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Planar and linear solitons in superfluid ^{3}He

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A means of classifying the planar and linear topological textures in superfluid phases of ^{3}He by taking into account the effects of both magnetic field and boundary conditions is described. The method of the conventional homotopic groups is applied. The structure and stability of various solitons is considered in detail. This method may be applied to the description of topologically stable solitons in other ordered systems.

I. INTRODUCTION

In our previous works[1,2] the singularities in the ^{3}He order-parameter fields (vortices, dysgyrations, disclinations, hedgehogs, and so on) were investigated.[3] It was shown that all the singularities may be divided into different classes with macroscopically large energy barriers for the transition between the classes. On the other hand, singularities may be continuously transformed into other singularities of the same class either without any energy barrier or with a small one caused by the change in gradient energy during the transformation. Every class of singularities is characterized by the element of the homotopy group of the order-parameter space (this space is named "the range of the degeneration parameter variation" in Ref. 2 and "the manifold of internal states" in Ref. 4). In the case of superfluid ^{3}He all the homotopic groups are Abelian. This means that every class of singularities may be characterized by one or several integers (topological invariants). When two singularities collapse forming another singularity the invariants are summed up (in case of linear singularities in ^{3}He-A a summation over modulus 4 takes place, e.g., $3+3=2$).

Spin-orbit coupling, the magnetic field, and boundary conditions change the structure of the order parameter and the topology of its space. There are several different scales of length: the coherence length ξ, the characteristic length of the dipole-dipole interaction ξ_{dip}, the magnetic length ξ_{magn}, and the size of the vessel. Every scale of length has its own topology of the order-parameter space and therefore its own homotopic groups and classes of singularities. Moreover there may exist textures with the mapping of the inner region of the texture into another order-parameter space.

One of the examples of such textures is the Maki planar soliton (domain wall) in ^{3}He-A.[5] Far from this soliton the order parameter space is restricted by the spin-orbit dipole-dipole interaction, but there is no such restriction inside the wall.

Another example of such a texture is the linear soliton in a nematic liquid crystal in a magnetic field [Fig. 1(a)]. Inside this soliton the order parameter space is S^2/Z_2 while outside the soliton the magnetic field reduces the order parameter space to one point $\vec{d}\parallel\vec{H}$. This soliton as well as in A-phase have no singularities in order parameter field. Nevertheless both of them are described by topological invariants. Actually the Maki wall may terminate on the singular line[2] and the linear soliton in nematics may terminate on the hedgehog in vector $\vec{d}$ field [Fig. 1(b)].

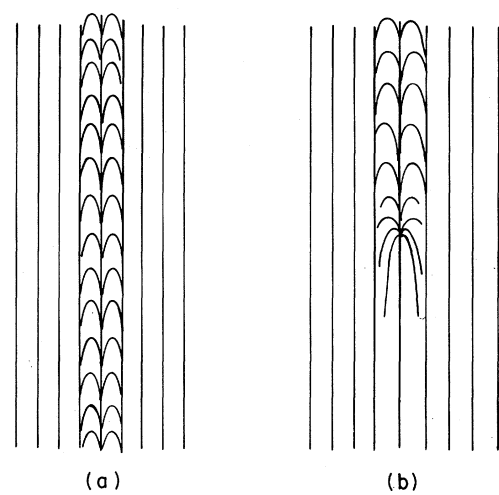

FIG. 1. (a) Distribution of the field lines of the director vector $\vec{d}(\vec{r})$ for the linear soliton texture in the nematic liquid crystal in the magnetic field. (b) The demonstration of the edge point of the same linear soliton—the hedgehog in the field of the director vector $\vec{d}(\vec{r})$.

Therefore, the soliton may be characterized by the same topological invariant as that of the singularity on which the soliton terminates.

However, in many cases it is more convenient to classify the solitons in a direct way without the investigations of the singularities. To classify them the so-called "conventional homotopic group method" is used. This method may be also applied to other ordered systems. In Sec. II the planar solitons in ^{3}He-A and ^{3}He-B are described. In Sec. III the linear solitons are considered. The particlelike solitons (point solitons) were investigated in Ref. 6. In Sec. III, the classification of the nonuniform states in a cylindrical vessel, taking into account the boundary conditions, is also described.

