r}_1$, $\vec{r}_2$.

The equations (2.9)–(2.12) are exact. In obtaining (2.13), however, factors of $(\tau^2 K_\tau)^2$ were ignored. This includes contributions to $\tau^2 K_\tau$ from short-ranged interactions among the vortices. A short-ranged interaction of the form $D p_i p_j (r_{ij})^{-X}$ $(X > 0)$ will be governed by a renormalization-group equation of the form

$$\tau \frac{d}{d\tau} D_\tau = -X D_\tau + O((\tau^2 K_\tau)^2) .$$

Thus it follows that, so long as $\tau^2 K_\tau$ is small and β_τ exceeds two, $\tau^2 K_\tau$ and short-ranged interactions will approach zero with increasing τ. We will, as is usual in renormalization-group work, assume that local stability properties are global. That is, even when K_1 is not arbitrarily small, it is assumed that, in the large τ limit for $\beta\tau > 2$, the locally stable approximation, (2.13), will be approached. What is interesting from the theorists point of view is that the Kosterlitz renormalization group is able to make statements about even the local stability properties of a fixed point. In most renormalization-group work one is forced to resort to asymptotic expansions or uncontrolled approximations to find the fixed points of interest.

Define τ_1 by

$$\tau_1 V_{\tau_1} = 1 . \tag{2.15}$$

For $\tau \ll \tau_1$, we use the small x limit of $I_m(x)$ to rewrite the exact equations (2.9)–(2.12):

$$\frac{d}{d\tau} \lambda_\tau = \frac{\lambda_\tau}{\beta_\tau} \frac{d\beta_\tau}{d\tau} \left[1 + O\left(\left| \frac{\tau}{\tau_1} \right|^2 \right) \right] , \tag{2.16}$$

$$\frac{d}{d\tau} V_\tau = \frac{V_\tau}{\beta_\tau} \frac{d\beta_\tau}{d\tau} \left[1 + O\left(\left| \frac{\tau}{\tau_1} \right|^2 \right) \right] , \tag{2.17}$$

$$\frac{d}{d\tau} J(\tau) = \rho_0 \frac{\lambda_\tau V_\tau}{2} \frac{1}{(\beta_\tau)^2} \frac{d\beta_\tau}{d\tau} \left[1 + O\left(\left| \frac{\tau}{\tau_1} \right|^2 \right) \right] , \tag{2.18}$$

For $\tau >> \tau_1$ the large x limit of $I_m(x)$ is approached and (2.9)–(2.12) reduce to

$$\frac{d}{d\tau}\lambda_\tau = 2\frac{\lambda_\tau}{\beta_\tau}\frac{d\beta_\tau}{d\tau} , \quad (2.19)$$

$$\frac{dV_\tau}{d\tau} = \frac{2}{\tau}\frac{1}{\beta_\tau}\frac{d\beta_\tau}{d\tau} , \quad (2.20)$$

$$\frac{dJ(\tau)}{d\tau} = \frac{\rho_0\lambda_\tau}{\tau(\beta_\tau)^2}\frac{d\beta_\tau}{d\tau} . \quad (2.21)$$

One does not carry the renormalization group out to arbitrarily large τ. For $\tau > r_0(v)$ there are no metastable bound pairs. That is $j_v = \rho_0 J(r_0(v))$. Combining (2.9)–(2.21) gives

$$j_v = \frac{\rho_0 V_1}{2(\beta_1)^2}(\beta_{\tau_1} - \beta_1) + \int_{\tau_1}^{r_0(v)} d\hat{\tau}\left(\frac{1}{\hat{\tau}\beta_1\beta_{\tau_1}}\right)\frac{d\beta_{\hat{\tau}}}{d\hat{\tau}}\rho_0 . \quad (2.22)$$

