Microscopic investigation of the proximity effect in a finite geometry

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The system of a finite superconductor S_1 in proximity with a weaker superconductor S_2 is considered. The order parameter is assumed to vary in space as $\Delta_1 \cos[q(z-D)]$ $(0 \leq z \leq D)$ in S_1, and as $\Delta_2 \cosh[K(z+L)]$ $(-L \leq z \leq 0)$ in S_2. The Bogoliubov-deGennes equations were solved for the two regions: $z<0$. The transition temperature $T_{c\,\mathrm{vs}}$ of the system is evaluated for the case where S_2 is a thick normal layer. The dependence of $T_{c\,\mathrm{NS}}$ on the thickness D is in agreement with experimental results.

I. INTRODUCTION

The Bogoliubov-deGennes[1] (BdG) equations for the two-component wave functions (U, V) of the excitations in an inhomogeneous superconductor were extensively treated in the last few years. In the absence of a magnetic field, a procedure was devised for an analytic solution of these equations.[2] This method was used to calculate the wave functions for order parameters of the form $\tanh \alpha z^2$ and of the form $\Delta_0 - \Delta_1 e^{-\alpha z}$.[3] These forms for the order parameter (or pair-potential) may describe the behavior of a semi-infinite superconductor. For a finite superconductor the boundary condition of the free interface is

$$\left.\frac{d\Delta}{dz}\right|_{\text{free interface}} = 0 \tag{1.1}$$

We consider here a system of a superconductor $S_1(0 \leqslant z \leqslant D)$ in contact with a weaker superconductor $S_2(-L \leqslant z \leqslant 0)$. Near the transition temperature of the system, $T_{cs_1s_2}$, the linearized Ginsburg-Landau equation provides us with the behavior of the order parameter.

$$\Delta(z) = \Delta_1 \cos[q(z-D)]\Theta(z) + \Delta_2 \cosh[K(z+L)]\Theta(-z), \tag{1.2}$$

where

$$\Theta(z) = 0(1) \quad \text{for} \quad z<0(z>0).$$

The order parameter of Eq. (1.2) is chosen for the present investigation. In Sec. II the BdG equations are solved within the Andreev[4] approximation (or WKBJ approximation). In Sec. III we consider the case where S_2 is a thick normal layer ($\Delta_2 = 0$). We construct the wave functions which match smoothly at $z=0$ and vanish at the free interfaces. The eigenenergies then follow. In Sec. IV the transition temperature of the $S-N$ system is calculated.

II. SOLUTION OF THE BdG EQUATIONS

The BdG equations in the absence of a magnetic field are

$$\begin{aligned} (E+E_\mathrm{F})u &= (\hbar^2/2m)\nabla^2 u + \Delta v, \\ (E-E_\mathrm{F})v &= (\hbar^2/2m)\nabla^2 v + \Delta v \end{aligned} \tag{2.1}$$

where E is the energy of the excitation relative to the Fermi energy, E_F, $\hbar$ is the Planck constant over 2π, and m is the electronic effective mass. Substituting $(u, v) = (\bar{u}, \bar{v}) \exp(i\vec{k}_\mathrm{F} \cdot \vec{r})$ and neglecting terms of the order of Δ/E_F (Andreev or WKBJ approximation) these equations become

$$\begin{aligned} E\bar{u} &= -i\hbar V_\mathrm{F} p \frac{d\bar{u}}{dz} + \Delta\bar{v}, \\ E\bar{v} &= i\hbar V_\mathrm{F} p \frac{d\bar{v}}{dz} + \Delta\bar{u}, \end{aligned} \tag{2.2}$$

where V_F is the Fermi velocity and $p = V_{Fz}/V_\mathrm{F}$. The functions

$$f_{1,2} = \bar{u} \pm i\bar{v} \tag{2.3}$$

obey[2]

$$\begin{aligned} Ef_1 &= -i\left(\hbar V_\mathrm{F} p \frac{d}{dz} + \Delta\right) f_2, \\ Ef_2 &= -i\left(\hbar V_\mathrm{F} p \frac{d}{dz} - \Delta\right) f_1. \end{aligned} \tag{2.4}$$

These equations were decoupled to give[2]

$$E^2 \underset{l=1,2}{f_l} = \left[-(\hbar V_\mathrm{F} p)^2 \frac{d^2}{dz^2} + \Delta^2 - (-1)^l \hbar V_\mathrm{F} p \left(\frac{d\Delta}{dz}\right)\right] f_l. \tag{2.5}$$

We realize that the substitution

$$f_{1,2} = \exp\left(\pm \int \frac{\Delta}{\hbar V_\mathrm{F} p}\, dz\right) g_{1,2}, \tag{2.6}$$

where g_l obeys

$$E^2 g_l = \left(-(\hbar V_F p)^2 \frac{d^2}{dz^2} + (-1)^l 2\Delta\hbar V_F p \frac{d}{dz} \right) g_l \qquad (2.7)$$

may be helpful especially for $E \ll \Delta\max$.