II. PLANAR TEXTURES

The order parameter for the A and B phases are

$$A_{ik}=\Delta(T)V_i(\Delta_k'+i\Delta_k''),\quad A_{ik}=\Delta(T)e^{i\phi}R_{ik}(\vec\omega;\theta). \tag{2.1}$$

First we consider the planar textures in R_{ik} field in magnetic field. R_{ik} is supposed to depend on one coordinate z. Far from the soliton the dipole-dipole interaction fixes the angle of the rotation $\theta=\theta_0=\arccos(-\frac{1}{4})$ and the magnetic field fixes the axis rotation $\vec\omega=\pm\vec h$ ($\vec h=\vec H/\vec H$). The problems are do there exist topologically stable textures in the $R_{ik}(z)$ field with these conditions at $|z|\to\infty$, what is the way of classifying them, what is the value of the barrier for the transition from one class into another, what is the way of destroying of these textures, what is the result of the collapsing of two planar textures.

To answer these questions the topological structure of the order-parameter space for different scales of length is to be known. The free-energy density of superfluid ^{3}He is as follows[7]

$$\begin{aligned}
&F=F_{\rm cond}+F_{\rm dip}+F_{\rm magn}+F_{\rm grad}\,,\\
&F_{\rm cond}=-\alpha A_{ik}A^*_{ik}+\beta(\text{fourth-order terms})\,,\\
&F_{\rm dip}\,(B\text{ phase})=\tfrac{4}{5}g_D\,(\cos\theta+2\cos^2\theta)\,,\\
&F_{\rm dip}\,(A\text{ phase})=-\tfrac{3}{5}g_D(\vec l\,\vec V)^2\,,\\
&F_{\rm grad}=\gamma\left[\left(\frac{\partial A_{ik}}{\partial z}\right)^2+2\left(\frac{\partial A_{iz}}{\partial z}\right)^2\right],\\
&F_{\rm magn}\,(B\text{ phase})\sim -g_D(\mu/\Delta)^2(\vec\omega\vec H)^2\,,\\
&F_{\rm magn}\,(A\text{ phase})\sim \chi_N(\vec H\,\vec V)^2\,.
\end{aligned}\tag{2.2}$$

The different scales of length are

$$\xi\sim(\gamma/\alpha)^{1/2},\quad \xi_{\rm dip}\sim(\gamma\Delta^2/g_D)^{1/2},\quad \xi_{\rm magn}(A)\sim\left(\frac{\gamma\Delta^2}{\chi_N H^2}\right)^{1/2},\quad \xi_{\rm magn}(B)\sim\xi_{\rm dip}\frac{\Delta}{\mu H} \tag{2.3}$$

with the following relations between them

$$\begin{aligned}
&\xi\ll\xi_{\rm dip}\ll\xi_{\rm magn}(B)\,;\\
&\xi_{\rm dip}<\xi_{\rm magn}(A),\quad H<50G\\
&\xi_{\rm magn}(A)<\xi_{\rm dip}\,,\quad H>50G\,.
\end{aligned}\tag{2.4}$$

In Table I it is shown the topological structure of the order-parameter space and its homotopic groups for every scale of length.

In case of the ^{3}He-B one has far from the soliton the space $R_B^H=Z_2$ (we do not consider the phase ϕ variation), consisting only of two points $\vec\omega=\pm\vec h$, $\theta=\theta_0$. Inside the soliton, the order-parameter space may be extended up to the $\tilde R_B=S^2$ (spherical surface with radius $\theta=\theta_0$) or up to the $R_B=SO_3$ (the solid sphere of the radius $\theta=\pi$, every point $\vec\alpha=\vec\omega\theta$ of this solid sphere describes the rotation around the axis $\vec\omega$ by an angle θ, every point on

TABLE I. Topological structure of the order-parameter space and its homotopic groups for every scale length.

