Critical behavior of the resistivity in magnetic systems. II. Below T_c and in the presence of a magnetic field

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The effect of critical fluctuations on the electrical resistivity of magnetic materials is discussed in detail. The temperature and magnetic field dependences of this critical resistivity $\rho(T,H)$ are given for both cases: when the critical temperature T_c is approached from above and when it is approached from below. The results obtained apply to the two basic classes of magnetic materials: ferromagnets and antiferromagnets, as well as to the two basic electronic systems: metals and semiconductors. The present work is founded on the conclusion that close enough to T_c the critical resistivity of all systems has a magnetic-energy-like behavior. This behavior extends farther away from T_c for all systems except for ferromagnetic semiconductors. It is shown that the critical magnetoresistance, $\Delta\rho = \rho(T,H) - \rho(T,0)$ is negative except for antiferromagnetic semiconductors. $\Delta\rho$ is found to be peaked at T_c for all systems but it never diverges as a function of temperature or field. The results are expressed in terms of power-law dependences of $\Delta\rho$ on $T-T_c$ and H for all the different material classes and for all interesting temperature regions. The corresponding powers, under various conditions, are combinations of the well-known critical exponents, α, β, γ, δ, and ν. In the mean-field regime the powers are predicted to be those of the critical regime except that the classical values of these exponents have to be used. The present results predict more details of the critical resistivity than can be deduced from the existing experimental data. However, the experimental data available are in accord with the results. In the case of antiferromagnetic metals, the present work explains the features of the magnetoresistance in the rare-earth metals, features that have not been understood before. It is suggested that by proper fit of critical resistance and magnetoresistance data to the power-law behaviors predicted here, critical exponents can be deduced. Recent demonstrations of such fits show that this is indeed feasible.

I. INTRODUCTION

In a previous paper,[1] hereafter Paper I, it was shown that the temperature dependence of the resistivity, in the close vicinity of the critical temperature T_c, is the same as that of the magnetic energy.[1-3] That work was essentially a generalization of the Fisher-Langer[2] result, for ferromagnetic metals, to the other magnetoelectronic systems (i.e., ferromagnets, antiferromagnets, metals, and semiconductors). However, for temperatures below T_c the critical resistivity was discussed very briefly[1,2] and only for the very close[1] vicinity of T_c. The magnetic field dependence and the temperature dependence of the critical resistivity $\rho(T,H)$ in the presence of a magnetic field have not been considered in I. The discussions of these dependences in the literature were concerned mainly with the magnetic field dependence of the measured magnetoresistance,[4-14] $\Delta\rho=\rho(T,H)-\rho(T,0)$, and the calculations were carried out in the molecular-field approximation.[14-17] Recently the critical resistivity of Gd was measured in the presence of a magnetic field,[18] and the results have shown that the proportionality of $d\rho/dT$ to the specific heat C_p is also maintained under these conditions.

In this paper we would like to extend the work reported in I by finding the temperature and magnetic field dependences of the critical resistivity below and above T_c. This is possible particularly now in view of the very recent group-renormalization results[19] for the magnetization-dependent correlation functions. Considerations of the antiferromagnetic correlation function and plausible correlation functions in the mean-field regime, enable a comprehensive study of the critical resistivity in all magnetoelectronic systems and in both the critical[20] and mean-field[21] regimes. The calculated $\rho(T,H)$ is shown to have power-law dependences on the parameters $T-T_c$ and H. These dependences are determined for asymptotic relations between the two parameters. The predicted power-laws show universal behaviors, the powers being combinations of the critical (or classical) exponents α, β, γ, δ, and ν. It is thus suggested that the measurements of the critical resistivity under the proper asymptotic conditions can yield the values of these exponents. The confirmation of many of the present predictions by the available experimental results indicates that the asymptotic conditions can be easily obtained in real magnetic systems. Further, this confirmation shows that our quantitative predictions that involve the leading temperature and magnetic field dependences are at least as accurate as the available experimental results, and the correction to the leading dependences under asymptotic conditions can be neglected.

We should stress that the present work is not intended to predict the absolute value of $\rho(T,H)$ or its exact behavior between the asymptotic regions. On the other hand, the results do indicate the qualitative features of the critical resistivity for all $T-T_c$ and H regions. In some cases these features explain the qualitative behavior of the magnetoresistance of materials for which this property has not been understood before.