For the order parameter of Eq. (1.2) in the region $z \geq 0$ we expand the solution in a Fourier series:

$$g_l = e^{i\rho q z} \sum_{n=-\infty}^{\infty} a_n^l e^{inq(z-D)} = e^{i\rho q z} G_l \qquad (2.8)$$

Here the coefficients are determined from the recurrence relations

$$[(n+\rho)^2 - \epsilon^2] a_n^l + (-1)^l i\delta[(\rho+n+1)a_{n+1}^l + (\rho+n-1)a_{n-1}^l] = 0 \qquad (2.9)$$

where

$$\delta = \Delta_1/\hbar V_F p q \quad \text{and} \quad \epsilon = E/\hbar V_F p q\,.$$

In order to deal with real quantities we use $b_n^l = (i)^n a_n^l$ determined by

$$[(n+\rho)^2 - \epsilon^2] b_n^l - (-1)^l \delta[(\rho+n+1)b_{n+1}^l - (\rho+n-1)b_{n-1}^l] = 0 \qquad (2.9')$$

The characteristic parameter ρ is determined so that the infinite determinant of coefficients vanishes. The evaluation of ρ may be done by the technique used for solving Matieu functions. Near $T_{CS_1 S_2}$, $\delta \to 0$, and we find

$$\epsilon^2 - \rho^2 \cong 1 - 2\delta\rho^2/(2\rho^2 + 2\epsilon^2 - 1)\,,$$

$$b_{\pm 1}^l \cong (-1)^l [(\delta\rho/(2\rho \pm 1)] b_0^l\,,$$

where

$$\rho \neq \tfrac{1}{2} + m\,, \quad m = \pm 1, \pm 2, \cdots\,. \qquad (2.10)$$

For a given value of E, ρ, and $-\rho$ are two independent solutions ($\rho \neq m - \frac{1}{2}$). Substituting the solutions back into Eq. (2.4) we find

$$a_0^1/a_0^2 = \mathrm{sgn}(\rho p)\,. \qquad (2.11)$$

In the regions around $\epsilon = m - \frac{1}{2}$ (unstable regions), ρ has an imaginary part. However, the halfwidth of these regions is of order $(\frac{1}{2}\rho)^m$ ($m > 0$) and are very narrow near $T_{cs_1 s_2}$.

The wave functions for a given E and $|p|$ are four:

$$\hat{\psi}_a^b = \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}_a^b = N e^{iabpqz} \begin{pmatrix} \exp\{a\,\delta\sin[q(z-D)]\} G_1(ab\rho) + \exp\{-a\delta\sin[q(z-D)]\} G_2(ab\rho) \\ -i\exp\{a\,\delta\sin[q(z-D)]\} G_1(ab\rho) + ib\exp\{-a\delta\sin[q(z-D)]\} G_2(ab\rho) \end{pmatrix}, \qquad (2.12)$$

where N is a normalization constant, $a = \mathrm{sgn}(V_F z)$, $b = 1(-1)$ corresponds to an electron (hole) like excitations, and ρ is taken as the characteristic parameter with $\mathrm{Re}\rho \geq 0$.

In the region $-L \leq z \leq 0$ the solutions are formally obtained by the coordinate change $z \to y = iz$, i.e.,

$$g_l = e^{i\rho k y} \sum_{n=-\infty}^{\infty} C_n^l e^{ink(y+iL)}\,, \qquad (2.13)$$

where the C_n^l and ρ are obtained from the set of equations

$$[(\rho+n)^2 + \epsilon'^2] C_n^l \mp \delta'[(\rho+n+1)C_{n+1}^l + (\rho+n-1)C_{n-1}^l] = 0\,, \qquad (2.14)$$

where $\delta' = \Delta_2/\hbar V_F k p$ and $\epsilon' = E/\hbar V_F p k$. For $\delta' \to 0$ we find

$$\rho^2 + \epsilon'^2 \cong 2\delta'^2 \rho^2/(4\rho^2 + 1)\,, \qquad (2.15)$$

with a pure imaginary ρ.

III. EIGENSOLUTIONS FOR SN GEOMETRY. STABLE ϵ REGIONS

We continue our investigation for a simpler geometry where the region $-L \leq z \leq 0$ is thick ($L \to \infty$) and is occupied by a normal metal (N). The solutions in the normal metal are

$$\hat{\psi}_a^b = \begin{pmatrix} 1+b \\ 1-b \end{pmatrix} \exp\left(\frac{iabE}{\hbar V_F p}\right)(z+L)\,. \qquad (3.1)$$

For simplicity it is assumed that the Fermi velocity and momentum of the electrons in the two regions $z \lessgtr 0$ are the same.