For $\tau << \tau_1$ we use the small x limit of I_m and reduce (2.9) and (2.13) to the equations of Kosterlitz:

$$\tau\frac{d}{d\tau}\beta_\tau = -(\beta_\tau)^2(2\pi\tau^2K_\tau)^2 , \quad (2.23)$$

$$\tau\frac{d}{d\tau}(\tau^2K_\tau) = (2-\beta_\tau)\tau^2K . \quad (2.24)$$

Let us denote the $\tau \to \infty$ limit of β_τ for (2.23) and (2.24) by $\beta^*(T)$, where T, the temperature determines the bare fugacity K_1, and coupling constant, β_1 for a given coverage and substrate. Kosterlitz has shown that, as the critical point is approached, $\beta^*(T)$ approaches two with a square-root cusp (1.2). To find β_τ at large τ, observe that

$$4/\beta_\tau + 2\ln\beta_\tau - (2\pi\tau^2K_\tau)^2$$

is independent of τ. (This generalizes an invariance relation used by Kosterlitz near T_c.) Hence to second order in $[\beta_\tau - \beta^*(T)]$:

$$[\beta^*(T)]^2(2\pi\tau^2K_\tau)^2 = 2[\beta^*(T)-2][\beta_\tau - \beta^*(T)] + [\beta_\tau - \beta^*(T)]^2 . \quad (2.25)$$

Since the quadratic term is important only near T_c, the critical point value of its coefficient has been used. Applying (2.25) to (2.23) gives

$$\beta_{\tau_1} = 2 + f(T)[(1+\tau_1^{-2f(T)})/(1-\tau_1^{-2f(T)})] , \quad (2.26)$$

$$\tau_1^2K_{\tau_1} = f(T)\tau_1^{-f(T)}/\pi(1-\tau_1^{-2f(T)}) , \quad (2.27)$$

where $f(T) \equiv \beta^*(T) - 2$. Equations (2.26) and (2.27) give the large τ behavior of β_τ, K_τ for all subcritical temperatures, a fact that permits relating velocity exponents to the superfluid density at zero velocity. Above the critical point Kosterlitz has shown that similar results apply with $\tau_1^{-[\beta^*(T)-2]}$ replaced with $\exp\{-iC(\ln\tau_1)[(T-T_c)/T_c]^{1/2}\}$ provided τ_1 is small compared to the correlation length ξ:

$$\tau_1 << \xi \sim \exp\left(\frac{1}{C[(T-T_c)/T_c]^{1/2}}\right) .$$

For $T > T_c$ and $\tau_1 >> \xi$, this renormalization group is no longer strictly appropriate since τ^2K_τ is growing with τ.

From (2.17) $V_{\tau_1} = (\beta_{\tau_1}/\beta_1)V_1$, so, by (2.15) and (2.26) $\tau_1 = (\beta_1/\beta_{\tau_1}V_1) \equiv 1/4v$ near the critical point (where $\beta_{\tau_1} \xrightarrow[\tau_1\to\infty]{} 2$). For $\tau_1 < \tau < r_0(v)$:

$$\tau\frac{d\beta_\tau}{d\tau} \approx -(\beta_{\tau_1}\tau_1^2K_{\tau_1})^2 2\pi^{3/2}\frac{e^{\tau/\tau_1}}{(\tau/\tau_1)^{1/2}} ,$$

$$r_0(v) \equiv 2\beta^*(T)/V_1 \equiv 4\tau_1 . \quad (2.28)$$

Hence

$$\int_{\tau_1}^{r_0(v)} d\hat{\tau}\frac{\rho_0}{\hat{\tau}\beta_1\beta_{\tau_1}}\frac{d\beta_{\hat{\tau}}}{d_{\hat{\tau}}} = -\frac{\rho_0 v}{\beta_1}\beta_{\tau_1}(\tau_1^2K_{\tau_1}\pi)^2A , \quad (2.29)$$

where

$$A \cong \frac{2}{\pi^{1/2}}\int_1^4 dx\frac{e^x}{x^{2.5}} \approx 5 . \quad (2.30)$$

Equations (2.22), (2.26), (2.29), and the definition for V_1 imply that the vortex backflow is

$$j_v = v(\rho_0/\beta_1)\{\beta_{\tau_1}[1 - A(\tau_1^2K_{\tau_1}\pi)^2] - \beta_1\} . \quad (2.31)$$

Since the superfluid density is simply the total current divided by v, $\rho_s v = \rho_0 v + j_v$ we obtain (1.1).