Phases	Scale of length	Scale of energy	Order-parameter space	π_0	π_1	π_2
	$r\lesssim\xi$	$F_{\rm grad}\gtrsim F_{\rm cond}$	R^{18}	0	0	0
B phase	$\xi\ll r\lesssim\xi_{\rm dip}$	$F_{\rm cond}\gg F_{\rm grad}\gtrsim F_{\rm dip}$	$R_B=SO^3\times S^1$	0	$Z+Z_2$	0
	$\xi_{\rm dip}\ll r\lesssim\xi_{\rm magn}$	$F_{\rm dip}\gg F_{\rm grad}\gtrsim F_{\rm magn}$	$\tilde R_B=S^2\times S^1$	0	Z	Z
	$\xi_{\rm magn}\ll r$	$F_{\rm magn}\gg F_{\rm grad}$	$R_B^H=Z_2\times S^1$	Z_2	Z	0
A phase	$\xi\ll r\lesssim\xi_{\rm dip},\xi_{\rm magn}$	$F_{\rm cond}\gg F_{\rm grad}\gtrsim F_{\rm dip},F_{\rm magn}$	$R_A=(SO_3\times S^2)/Z_2$	0	Z_4	Z
	$\xi_{\rm dip}\ll r\lesssim\xi_{\rm magn}$	$F_{\rm dip}\gg F_{\rm grad}\gtrsim F_{\rm magn}$	$\tilde R_A=SO_3$	0	Z_2	0
	$\xi_{\rm magn}\ll r\lesssim\xi_{\rm dip}$	$F_{\rm magn}\gg F_{\rm grad}\gtrsim F_{\rm dip}$	$\tilde{\tilde R}_A=(SO_3\times S^1)/Z_2$	0	$Z+Z_2$	0
	$\xi_{\rm dip},\xi_{\rm magn}\ll r$	$F_{\rm magn},F_{\rm dip}\gg F_{\rm grad}$	$R_A^H=S^1\times S^1$	0	$Z+Z$	0

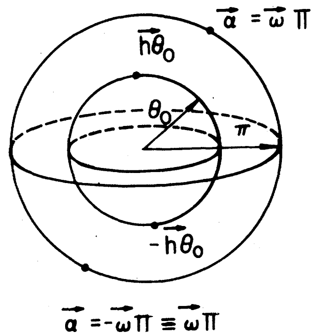

FIG. 2. Order parameter spaces for the B phase: $R_B = SO_3$; $\tilde{R}_B = S^2$, the spherical surface of the radius $\theta_0 \simeq 104°$; $R_B^H = Z_2$, two points on this surface.

a spherical surface $\vec{\alpha} = \vec{\omega}\pi$, describing the rotation around $\vec{\omega}$ by angle π, has its equivalent point $\vec{\alpha} = -\vec{\omega}\pi \equiv \vec{\omega}\pi$, see Fig. 2).

To find the classes of topologically stable planar textures one must find the classes of mapping of line $-\infty < z < +\infty$ into S^2 or SO_3 with the infinite points $-\infty$ and $+\infty$ being mapped into Z_2. All the classes of such a mapping are shown in Fig. 3 (the wave line is the image of the line $-\infty < z < +\infty$). It may be easily seen, that the contours (wave lines) drawn in Fig. 3 cannot be continuously transformed into one another. So we have eight classes of different domain walls. Each class is denoted by signs(+) or (−) and by the index N which takes values 0 and 1. The first sign shows the image of $-\infty$ $[\vec{\omega}(-\infty) = +\vec{h}$ or $-\vec{h}]$, the second sign shows the image of $+\infty$. The index N is the same as the one which describes the singular lines in ^{3}He-B. Two indices are summed up over modulo 2 ($1+1=0$). Note that these classes do not form a group. In fact for example the soliton (+ −1) and the soliton (+ −0) cannot be combined together. This is the result of the nonconnectivity

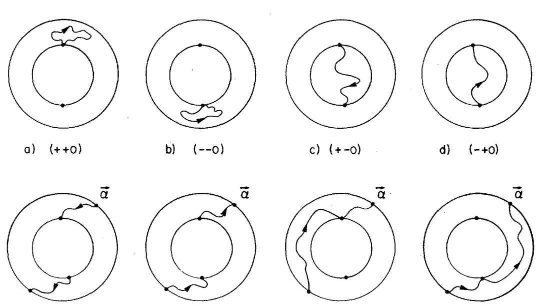

FIG. 3. Eight different classes of contours in SO_3 which correspond to eight classes of domain walls in B phase.

of the space $R_B^H = Z_2$ $[\pi_0(R_B^H) = Z_2]$.

Now let us write down some solutions of Ginzburg-Landau equation for each class of the planar solitons.

(+ + 0); (− − 0). The solution with minimal energy in these classes is the uniform distribution $\theta = \theta_0, \vec{\omega} = \vec{h}$ or $\theta = \theta_0, \vec{\omega} = -\vec{h}$. Any planar soliton of these two classes may be continuously transformed into uniform state.