The linear combinations which satisfy the boundary conditions at $z = 0, -L, D$ can be written

$$\begin{pmatrix} U \\ V \end{pmatrix} = e^{i\vec{k}_{F\perp}\cdot\vec{r}} \sum_{a=\pm} a\hat{\Psi}_a e^{iak_F p(z+L)}\,. \qquad (3.2)$$

In the normal region ($z \leq 0$)

$$\hat{\Psi}_a = \begin{matrix} A\exp\left(\dfrac{iaE}{\hbar v_F p}\right)(z+L) \\ B\exp\left(\dfrac{-iaE}{\hbar v_F p}\right)(z+L)\,. \end{matrix} \qquad (3.3)$$

In the superconducting region

$$\hat{\Psi}_a = N_a(\hat{\psi}_a^{+1} e^{-\rho q D} + C_a \hat{\psi}_a^{-1} e^{i\rho q D})\,. \qquad (3.4)$$

From Eq. (2.8) we find ($\epsilon \neq m-\frac{1}{2}$),

$$a_n^1/a_0^1=(a_n^2/a_0^2)^*. \tag{3.5}$$

Therefore at $z-D=0$ we have in the stable regions of ϵ

$$\psi_a^+=\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}e^{ia\rho qD};\quad \psi_a^-=i^{-1}\begin{pmatrix}\sin\phi\\ \cos\phi\end{pmatrix}e^{-ia\rho qD}, \tag{3.6}$$

where the upper ± sign stands for $b=\pm1$, and $\rho>0$. For $T\simeq T_{cS_1S_2}$ $(\delta\to0)\phi=2\delta\epsilon/(4\epsilon^2-1)$.

The coefficients can be chosen as $C_+=C_-=C$ and $N_\pm=e^{\pm i\gamma}N$, where

$$\exp\{i[\gamma+k_F(D+L)]\}=1.$$

For a thick normal layer it is useful to allow k_F a slight p dependence [of order $1/(L+D)$] so that we can let $\gamma=0$.

We write now the components of $\psi_+^+(z=0)$ in terms of exponents

$$\exp\left[\xi_{20}+\left(\frac{i\eta_{10}}{2}\right)\right]\begin{pmatrix}e^{\eta_{20}/2+i\xi_{10}}\\ e^{-\eta_{20}/2-i\xi_{10}}\end{pmatrix}=\begin{pmatrix}\overline{u}(0)\\ \overline{v}^*_{(0)}\end{pmatrix}_+^+. \tag{3.7}$$

Matching the solutions and their derivatives at $z=0$, one can verify[5] that $C=\pm1, B=\pm A$ are real constants, and that the respective eigenenergies are determined by

$$\tan(\rho qD-\xi_{10})+\frac{1\pm e^{-\eta_{20}}}{1\mp e^{-\eta_{20}}}\tan\left(\frac{E}{\hbar V_F p}L-\tfrac{1}{2}\eta_{10}\right)=0. \tag{3.8}$$

Here the ± sign corresponds to solutions with $C=\pm1$. The coefficients of the wave functions at the normal region are given by

$$(A/N)_\pm^2=e^{2\xi_{20}}\,\mathrm{Sinh}\,\eta_{20}/Z_\pm(E), \tag{3.9}$$

where

$$Z_\pm(E)=\frac{(1\mp e^{-\eta_{20}})/(1\pm e^{-\eta_{20}})}{\cos^2(\rho qD-\xi_{10})+[(1\mp e^{-\eta_{20}})/(1\pm e^{-\eta_{20}})]^2\sin^2(\rho qD-\xi_{10})}. \tag{3.10}$$

In Appendix A it is shown that

$$e^{\xi_{20}}\sinh\eta_{20}=\frac{v_{Sz}}{v_{Nz}}=\left(\frac{\partial\rho}{\partial\epsilon}\right)^{-1}, \tag{3.11}$$

where v_{Sz} and v_{Nz} is the z component of the excitation velocity in the superconductive and normal metal, respectively.