Although this renormalization group is not appropriate above the critical point, it is worth noting that these equations do give the expected result, $\rho_s = 0$ for zero velocity films above T_c. Above T_c (2.9)–(2.13) imply that the large τ limiting behavior is

$$\tau^2K_\tau \to \hat{K}\tau^2 , \quad (2.32)$$

$$\beta_\tau \to 4/(2\pi)^2\hat{K}^2\tau^4 . \quad (2.33)$$

In addition

$$\tau V_\tau = V_1\tau(\beta_\tau/\beta_1) , \quad (2.34)$$

$$\lambda_\tau = \beta_\tau/\beta_1 , \quad (2.35)$$

$$J(\tau) = \rho_0(\beta_\tau - \beta_1)\frac{V_1}{2(\beta_1)^2} = \rho_0(\beta_\tau - \beta_1)\frac{v}{\beta_1} . \quad (2.36)$$

The last three results [(2.34)–(2.36)] depend on those renormalization-group equations (2.9)–(2.12) that are valid at all temperatures. The equation K_τ, (2.13) was needed only to obtain the results $\lim_{\tau\to\infty} \beta_\tau$, $\tau V_\tau = 0$ above T_c. Granted these then, in the $\nu \to 0$ limit, $r_0(\nu) = \infty$, $\beta_{r_0}(\nu) = 0$, and

$$\rho_s = \rho_0 + (1/\nu) J(\infty)$$
$$= \rho_0 + (1/\nu)\rho_0(-\beta_1)(\nu/\beta_1) = 0$$

above T_c, as one would expect.

The calculation of ρ_s is completely analogous to finding the polarizability of the two-dimensional Coulomb gas. The Hamiltonian for a two-dimensional Coulomb gas with a uniform electric displacement vector is

$$\mathcal{H} = q^2 \sum_{i,j} p_i p_j \ln r_{ij} - \sum_i q p_i \mathfrak{D} y_i \ ,$$

where q is the unit of electric charge and the displacement vector $\mathfrak{D}$ points in the y direction. The polarizability is, by direct analogy to (2.31):

$$\langle \sum_i p_i y_i \rangle / \mathfrak{D}$$
$$= \frac{1}{4\pi} \left[1 - \frac{\beta_{\tau_1}}{\beta_1} [1 - A(\tau_1^2 K \tau_1)^2] \right] \Big|_{\tau_1 = \mathfrak{D}^{-1}} \ , \quad (2.37)$$

where $\beta^*(T)$ is given by (1.2) near the critical point and, for the Coulomb gas problem

$$\beta_1 = q^2 / k_B T \ . \quad (2.38)$$

Renormalization-group trajectories are such that $1 > \beta^*(T)/\beta_1$. This means that the polarizability is positive. Equations (2.37) and (2.38) imply that the dielectric constant is given by

$$\epsilon = \beta_1 / \beta^*(T) = \rho_0 / \rho_s \quad (2.39)$$

in the zero $\mathfrak{D}$ limit. Note that ϵ is finite on and below the critical point. Since ϵ is equivalent to ρ_s^{-1} the experimental evidence for a finite ρ_s on and below T_c is also support for the prediction of a finite ϵ in the two-dimensional Coulomb gas for $T \leq T_c$. Near the critical point (2.39) differs from the result of Zittartz and Huberman.[20] The discrepancy should not be surprising, since their calculation was to first order in the bare fugacity K_1. Near the critical point a fugacity expansion carries powers of $K_1/(T_c - T)$ and is not expected to be appropriate; the dielectric constant is nonanalytic on the critical point. Equivalently, near the critical point the successive powers of $K_1/(T_c - T)$ signal that multiple vortex interactions are important and one must use the renormalization group.

III. TIME DEPENDENCE OF THE SUPERFLUID VELOCITY

In this section we will enunciate the rather elementary considerations that lead from the renormalization-group results of Sec. II to the relation (1.4) for time dependence of superfluid velocity. Factors which are independent of velocity or temperature and which ultimately feed into the substrate dependent coefficient ν_0 are of no importance here.