(+ − 0); (− + 0). One of the solutions of the Ginzburg-Landau (GL) equation for the solitons of these classes was obtained by Maki and Kumar[5]

$$\theta = 2\arctan\left(\sqrt{\tfrac{5}{3}}\tanh\frac{|z|}{\xi_{\text{dip}}}\right), \quad \vec{\omega} = \mp\vec{h}\,\text{sgn}\,z\,. \quad (2.5)$$

The size of this soliton $\sim\xi_{\text{dip}}$ and the energy per unit area $E/S \sim g_D\xi_{\text{dip}}$. This soliton realizes the mapping drawn on Fig. 4(a). There is another solution in these classes with the essentially smaller energy:

$$\theta = \theta_0, \quad \vec{\omega} = \vec{h}\cos\beta(z) + \vec{h}\times\hat{z}\sin\beta(z), \quad \vec{h}\perp\hat{z}$$

$$\beta(z) = 2\arctan\,[\exp(\mp z/\xi_{\text{magn}})]\,. \quad (2.6)$$

This soliton with the size $\sim\xi_{\text{magn}}(B)$ and the energy $E/S \sim F_{\text{magn}}\xi_{\text{magn}} \sim g_D\xi_{\text{dip}}(\xi_{\text{dip}}/\xi_{\text{magn}}) \ll g_D\xi_{\text{dip}}$ realizes the mapping drawn on Fig. 4(b). Thus the soliton (2.5) may be continuously transformed into (2.6), the energy of the soliton being decreased.

(+ − 1); (− + 1). One of the solutions was obtained by Maki and Kumar (5):

$$\theta = 2\arctan\left(\sqrt{\tfrac{5}{3}}\coth\frac{|z|}{\xi_{\text{dip}}}\right), \quad \vec{\omega} = \mp\vec{h}\,\text{sgn}\,z\,. \quad (2.7)$$

The size of this soliton is $\sim\xi_{\text{dip}}$ and the energy $\sim g_D\xi_{\text{dip}}$. If there is another solution with smaller energy, it also has a size $\sim\xi_{\text{dip}}$ and energy $\sim g_D\xi_{\text{dip}}$. It may be seen from the following consideration. Given the size of the soliton being equal to R, then in the region of the order of R angle θ change from θ_0 to π, therefore the dipole energy is of the order of $E/S \sim g_D R$ and the gradient energy $\sim\gamma(\Delta^2/R^2)R$. Minimizing the total energy

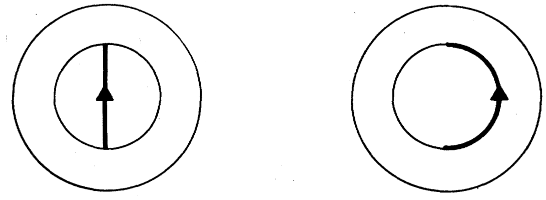

FIG. 4. (a) Solution by Maki and Kumar for the soliton (− + 0) as the path in SO_3. (b) Solution with the smaller energy for the soliton of the same class (− + 0).

$$E/S \sim g_D R + \gamma \Delta^2 / R,$$

one may find that in equilibrium $R \sim \xi_{\text{dip}}$.

(+ + 1); (− − 1). The solution in the class (+ + 1) may be constructed for example from the solutions (+ − 1) and (− + 0), or (+ − 0) + (− + 1). Now let us discuss a means of destroying the solitons. To destroy solitons with different signs: (+ − 0), (+ − 1), (− + 0), (− + 1), one must change the direction of $\vec{\omega}$ in one of the half spaces and therefore expend the macroscopically large energy $\sim V F_{\text{magn}}$ (V is the volume of the half space). Thus these solitons may be eliminated only on the boundary of the vessel, or by turning off the magnetic field, or by collapsing two solitons. The existence of such a large barrier for destroying solitons of this type is a consequence of the nonconnectivity of the space R_B^H $[\pi_0(R_B^H) = Z_2]$.

The solitons of the type (+ + 1), (− − 1) have different level of stability. As it was shown in Ref. 2 these solitons may terminate on a singular line with $N = 1$. Therefore to destroy this solution the closed singular line (ring) with the radius $R \sim \xi_{\text{dip}}$ must be created. After the creation the radius of the ring begins to increase because if $R > \xi_{\text{dip}}$ the energy of the singular core of the singular line $\sim F_{\text{cond}}\, \xi^2 R \ln(R/\xi)$ is compensated by the surface energy of soliton $g_D \xi_{\text{dip}} R^2$. The value of the barrier for destroying the soliton is equal to the energy of the created ring $\sim g_D \xi^3_{\text{dip}} \ln(\xi_{\text{dip}}/\xi)$. The existence of the solitons of this type is the consequence of the fact that $\pi_1(SO_3) = Z_2$.