As in I we consider here the critical regime[20,22] as well as the mean-field regime,[21,23] although the transition point between these regimes cannot be predicted by the present calculation. On the other hand, with the present predictions one can analyze experimental data and find exactly the critical parameters of the corresponding system as well as the temperature at which such a transition is taking place. This has been demonstrated recently by nonlinear least-squares analysis[24] of the critical resistivity of dysprosium.[25]

The present work has more predictions than can be gathered from the characteristic results of the experimental data. As far as the qualitative temperature dependence and the power laws of the magnetic field dependence, we do not know about experimental results that are not in accord with the present results. However, probably due to the absence of adequate theory, the experimental power-law determination of the temperature dependence of the magnetoresistance in the close vicinity of T_c has not been reported. The only relevant data[18] just confirm the general conclusion that the temperature derivative of the resistivity is proportional to the specific heat.

The calculation here is based on the well-known[1,2] relation between the critical resistivity (i.e., the inverse of the carriers mean free time) and the correlation function, $\Gamma_{|\vec{q}-\vec{K}|}$. Here $\vec{q}$ is the wave vector of the momentum transfer, $\vec{K}$ is the point of instability, and $q=|\vec{q}|$. This relation, in the case of a spherical Fermi surface and when plane-wave-like eigenfunctions are assumed for the charge carriers, can be written as[1,2,26]

$$\rho(T,H)\propto I_3\equiv\int[\Gamma_{|\vec{q}-\vec{K}|}+\langle\vec{S}\rangle^2\delta(\vec{q})]q\,d^3q\,, \tag{1.1}$$

where the integration is over the Fermi surface (assumed to be of diameter Σ) of the carriers participating in the interaction with the spin fluctuations. In this equation $\langle\vec{S}\rangle$ represents the thermodynamical average[21] of the spin operator. In the case of a ferromagnet this average is simply the magnetization per spin,[26] m, while for antiferromagnets, as will be discussed below, the situation is not as simple. Extensions to other, more complicated, Fermi surfaces are possible, but then the integral (1.1) has more than one cutoff Σ for the lower-symmetry cases.[1]

As is immediately apparent from (1.1) the heart of the problem is the finding of a proper correlation function. By "proper" we mean a function that accounts for both elastic[1] and inelastic scattering processes[27] due to spin fluctuations.[28] Further, its $\vec{q}$ dependences should be given explicitly for all momentum transfers in the range $0\leq q\leq\Sigma$. Unfortunately, the correlation functions available are far from satisfying these requirements.[19-23] In the critical regime one has an explicit expression only for the asymptotic cases[19,29] $q/\kappa\ll1$, and $q/\kappa\gg1$, where $1/\kappa$ is the correlation length.[20] These expressions are for the Ising model and in the quasistatic approximation. Hence scattering by transverse modes[16,30] (spin waves) as well as inelastic spin fluctuations scattering processes are not described by these expressions. The situation is even worse in the mean-field regime since for this regime only the correlation function for the $q/\kappa\ll1$ limit, i.e., the Ornstein-Zernike correlation function,[21] is available.

The fact that only asymptotic expressions are available for the static correlation function was discussed in detail in I. It was concluded then that for the interaction of charge carriers with spin fluctuations this limitation is of minor importance, especially in the critical regime, and that energy-like behavior of the critical resistivity should always be observed. We should note that this latter conclusion indicates that the specific forms (e.g., Ising-model result) of the correlation functions used there (and to be used here) are not important. This is the conclusion of the fact that the q, the $T-T_c$, and the m dependences of Γ_q are determined by general consideration of the monotonicity of the internal energy and by the scaling hypothesis.[1,2]

Before extending the use of the static correlation function for $T<T_c$ and for the case of an applied field, we should remark on the use of the quasi-elastic approximation for determination of the critical resistivity. This problem has not been considered previously in the literature. Hence we have studied its relevance, and have come to the conclusion that while the contribution of the dynamical scattering may be significant in the case of large momentum transfers, its temperature and magnetic-field dependences will be the same as those obtained from the present quasi-elastic approximation. Our study of this problem, in regard to all carrier transport properties, will be published elsewhere.

For systems in which the spin has more than one component, transverse modes[30] may be exited below T_c. However, such modes are expected to be damped in real materials since the latter do not have a spatial rotational isotropy.[30] Even if these modes do exist, they should be ineffective for carrier scattering[1,2] (see below) due to the fact that they are zero-momentum modes. Hence, for the present problem the effect of these modes can be neglected and the Ising-model results can be used.