The rate at which thermal fluctuations produce destabilized vortex pairs is porportional to the number of pairs near the threshold of instability, at separations comparable, that is, to $r_0(\nu)$. This, in turn, is simply the square of the unscaled fugacity K_τ:

$$K_\tau^2 \big|_{\tau = r_0(\nu)} = \frac{[\beta^*(T) - 2]^2 \nu^{2\beta^*(T)}}{\pi^2 (1 - \nu^{2[\beta^*(T) - 2]})^2} \ .$$

R will be used here to denote the rate per unit area at which destabilized vortex pairs are produced

$$R \sim (K_\tau)^2 \tau = \nu^{-1} \ . \quad (3.1)$$

Next observe that an isolated vortex need not remain that way indefinitely. If an isolated vortex comes sufficiently close to another free vortex of opposite circulation they will rebind. This process is illustrated in Fig. 1. It shows a system which initially consists of two destabilized pairs superimposed on a gas of metastable bound vortices. Under the influence of substrate induced thermal fluctuations the separation of the destabilized pairs will increase in the direction normal to the flow of the fluid—this is the motion that tends to minimize energy as seen in the rest frame of the substrate. In the case illustrated, however, the positive and negative free vortices in the center of Fig. 1 will pass sufficiently close to bind. The result is that, in the case of Fig. 1, one ends up with a single unbound pair of vortices instead of the two that one started with. The meaning of "sufficiently close to bind" will depend on the dynamics of the system. For vortices in a two-dimensional fluid the dynamical equations are first order in time. Thus the rate at which the location $\vec{r}$, of a vortex changes is of the form $d\vec{r}/dt = \vec{F}(\vec{r}) + \vec{g}$, where $\vec{g}$ is a random term. The question of whether two vortices will coalesce into a metastable bound state is determined by their locations. Roughly, if they pass within $r_0(\nu)$ of each other, they will bind. For the analogous two-dimensional Coulomb gas problem of charges moving in a uniform electric field, the dynamics are a bit different. The charges are accelerated by the electric field. The difference between the second-order differential equation that governs the Coulomb gas dynamics and the first-order differential equation of vortex motion in a fluid is, of course, that an isolated charge in the Coulomb gas can acquire sufficient kinetic energy to avoid being trapped when it passes close to an opposite charge. Because of this complication, electric current in the two-dimensional Coulomb gas (this is the analog of free vortex motion in helium) will not be discussed in this paper.