We have enumerated all the types of the planar solitons in ^{3}He-B in magnetic field. As was noted these classes of solitons do not form a group due to the nonconnectivity of the space $R_B^H = Z_2$.

In the A phase, the order-parameter space is connected for all scales of length [there is a mistake in our previous work[2] concerning the definition of $\tilde{R}_A$, in fact $\tilde{R}_A = SO_3$ and $\pi_0(\tilde{R}_A) = 0$].[8] In case of the planar solitons in the A phase in a magnetic field (with the following relations between the scales of length: $\xi \ll \xi_{\text{dip}} \ll \xi_{\text{magn}}$) the solitons are described by the classes of mappings of a

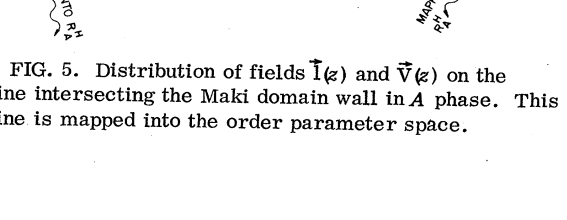

FIG. 5. Distribution of fields $\vec{l}(z)$ and $\vec{V}(z)$ on the line intersecting the Maki domain wall in A phase. This line is mapped into the order parameter space.

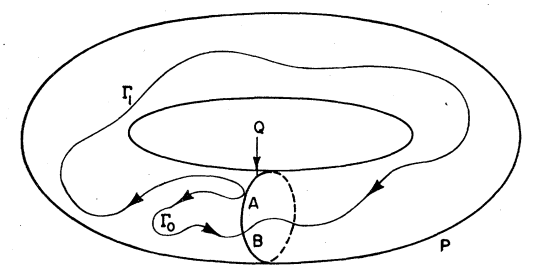

FIG. 6. Space $P = S^1 \times S^1$, torus; the subspace $Q = S^1 \subset P$, the small circumference on the torus. Γ_0 and Γ_1 are the contours which begin at point $A \in Q$ and finish at point $B \in Q$. Γ_0 belongs to the class of contours which corresponds to zero element of the group $\pi_1(P, Q)$, Γ_1 belongs to the element $N = 1$ of the same group.

line $-\infty < z < +\infty$ into R_A and $\tilde{R}_A$ with the ends of this line being mapped into R_A^H $(R_A^H \subset \tilde{R}_A \subset R_A)$ (see Fig. 5).

Two different mappings are of the same class if they may be obtained from each other by continuous transformation. The classes of mappings of the line into some space P with the ends of the line being mapped into one point $A \in P$ form the fundamental group[2] $\pi_1(P, A) = \pi_1(P)$. In the case of mapping of the ends of the line into some connected subspace $Q \subset P$ the classes of such mappings form the so called "conventional homotopic group" $\pi_1(P, Q)$. Let us for example consider $P = S^1 \times S^1$ (torus) and $Q = S^1 \subset P$ [the small circumference on the torus (see Fig. 6)]. It may be seen that there are such classes of mapping: (a) The line maps into the contour Γ_0 which may be deformed into the point. (b) The line maps into the contour which N times encircles the large circumference of the torus. This contour cannot be contracted into the point. These classes of mapping form a group

$$\pi_1(P, Q) = Z.$$

Unlike the fundamental group $\pi_1(P) = Z \times Z$ which contains the contours embracing the large and small circumference on the torus, this group does not contain the contours which encircle the small circumference of the torus, because one may contract these contours into the point by moving the ends A and B along the subspace Q.

Generalizing the procedure of the calculation of the conventional homotopic group $\pi_1(P, Q)$, we may state that to find $\pi_1(P, Q)$ one must find $\pi_1(P)$ and then exclude from $\pi_1(P)$ the elements which corresponds to $\pi_1(Q)$. From a mathematical point of view one must find the factor group of $\pi_1(P)$ by its subgroup, this subgroup being the image of the homomorphism

$$\pi_1(Q) \to \pi_1(P),$$

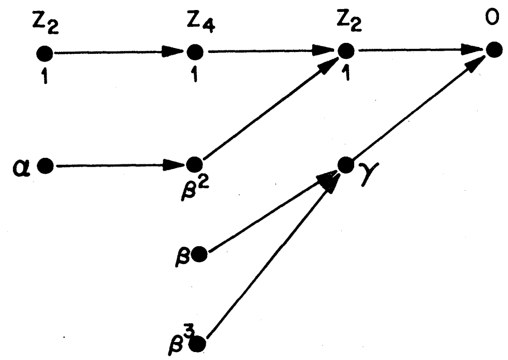

FIG. 7. Illustration of the application of the exact sequence of homomorphisms to the calculation of the homotopic group $\pi_1(R_A, \tilde{R}_A)$

$$\begin{array}{ccccccc} \pi_1(\tilde{R}_A) & \to & \pi_1(R_A) & \to & \pi_1(R_A\tilde{R}_A) & \to & 0 \\ \| & & \| & & \| & & \| \\ Z_2 & \to & Z_4 & \to & Z_2 & \to & 0 \end{array}$$