Therefore,

$$\left(\frac{A}{N}\right)_\pm^2=\frac{1}{Z_\pm(E)}\left(\frac{\partial\rho}{\partial\epsilon}\right)^{-1} \tag{3.9$'$}$$

For $L\gg\xi(T)$ it is convenient to rewrite Eq. (3.8) in the form

$$\pi\eta=EL/\hbar v_F p-\tfrac{1}{2}\eta_{10}+\delta_{0\pm}, \tag{3.8$'$}$$

where

$$\delta_{0\pm}=\arctan\left(\frac{1\mp e^{-\eta_{20}}}{1\pm e^{-\eta_{20}}}\tan(\rho qD-\xi)\right).$$

The density of states for a given energy and $|p|$ is given by dn/dE,

$$\Xi^\pm(E,p)=\frac{1}{\pi}\left[\left(\frac{L}{\hbar v_F p}-\frac{1}{2}\frac{d\eta_{10}}{dE}\right)+Z(E)\frac{\partial\rho}{\partial\epsilon}\left(\frac{D}{\hbar v_F p}-\frac{\partial\xi_{10}}{\partial\epsilon}\right)\right]. \tag{3.12}$$

It is important to note that there is no divergence in the density of states at $E=\Delta_1$. For $\delta\ll1$ we find that $e^{-\eta_{20}}\alpha\delta\rho/(4\rho^2-1)$ and therefore $Z_\pm(E)\simeq1$. Only when $\rho\to m-\frac{1}{2}$, does $e^{-\eta_{20}}\to1$, and as a result $Z_\pm(E)\to0$.

The product $Z_\pm(E)\partial\rho/\partial\epsilon$, however, remains finite. (See Appendix B).

IV. EIGENSOLUTIONS IN THE UNSTABLE REGION

In the unstable regions the characteristic exponent becomes complex

$$\rho=\pm(m-\tfrac{1}{2}+ix),\quad m=1,2,\ldots. \tag{4.1}$$

As mentioned above the half width of these regions are of the order of $(\frac{1}{2}\delta)^m$. Therefore, for the calculations of the transition temperature of the system only the first, unstable region is important. In this region we find that when $\delta\to0$,

$$x=[(\tfrac{1}{2}\delta)^2-y^2]^{1/2}, \tag{4.2}$$

where $y=|\epsilon|-\frac{1}{2}$.

The four solutions for a given E and $|p|$ are

$$\hat{\phi}_+^+=\begin{matrix}g^{(1)}\\ g^{(2)}\end{matrix}\,{}_+^+ =e^{-xq(z-D)}\begin{pmatrix}\cos[\frac{1}{2}q(z-D)-\phi_0]\\ i\sin[\frac{1}{2}q(z-D)-\phi_0]\end{pmatrix}+O(\delta),$$

$$\hat{\phi}_+^-=e^{xq(z-D)}\begin{pmatrix}i\sin[\frac{1}{2}q(z-D)+\phi_0]\\ \cos[\frac{1}{2}q(z-D)+\phi_0]\end{pmatrix}+O(\delta), \tag{4.3}$$

$$\hat{\phi}_-^+=e^{xq(z-D)}\begin{pmatrix}\cos[\frac{1}{2}q(z-D)+\phi_0]\\ -i\sin[\frac{1}{2}q(z-D)+\phi_0]\end{pmatrix}+O(\delta),$$

$$\hat{\phi}_-^-=e^{-xq(z-D)}\begin{pmatrix}-i\sin[\frac{1}{2}q(z-D)-\phi_0]\\ \cos[\frac{1}{2}q(z-D)-\phi_0]\end{pmatrix}+O(\delta), \tag{4.4}$$

where

$$e^{i\phi_0}=2(x-iy)/\delta\ .$$

At $z=D$ the wave functions constructed from the solutions are

$$\hat{\psi}_a^+=\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}_a^+=\begin{pmatrix}e^{-i\phi_0}\\ -i\,e^{i\phi_0}\end{pmatrix}\ ,$$

$$\hat{\psi}_a^-=\begin{pmatrix}e^{i\phi_0}\\ i\,e^{-i\phi_0}\end{pmatrix}\ . \tag{4.5}$$

The linear combinations

$$\psi_a'^{\pm}=(\psi_a^+\pm\psi_a^-)e^{iap qD} \tag{4.6}$$

are of the form given in Eq. (3.6) (for $z=D$) at least in the present approximation. The calculation of the density of states and the linear combinations which satisfy the boundary conditions can be done on the same lines as in the former section.