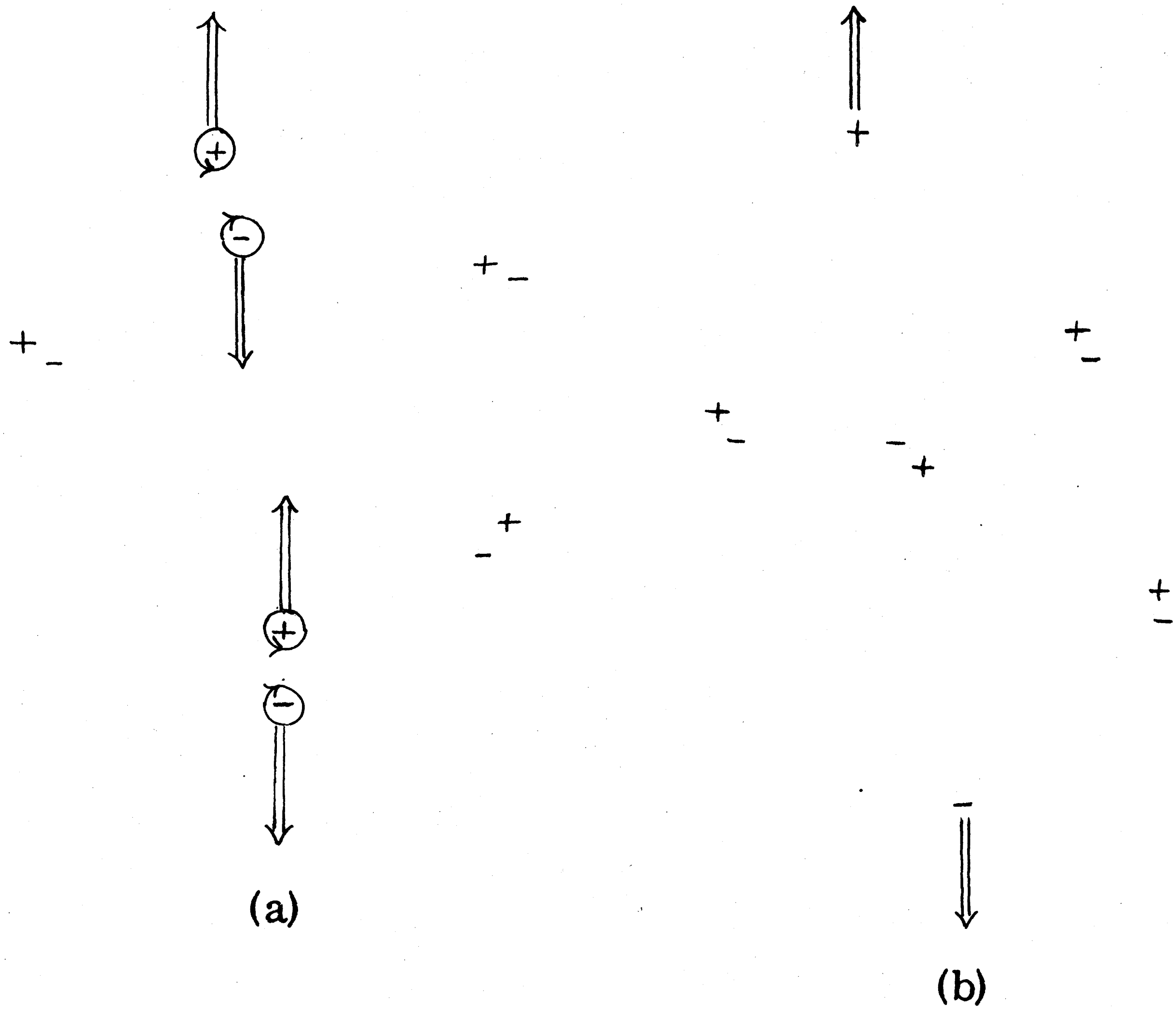

FIG. 1. (a) Motion (due to thermal fluctuations) of two destabilized vortex pairs in a moving two-dimensional fluid. Film flow is from right to left, thermally induced drift of the free vortices is up and down, depending on the sign of the vorticity. The free vortices approaching the center of the figure will pass close enough to bind. (b) Motion after the positive and negative vortex in the center drop into a metastable bound configuration.

Let us use t_0 to denote the mean time that a destabilized vortex pair spends moving apart before one member of the pair comes within $r_0(\nu)$ of a vortex from another destabilized pair. The mean distance traveled before rebinding occurs will be denoted by λ. λ and t_0 are related by the rate v_D at which thermal fluctuations cause a destabilized pair to move apart

$$\lambda/\tau \sim v_D \ . \tag{3.2}$$

In addition, the product of the mean free path λ and the binding radius $r_0(\nu)$ is equal to the inverse of the mean number of destabilized pairs per unit area. But the mean number of destabilized pairs per unit area is simply the rate per area at which they are being produced, R, multiplied by the time they endure t_0:

$$\lambda r_0(\nu) \sim (R t_0)^{-1} \ . \tag{3.3}$$

from (2.27), $r_0(\nu) \sim 1/\nu$. Given v_D one may use (3.2), (3.3), and (3.4) to find λ. But thermal fluctuations cause the separation $\vec{r}$ of a vortex pair to evolve with time according to

$$\frac{d\vec{r}}{dt} = -f_0\left(\frac{\vec{\nabla}E(\vec{r})}{k_B T} + \vec{g}(t)\right) \ , \tag{3.4}$$

where $\vec{\nabla}E(\vec{r})$ denotes the spatial gradient of the energy at separation $\vec{r}$ and $\vec{g}(t)$ is a random term with appropriate autocorrelation. For purposes of this section, there is no significant difference between the bare energy of an isolated pair, with parameters β_1 and V_1, and the dressed energy, with parameters β_τ

and V_τ, that one obtains after integrating out the ensemble of metastable bound vortices. In either case one obtains

$$(1/k_B T)\vec{\nabla} E(\vec{r}) \sim v\hat{y} \tag{3.5}$$

for $r >> r_0(v)$. $\hat{y}$ is a unit vector perpendicular to the direction of flow.