α is the nontrivial element of the group $\pi_1(\tilde{R}_A) = Z_2$ $(\alpha \cdot \alpha = 1)$, it describes the vortices with one quantum of circulation or disgyrations; β is the element of the group $\pi_1(R_A) = Z_4$ $(\beta^4 = 1)$; γ is the nontrivial element of the group $\pi_1(R_A, \tilde{R}_A) = Z_2$, it describes the Maki soliton.

that is

$$\pi_1(P, Q) = \pi_1(P)/\mathrm{Im}[\pi_1(Q) \to \pi_1(P)]. \qquad (2.8)$$

In the case of $P = S^1 \times S^1$ and $Q = S^1$

$$\mathrm{Im}[\pi_1(Q) \to \pi_1(P)] = \pi_1(Q) = Z$$

because all the contours of the group $\pi_1(Q) = Z$ form the same group in the space P, thus

$$\pi_1(P, Q) = Z \times Z/Z = Z.$$

On the other hand the Eq. (2.8) is the abbreviation of the exact sequence of homomorphisms[9]

$$\pi_1(Q) \to \pi_1(P) \to \pi_1(P, Q) \to 0. \qquad (2.9)$$

As is known the exact sequence of the homomorphisms of the groups A, B, C, D

$$\cdots \to A \to B \to C \to D \to \cdots$$

is such a sequence of groups that an image of any given homomorphism, for example $A \to B$ (i.e., the set of the elements of the group B into which the elements of the group A are transformed) is simultaneously the kernel of the next homomorphism $B \to C$ (the set of the elements of B which are transformed into zero element of C). For example of $P = R_A$, $Q = \tilde{R}_A$ and $\pi_1(P) = Z_4$, $\pi_1(Q) = Z_2$ the Eq. (2.9) is shown on Fig. 7. From this Fig. 7 we have

$$\pi_1(R_A, \tilde{R}_A) = Z_4/Z_2 = Z_2. \qquad (2.10)$$

Applying this procedure for the plane solitons in ^{3}He-A one obtains [compare (2.10)]

$$\pi_1(\tilde{R}_A, R_A^H) = Z_2/Z_2 = 0,$$
$$\pi_1(R_A, R_A^H) = Z_4/Z_2 = Z_2. \qquad (2.11)$$

It means that there are only two classes of planar solitons in A phase. The solution of the GL equation for the soliton of the nontrivial class was given by Maki and Kumar.[5] When two Maki's solitons collapse the uniform state is formed due to the summation of indices N (N takes values only 0 and 1) by modulo 2. To destroy this soliton a disclination loop in vector $\vec{V}$ field with the radius $\sim\xi_{\mathrm{dip}}$ must be created. Therefore the energy barrier for destroying the soliton $\sim g_D \xi_{\mathrm{dip}}^3 \ln(\xi_{\mathrm{dip}}/\xi)$.

III. LINEAR SOLITONS

Let us consider nonuniform distributions of the order parameter fields depending on two coordinates (x, y). In the case of ^{3}He-B the linear solitons are described by the classes of mappings of the plane crossing the soliton (see Fig. 8) into the space R_B and $\tilde{R}_B$ with the boundary of the plane being mapped into R_B^H $(R_B^H \subset \tilde{R}_B \subset R_B)$. The classes of such mappings form the conventional homotopy groups $\pi_2(R_B, R_B^H)$ and $\pi_2(\tilde{R}_B, R_B^H)$. In the simple case of Q being one point A

$$\pi_2(P, A) = \pi_2(P).$$

In general case, the group $\pi_2(P, Q \subset P)$ consists of the elements of the factor group [as in the case of $\pi_1(P, Q)$]