V. CONTRIBUTION OF "STABLE REGIONS" TO Δ

The self-consistency condition for the order parameter is

$$\Delta=g\mathfrak{F}\equiv g\sum_n U_nV_n^*\tanh\left(\frac{E_n}{2k_BT}\right)\ . \tag{5.1}$$

The contribution of the stable states [ϵ is not in the region $m-\frac{1}{2}\pm(\frac{1}{2}\delta)^m$] is

$$\begin{aligned}(UV^*)^{\pm}=&\operatorname{Im}(g^{(1)}g^{(2)*})\pm\operatorname{Re}(e^{2ipqD}g^{(1)}g^{(2)}\\ &\pm\tfrac{1}{2}\cos 2k_Fp(L+z)\\ &\times[\operatorname{Im}[(g^{(1)^2}-g^{(2)^2})e^{2ipq}]\\ &+|g^{(1)}|^2+|g^{(2)}|^2]\end{aligned} \tag{5.2}$$

where the ± sign refers to the sign of the coefficient C, and $g^{(1,2)}$ are those with $a=b=1$. Using Eq. (3.8) we find

$$e^{2ipqD}=2e^{2i\xi_{10}}\left\{\left[1-i\,\frac{1\pm e^{-\eta_{20}}}{1\mp e^{-\eta_{20}}}\times\tan\left(\frac{EL}{\hbar V_Fp}-\tfrac{1}{2}\eta_{10}\right)\right]^{-1}-1\right\}. \tag{5.3}$$

Since we assume $L\gg\xi(T)$, the tangent function oscillates very rapidly. Averaging over these oscillations by integrating over $\Theta=EL/\hbar V_Fp-\frac{1}{2}\eta_{10}$ from 0 to 2π assuming all other quantities as constants one finds[5]

$$\langle e^{2ipD}\rangle_{\text{osc}}^{\pm}=\mp e^{2i\xi_{10}-\eta_{20}}=\mp\bar{v}^*(0)/\bar{u}(0)\equiv\mp iR\ . \tag{5.4}$$

Therefore

$$\langle UV^*\rangle_{\text{osc}}^{\pm}=\operatorname{Im}[g^{(1)}g^{(2)^*}+Rg^{(1)}g^{(2)}]\ . \tag{5.5}$$

Expanding the functions $g^{(1,2)}$ in a power series of δ near $T_{c\,\mathrm{NS}}$

$$\begin{aligned}\langle UV^*\rangle_{\text{osc}}=[2\delta\epsilon/(4\epsilon^2-1)]\,&[\cos q(z-D)\\ &+\operatorname{Re}\{e^{2ipqz}[-\cos(qD)\\ &+i\sin(qD)/2\epsilon\,]\}]+O(\delta^2).\end{aligned} \tag{5.6}$$

The contribution of the excitations in the stable regions is then calculated to be (See Appendix B for the details of the summation procedure)

$$\begin{aligned}\mathfrak{F}&=N(0)g\int_0^1dp\int_0^{\omega_D}dE\langle UV^*\rangle_{\text{osc}}\tanh\left(\frac{E}{2k_BT}\right)\\ &=N(0)g\Delta_1\int_0^1dp\,\pi T\sum_{n=0}^{\omega_D/\pi T}\frac{2|\omega_n|}{4|\omega_n|^2+(\hbar V_Fqp)^2}\\ &\quad\times\left[\cos q(z-D)+e^{\frac{-2|\omega_n|z}{\hbar V_Fp}}\left(\cos qD-\frac{\sin qD}{2|\omega_n|}\right)\right],\end{aligned} \tag{5.7}$$

where

$$\omega_n=i\pi k_BT(2n+1),\quad n=1,2,\ldots.$$

VI. CONTRIBUTION OF UNSTABLE REGIONS

As mentioned in Sec. IV. only the contribution of the first unstable region is important. The contribution from states of this region in ϵ has the same form as that of the stable regions provided that the primed wave functions of Eq. (4.6) are used. We find

$$\begin{aligned}\operatorname{Im}g'^{(1)}g'^{(2)*}&=\sin 2\phi_0\cos q(z-D)\\ &=(2y/\delta)\cos q(z-D)\ ,\end{aligned} \tag{6.1}$$

$$\begin{aligned}iR&=\bar{v}'^*(0)/\bar{u}'(0)=\tan\phi_0\\ &=-y/[\tfrac{1}{2}\delta+(\tfrac{1}{2}\delta^2+y^2)^{1/2}]+O(\delta)\ ,\end{aligned} \tag{6.2}$$

$$g'^{(1)}g'^{(2)}=\cos q(z-D)+O(\delta) \tag{6.3}$$

where

$$g'^{(1,2)}=(u'\pm iv')_+^+\ .$$

Both $\operatorname{Im}g'^{(1)}g'^{(2)*}$ and $Rg'^{(1)}g'^{(2)}$ are odd functions of y and vanish under summation. We therefore conclude that the contribution of the unstable states is at most of the order δ^2.