The coefficient f_0 in (3.4) may, in general, vary with time and $\vec{r}$; here we will assume it to be uniform. Equation (3.4) may be derived in standard ways (see, for example, Ref. 4) from thermal fluctuation theory. One needs no more than detailed balance and the assumption that thermal fluctuations cause r to change through a series of small hops. Given (3.4) and (3.5) it follows that the rate at which a destabilized pair moves apart is:

$$v_D \sim v \quad . \tag{3.6}$$

This means that the mean distance, λ, that a free vortex travels before it rebinds with some isolated vortex of opposite vorticity is [from (3.1)–(3.4), and (3.6)]:

$$\lambda \sim (v v_D / R)^{1/2}$$

or

$$\lambda \sim v/K_{\tau_1} \quad . \tag{3.7}$$

From (3.7) it is a straightforward matter to obtain the quantity of interest, the rate (1.4) of decay of the superfluid velocity. The rate at which the film's velocity diminishes with time is given by the product of $2\pi/L$ and the rate at which isolated vortices are produced. (L here refers to the length of the film in the direction of flow.) The net rate at which free vortices are produced is R, the rate per area at which pairs destabilize, multiplied by $L\lambda$. This gives

$$\frac{dv}{dt} \sim -\frac{1}{L} R L \lambda$$

or (1.4):

$$\frac{dv}{dt} \propto -vK_\tau\big|_{\tau - v^{-1}} \quad .$$

IV. SUGGESTIONS FOR FUTURE RESEARCH

The basic results of this paper, reviewed in Sec. I have been derived. Because of the virtual rigor of the Kosterlitz renormalization-group approach, these results are sound, if the vortex nucleation theory is sound. Preliminary comparisons with experiment are quite satisfactory, with near critical anomalies understandable as finite velocity effects. It would be interesting to attempt to extend this work to three-dimensional helium. Some preliminary thoughts as to how to treat this problem were discussed in Banks, Myerson, and Kogut.[21] One difficult feature is that the three-dimensional renormalization-group fixed point is believed to be a "soft spin" fixed point rather than the "stiff spin" fixed point of two dimensions. Perhaps the most efficient way to deal with the three-dimensional problem is through careful work with the dimensional expansion (ϵ expansion) about four dimensions, in the manner of Rudnick and Jasnow.[22] As a preliminary guess, it seems reasonable to predict that, at nonzero velocities, strong scaling theory will hold, with $T_c - T$ scaling in the usual way and v scaling as the inverse of distance. In particular ρ_s for bulk helium at finite velocity should be consistent with the Josephson[23] scaling relation $\rho_s \sim \xi^{-d+2}$, with the correlation length ξ being the lesser of $(T_c - T)^{-\nu}$ and $1/v$ roughly $\xi^{-1} \sim v + (T_c - T)^{\nu}$.

Supplementary note

Since preparing the original draft of this paper the author has learned of the excellent work of Ambegaokar *et al.*[24] on dissipation in an Andronikashvilli experiment at high frequencies. The criterion for high-frequency effects becoming important is $\sqrt{D/\omega} \lesssim 1/v$, where D is the vortex diffusivity and $\sqrt{D/\omega}$ is the characteristic distance travelled by a diffusing vortex in one period of oscillation of the substrate. Bishop and Reppy have fitted their low drive strength (low v) data to the high-frequency theory. As drive strength (and therefore v) is increased, however, the temperature range of the observed dissipation peak broadens. This is not accounted for by the high-frequency theory, but certainly would be expected in the low-frequency limit discussed in this paper. Work is currently in progress on the crossover from the high to low frequency versus drive strength regimes.

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