$$\pi_2(P)/\mathrm{Im}[\pi_2(Q) \to \pi_2(P)]$$

and besides these elements the group also con-

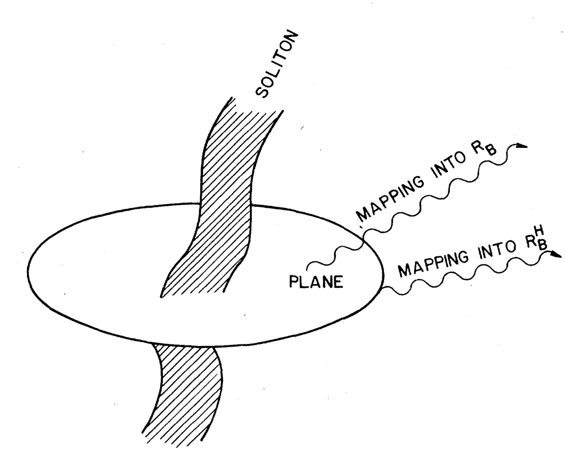

FIG. 8. Mapping of the plane intersecting the linear soliton into the order parameter space P with its boundary being mapped into the subspace $Q \subset P$ (here $P = R_B$, $Q = R_B^H$).

tains those elements of $\pi_1(Q)$ (classes of contours) which may be contracted into a point in the space P. It may be seen also from the exact sequence of homomorphisms

$$\pi_2(Q) \to \pi_2(P) \to \pi_2(P, Q) \to \pi_1(Q) . \tag{3.1}$$

If P is a topologically trivial space $[\pi_1(P) = \pi_2(P) = 0]$ then

$$\pi_2(P, Q) = \pi_1(Q) .$$

In the case of the B phase

$$\pi_2(\tilde{R}_B, R_B^H) = \pi_2(\tilde{R}_B) = Z, \quad \pi_2(R_B, R_B^H) = 0 .$$

So we have N classes of linear solitons in the vector $\vec{\omega}(x, y)$ field, where N takes the values from $-\infty$ to $+\infty$ and has the same analytical expression as in the case of point singularities in $\vec{\omega}$ field

$$N = \frac{1}{4\pi} \int dx\, dy \left(\vec{\omega}, \frac{\partial \vec{\omega}}{\partial x} \times \frac{\partial \vec{\omega}}{\partial y}\right), \tag{3.2}$$

where the integration is over the cross section of the soliton. The soliton with $N = 1$ is drawn in Fig. 1(a). The solutions of G.L. equations for arbitrary N may be found.[10] The size R of the soliton is defined by minimizing of the gradient and magnetic energy per unit length of soliton

$$E/L \sim \gamma \Delta^2 (1 + \xi_0^2/R^2) + F_{\text{magn}} R^2 .$$

Here we take into consideration the fourth-order terms in gradient energy, because the second-order terms do not depend on size R. So we have

$$R \sim (\xi_0 \xi_{\text{magn}})^{1/2}, \quad E/L \sim \gamma \Delta^2 . \tag{3.3}$$

A linear soliton with a topological invariant N may terminate on a hedgehog with the same N (3.2) where the integration now is around the hedgehog [see Fig. 1(b) for the case of $N = 1$].

$$N = \frac{1}{4\pi} \int d\theta\, d\phi \left(\vec{\omega}, \frac{\partial \vec{\omega}}{\partial \theta} \times \frac{\partial \vec{\omega}}{\partial \phi}\right).$$

Now let us consider the way of classifying the possible topologically stable nonuniform states in a cylindrical vessel with ^{3}He-A. The radius of the vessel for example $R \gg \xi_{\text{magn}} \gg \xi_{\text{dip}} \gg \xi$. On the boundary of the vessel $\vec{l} \parallel \vec{\nu}$ ($\vec{\nu}$ is the normal vector of the surface of the vessel). The order parameter space on the boundary is S^1—the range of the condensate phase variation (the phase ϕ is well defined on the boundary).[11] We must consider the mapping of the vessel into spaces $R_A^H, \tilde{R}_A, R_A, R^{18}$ with the boundary of the vessel being mapped in S^1. We have

$$\pi_2(R_A^H, S^1) = 0 , \tag{3.4}$$

$$\pi_2(\tilde{R}_A, S^1) = Z . \tag{3.5}$$

The elements of the last group are characterized

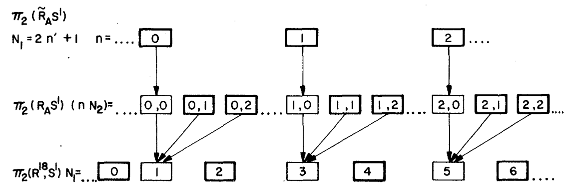

FIG. 9. Map of the homomorphisms.