VII. DETERMINATION OF q AND $T_{c\,\mathrm{NS}}$

The first term in the recalculated order parameter (5.7) has the same z dependence as the "input" order parameter (1.2). We demand that the other contributions are vanish "on the average." This demand yields

$$q\tan qD=1/b(T)\ , \tag{7.1}$$

where the extrapolation length b is calculated in

Appendix C. The transition temperature is then fixed by the requirement that the coefficient of the $\cos q(z-D)$ term in Eq. (5.7) would be equal to $[N(0)g]^{-1}$. Performing the integration over p and using

$$[N(0)g]^{-1} = \ln\frac{1.14\omega_D}{k_B T_{cS}},$$

$$\ln\frac{1.14\omega_D}{k_B T} = \sum_{n=0}^{\omega_D/T}(n+\tfrac{1}{2})^{-1},$$

the transition temperature is fixed according to

$$\ln\left(\frac{T_{c\,\mathrm{NS}}}{T_{cS}}\right) = \sum_{n=0}^{\infty}\left[\tan^{-1}\left(\frac{y}{n+\frac{1}{2}}\right) - \frac{y}{n+\frac{1}{2}}\right]y^{-1} = y^{-1}\,\mathrm{Im}\,\ln\Gamma(\tfrac{1}{2}-iy) + \psi(\tfrac{1}{2}), \tag{7.2}$$

where T_{cS} is the transition temperature of the bulk superconductor, $\Gamma(z+1) = z!$, $\psi(z) = \partial \ln\Gamma(z)/\partial z$, and $y = \hbar V_F q/4\pi k_B T_{c\,\mathrm{NS}}$. For large values of D, $T_{c\,\mathrm{NS}}$ is close to T_{cS} and approximately $\ln(T_{c\,\mathrm{NS}}/T_{cS}) = 1 - T_{c\,\mathrm{NS}}/T_{cS}$. In this limit we can approximate the arctangent by the first two terms of its Taylor series, getting

$$1 - T_{c\,\mathrm{NS}}/T_{cS} \simeq (\pi\hbar V_F/8Dk_B T_{c\,\mathrm{NS}})^2, \tag{7.3}$$

where we use the value $q = \pi/2D$ obtained in Appendix C for $T_{c\,\mathrm{NS}} \to T_{cS}$. The behavior of $T_{cS} - T_{c\,\mathrm{NS}} \propto D^{-2}$ for large values of D has been verified experimentally.[6]

On the other extreme, where $T_{c\,\mathrm{NS}} \to 0$, the argument of the function is large in absolute value and we can use Sterling's formula to get

$$\ln\left(\frac{T_{c\,\mathrm{NS}}}{T_{cS}}\right) = \ln\left(\frac{c\,4\pi k_B T_{c\,\mathrm{NS}} D}{0.825\hbar V_F}\right) - \frac{5}{24}\left(\frac{4\pi D k_B T_{c\,\mathrm{NS}}}{0.825\hbar V_F}\right)^2 + O(T_{c\,\mathrm{NS}}^3), \tag{7.4}$$

where $c = \exp[-\psi(\frac{1}{2}) - 1] = 2.62\ldots$, and we use $q = 0.825/D$ (Appendix C).

There is a minimum value of D for which the system turns superconductive,

$$D_m = 0.825\hbar V_F/c\,4\pi k_B T_{cS}. \tag{7.5}$$

Near this thickness $D \gtrsim D_m$, the transition temperature increases with increasing D as

$$(T_{c\,\mathrm{NS}}/T_{cS})^2 \cong (24/5c^2)(1 - D_m/D). \tag{7.6}$$

VIII. SUMMARY AND CONCLUSIONS

The system of two superconductors in proximity is investigated.

A solution is found for the Bogoliubov-deGennes equations with the order parameter of Eq. (1.2). We then turn to investigate the system of a finite superconductor in proximity with semi-infinite normal metal. This investigation is valid in the clean limit. The eigenfunctions and eigenenergies were calculated. It is shown that there is not divergence in the density of states at $E = \Delta_{\max}$ in contrary to the case of homogeneous and semi-infinite inhomogeneous superconductor. The quantity $\delta = \Delta(T)/\hbar V_F q$ is found to be a convenient expansion parameter, near the transition temperature of the system. The transition temperature of the system, $T_{c\,\mathrm{NS}}$, as a function of D is calculated from the self-consistent condition. It is found that $T_{c\,\mathrm{NS}} = 0$ for thicknesses less than $D_m \approx \xi_0 = \hbar V_F/2\pi k_B T_{cS}$. The dependence of $T_{c\,\mathrm{NS}}$ on D for relatively thick superconducting layer is in agreement with experimental results.