by the integer n. These elements describe the superflow along the boundary only with odd $(2n+1)$ quanta of the superfluid velocity $\vec{v}^s$ circulation around the vessel. (Even quanta of circulation result in the singular vortex line in the vessel with the space R^{18} in a core.)

$$\pi_2(R_A, S^1) = Z + Z . \tag{3.6}$$

The elements of this group are characterized by two integers (n, N_2), n is the same as in (3.5) and N_2 is the topological invariant of the type (3.2) describing the linear solitons in vector $\vec{V}(x, y)$ field

$$N_2 = \frac{1}{4\pi} \int dx\, dy \left(\vec{V}, \frac{\partial \vec{V}}{\partial x} \times \frac{\partial \vec{V}}{\partial y}\right) .$$

The next group

$$\pi_2(R^{18}, S^1) = \pi_1(S^1) = Z \tag{3.7}$$

describes the superflow along the boundary with any integer quantum of circulation of the superfluid velocity.

One may see that some elements of one group [for example $\pi_2(\tilde{R}_A, S^1)$] are simultaneously the elements of the other groups $[\pi_2(R_A, S^1), \pi_2(R^{18}, S^1)]$. It means that there are such mappings described by the elements of $\pi_2(R_A, S^1)$ in which the space R_A may be continuously contracted into the subspace $\tilde{R}_A$. To make the classification in a nonambiguous way one must exclude the elements of the group $\pi_2(\tilde{R}_A, S^1)$ from the elements of $\pi_2(R_A, S^1)$ and so on. For this purpose one must consider the homomorphisms $\pi_2(\tilde{R}_A, S^1) \to \pi_2(R_A, S^1) \to \pi_2(R^{18}, S^1)$ which take place when the space $\tilde{R}_A$ extends up to the space R_A and then up to the R^{18}. The map of these homomorphisms is constructed in Fig. 9.

All possible topologically stable states in cylindrical vessel taking into account the boundary conditions are described by the elements in heavy rectangles. Why is it necessary to leave the elements only in heavy rectangles? The case is that the sequence of the order parameter subspaces

$$S^1 \subset R_A^H \subset \tilde{R}_A \subset R_A \subset R^{18}$$

corresponds to the sequence of decreasing scales of length

$$R > \xi_{\text{magn}} > \xi_{\text{dip}} > \xi .$$

Every soliton described by the given class of mappings tends to enlarge its size in order to diminish the gradient energy (in some cases one must take into consideration the fourth order terms in gradient energy). For the soliton described by the element in thin rectangles the topology allows to deform its mapping into the next smaller subspace and therefore to enlarge its size and diminish its energy. But the topology does not allow this for the elements in heavy rectangles.

Let us enumerate all the textures and their size. The element n of the first line describes the state in a vessel with n coreless vortices (the size of the core $\sim\xi_{magn}$). This state takes place in the rotating ^{3}He-A. The value of n depends on the angular velocity ω of the rotation

$$n = 4m\,\omega S/\hbar\,,$$

S is the area of the cross section of the vessel[12] n may be written

$$2n+1 = \int \frac{dx\,dy}{2\pi}\left(\vec{1}, \frac{\partial\vec{1}}{\partial x}\times\frac{\partial\vec{1}}{\partial y}\right).$$

The elements of the second line (0.1), (0.2), ..., $(0, N_2)$ describe the states in the vessel with the solitons in vector field $\vec{V}$, the radius of soliton being equal $(\xi\xi_{dip})^{1/2}$. The other elements of this line in the heavy rectangles (n, N_2), $n\neq 0$, $N_2\neq 0$, are the composition of the n coreless vortices and the solitons with the total topological invariant N_2.

The elements in heavy rectangles of the third line describe states in the vessel with an even number of quanta of circulation of the superfluid velocity around the boundary of the vessel. In this case there must be at least one vortex line with a singular core (the radius of the core $\sim\xi$) inside the vessel.

The classification of the states in a cylindrical vessel with other relations between the scales of length (for example $R \gg \xi_{dip} \gg \xi_{magn}$ or $\xi_{magn} \gg R \gg \xi_{dip}$) or in the vessel of the form of the torus[11] may be done in analogical way. In the case of the torus one must find the mapping of the space inside the torus into $R_A^H, \tilde{R}_A, R_A, R^{18}$ with the boundary of the torus being mapped into S^1.

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