The result $b = 7\tau(3)\hbar V_F/3\pi^3 T$ $(D \to \infty)$ obtained here is in agreement with the result obtained for a semi-infinite superconductor in contact with semi-infinite normal metal.[7]

APPENDIX A

To calculate the quantity

$$e^{\xi_{20}}\sinh\eta_{20} = |\bar{u}(0)|^2 - |\bar{v}(0)|^2 \tag{A1}$$

we first show that

$$|\bar{u}(z)|^2 - |\bar{v}(z)|^2 = (\text{const}). \tag{A2}$$

This is a general property of wave functions in an inhomogeneous superconductor. To prove Eq. (A2), we multiply Eq. (2.2a) by u^* and Eq. (2.2b) by v^*. Taking the imaginary part of both sides of the equations then leads to

$$\hbar V_F p\,\frac{\partial|\bar{u}(z)|^2}{\partial z} = \mathrm{Im}\Delta\bar{u}^*\bar{v}, \tag{A3a}$$

$$\hbar V_F p\,\frac{\partial|\bar{v}(z)|^2}{\partial z} = \mathrm{Im}\Delta\bar{u}^*\bar{v}, \tag{A3b}$$

and Eq. (A2) follows. The quantity

$$V_F p(|\bar{u}|^2 - |\bar{v}|^2) = V_{sz} \tag{A4}$$

is the velocity of the excitation in the z direction.

To show this we consider an electron from the normal region incident at the SN interface $(z=0)$. It is partially reflected as a hole[4] and partially transmitted as a quasielectron. The corresponding wave functions are

$$\hat{\psi}_{\mathrm{in}} = \begin{pmatrix}1\\0\end{pmatrix}\exp\left(i\vec{k}_F\cdot\vec{r} + i\frac{E}{\hbar V_F p}z\right),$$

$$\hat{\psi}_{\mathrm{tran}} = t\begin{pmatrix}\bar{u}(z)\\ \bar{v}(z)\end{pmatrix}e^{i\vec{k}_F\cdot\vec{r}},$$

$$\hat{\psi}_{\mathrm{ref}} = r\begin{pmatrix}0\\1\end{pmatrix}\exp\left(i\vec{k}_F\cdot\vec{r} - \frac{iE}{\hbar V_F p}z\right). \tag{A5}$$

The reflection and transmission coefficients are

$$R=|r|^2=|\bar{v}(0)/\bar{u}(0)|^2,$$

$$T=(V_{Sz}/V_{Nz})|t^2|=(V_{Sz}/V_{Nz})|1/\bar{u}(0)|^2. \tag{A6}$$

Here $V_{Nz}=V_{\mathrm{F}}p$. The conservation law

$$R+T=1 \tag{A7}$$

then leads to Eq. (A4).

For the wave functions under investigation,

$$\frac{V_{Sz}}{V_{Nz}}=\frac{1}{V_{\mathrm{F}}p}\,\frac{\partial E}{\partial\rho}=\left(\frac{\partial\rho}{\partial\epsilon}\right)^{-1} \tag{A8}$$

Equation (3.9′) is then proved by combining Eqs. (A1), (A4), (A8), and (3.9).

APPENDIX B

The normalization of the wave functions is fixed by $\int_{-L}^{D}(|U|^2+|V|^2)\cong 4(A^2L+N^2D)=1$ or

$$N^2=\tfrac{1}{4}[D+(A/N)^2L]^{-1}. \tag{B1}$$

Using the value of $(A/N)^2$ [Eq. (3.9)] neglecting $\partial\eta_{10}/\partial E$ and $\partial\xi_{10}/\partial\epsilon$ compared to $(L+D)/\hbar V_{\mathrm{F}}p$, we find

$$(\Xi N^2)^{\pm}=\frac{1}{\pi\hbar V_{\mathrm{F}}p}\,Z_{\pm}\frac{\partial\rho}{\partial\epsilon} \tag{B2}$$

and (B2)

$$(\Xi A^2)^{\pm}=1/\pi\hbar V_{\mathrm{F}}p.$$

To estimate ΞN^2 near the transition temperature we calculate $e^{-\eta_{20}}=|\bar{v}(0)/\bar{u}(0)|$ to find in leading terms

$$e^{-\eta_{20}}=|a_1e^{-iqD}+a_{-1}e^{iqD}|/|a_0| \tag{B3}$$

where the a's are the coefficients appearing in the expansion of the wave-function component $g^{(1=1)}$.

For most of the stable region $(a_{\pm1}/a_0)\propto\delta$ and $e^{-\eta_{20}}\ll 1$. Therefore $Z_{\pm}(E)=1+0(\delta)$. In this region of ϵ, $\partial\rho/\partial\epsilon=1+0(\delta^2)$, and therefore $(\Xi N^2)^{\pm}=1/\pi\hbar V_{\mathrm{F}}p$ is constant. Only very close to the stable regions edges do the above considerations not hold. Near the first unstable region one finds

$$\frac{\partial\rho}{\partial\epsilon}\simeq\frac{y}{[y^2-(\frac{1}{2}\delta)^2]^{1/2}}, \tag{B4}$$

where

$$y=|\epsilon|-\tfrac{1}{2}\geq\tfrac{1}{2}\delta.$$

At the same time, $a_{-1}/a_0\to 1$, and we find

$$e^{-\eta_{20}}=1-[y^2-(\tfrac{1}{2}\delta)^2]^{1/2}/y. \tag{B5}$$

As $Z_{\pm}(E)=0$ for $e^{-\eta_{20}}=1$, we deduce that

$$Z_{\pm}(E)\propto[y^2-(\tfrac{1}{2}\delta)^2]^{1/2}/y. \tag{B6}$$

The product $Z_{\pm}(E)\partial\rho/\partial\epsilon$ remains of order 1 also in the vicinity of the stable region edge. The same result can be found for $m>1$. We therefore conclude that near the transition temperature $(\Xi N^2)_{\pm}$ can be taken as constant.

The summation procedure is then

$$\sum_{n}^{E_n=\omega_0}N^2\left(U_nV_n^*\tanh\frac{E_n}{2k_BT}\right)=\int_0^1 p\,dp\int_0^{\omega_D}\Xi N^2(\ldots)\,dE\cong N(0)\int_0^1 dp\int_0^{\omega_D}dE\,(\ldots), \tag{B7}$$

with a correction of at most of order δ.

APPENDIX C

The order parameter $g\mathfrak{F}$, calculated from Eq. (5.7), contains terms with spatial behavior different from that of the "input" order parameter (1.2). Therefore, we can require only self-consistency on the average,

$$\int_0^D(g\mathfrak{F}-\Delta)\,dz=0. \tag{C1}$$

Integrating over z and requiring that the sum of the last two terms on the right-hand side of Eq. (5.7) vanish, we find

$$q\tan(qD)=1/b(T), \tag{C2}$$

where

$$b(T)=\frac{\hbar V_{\mathrm{F}}}{4\pi Tk_B}\,\frac{\int_0^1 p^2\,dp\sum_{n=0}^{\infty}(1-e^{-4\pi T(n+1/2)D})/\{(n+\frac{1}{2})\,[(n+\frac{1}{2})^2+(\hbar V_{\mathrm{F}}q/4\pi k_BT)^2]\}}{\int_0^1 p\,dp\sum_{n=0}^{\infty}(1-e^{-4\pi T(n-1/2)D})/[(n+\frac{1}{2})^2+(\hbar V_{\mathrm{F}}q/4\pi Tk_B)^2]}. \tag{C3}$$

To obtain Eq. (C3) we have performed the integration over energy by closing the path of integration to a semicircle in the upper half of the complex E plane.

For thick superconducting layers $T_{c\,\mathrm{NS}}$ is close to T_{cs}. In this case $q(\alpha D^{-1})$ is small. Hence we can neglect $\hbar V_{\mathrm{F}}q/4\pi k_BT$ compared to $(n+\frac{1}{2})^2$, and $e^{-4\pi T(n+1/2)D}$ compared to 1.

We then obtain

$$b(T_{c\,\mathrm{NS}})=\frac{2}{3}\,\frac{\hbar V_{\mathrm{F}}}{4\pi k_BT}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})^{-3}\Big/\sum_{n=0}^{\infty}(n+\tfrac{1}{2})^{-2}=7\zeta(3)\hbar V_{\mathrm{F}}/3\pi^2k_BT, \tag{C4}$$

where $\zeta(3)=1.2\cdots$.

This result is in agreement with a previous calculation for b, for infinite system.[7]

In the other limit, when $T_{c\,\mathrm{NS}}=0$ and D is finite in Eq. (5.1), we put $\tan(E/2k_BT)=1$. The energy integration now yields

$$qb(0)=\frac{\gamma+\ln(qD)-[\cos(qD)\,\mathrm{Ci}(qD)+\sin(qD)\,\mathrm{Si}(qD)]}{\sin(qD)\,\mathrm{Ci}(qD)-\cos(qD)\,\mathrm{Si}(qD)} \tag{C5}$$

where

$$\gamma=0.577\,21\cdots,\quad \mathrm{Si}(x)=\int_0^x\frac{\sin t}{t}\,dt,\quad \mathrm{Ci}(x)=-\int_x^\infty\frac{\cos t}{t}\,dt.$$

In the first limit, $D\gg b(T_{cs})$ of Eq. (C4) we find from (C2) that

$$q\cong\pi/2D, \tag{C6}$$

while in the other limit, $T_{c\,\mathrm{NS}}\to 0$, we find by inserting (C5) into (C2) that

$$q=0.825/D. \tag{C7}$$

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