A justification for the use of the static-Ising correlation function is also borne out by all available experimental data. This is because semiquantitative analyses[1,18,31,32] of these data have shown that the relation $d\rho/dT \propto C_p$ is fulfilled for both $T > T_c$ and $T < T_c$ and in the absence[32] or presence[18] of magnetic field. A recent quantitative approach, that utilizes a nonlinear least-squares analysis[24,25] for the critical resistivity, has yielded[25,33,34] the same critical exponents below (α') and above (α) T_c. Further, the critical amplitude ratios (A/A') were found to be in excellent agreement with the theoretical predictions for the specific heat. For example, analysis[33] of Shacklett's data[31] on the critical resistivity of the Heisenberg ferromagnet, iron, has shown that $\alpha = \alpha' = -0.21 \pm 0.02$ and $A/A' = 1.48 \pm 0.09$, while the corresponding theoretical values[24] are $\alpha = \alpha' = -0.14 \pm 0.06$ and $A/A' = 1.36 \pm 0.06$. Following the above discussion it appears that the correlation function used in I and in the present work accounts for the dominant contribution to the critical resistivity in real magnetic solids.

In view of the importance of the correlation functions to the problem at hand, we shall present their definitions and expressions in some detail in Sec. II. The correlation functions of ferromagnets are presented in Sec. II A and those of antiferromagnets are presented in Sec. II B. The results for the critical resistivity are obtained then by using Eq. (1.1) and the dominant terms of the correlation functions. These results are given in Sec. III A for ferromagnets and in Sec. III B for antiferromagnets. The predictions are easily amenable to experimental examination, and thus a comparison with the available experimental data will be given in Sec. IV. We shall see that in some cases the present work gives the first explanation of experimental results. We do not discuss here the effect of the change in the number of carriers on the critical resistivity since this is apparent from our work on the critical band shifts.[1,35] We thus restrict our discussion, that will be summarized in Sec. V, to the effect of critical scattering on the carrier's mean free time.

II. THE SPIN CORRELATION FUNCTION

The spin-spin spatial correlation function is defined by[2]

$$\Gamma(\vec{R}) = \langle (\vec{S}_{\vec{0}} - \langle \vec{S} \rangle) \cdot (\vec{S}_{\vec{R}} - \langle \vec{S} \rangle) \rangle , \tag{2.1}$$

where $\vec{S}_{\vec{R}}$ is the spin of the ion located at the lattice site $\vec{R}$ measured with respect to a chosen lattice site $\vec{0}$, $\langle \vec{S} \rangle$ is the thermodynamical average of the spin in the system, and S is the spin eigenvalue. The Fourier transform of $\Gamma(\vec{R})$, the correlation function, is defined by

$$\Gamma_q = \sum_{\vec{R}} \Gamma(\vec{R}) e^{i\vec{q}\cdot\vec{R}} , \tag{2.2}$$

where $\vec{q}$ is the wave vector of the momentum transfer[1,2] and $q = |\vec{q}|$.

A. The correlation functions of ferromagnets

In the case of a ferromagnet, $|\langle \vec{S} \rangle|$ is the reduced magnetization $m = M(T,H)/M(0,0)$, where $M(T,H)$ is the magnetization at the temperature T when a magnetic field H is applied and $M(0,0)$ ($= g\mu_B SN$) is the saturation magnetization.[26] Hence in this case (2.2) can be written as

$$\Gamma_q = \langle \vec{S}_{\vec{q}} \cdot \vec{S}_{-\vec{q}} \rangle - m^2 \delta(\vec{q}) , \tag{2.3}$$

where $\vec{S}_{\vec{q}}$ is the Fourier transform of $\vec{S}_{\vec{R}}$ and where $\vec{q}$ is defined within a reciprocal lattice vector in our effective-mass approximation.

In the above expression the correlation function depends on the magnetization m. Since in the experiments the measurable quantities are the temperature T and the magnetic field H, we express all correlation functions in terms of the reduced temperature $t \equiv (T - T_c)/T_c$ and the reduced magnetic field $h \equiv \mu H / k_B T_c$, where μ is the magnetic moment of an ion and k_B is Boltzmann's constant. The relation $m = m(h,t)$ for ferromagnets is given by the Widom equation[19]

$$h/m^\delta = f(z) , \tag{2.4}$$

where $f(z)$ is the properly normalized Widom function, δ is the critical isothern exponent, $z = t/m^{1/\beta}$, and β is the spontaneous-magnetization exponent. The expressions for the functions $f(z)$ in the critical regime are quite cumbersome[19] and seem to be of little use if results that are comparable with experimental data are desired. Since for such a comparison a simple power-law dependence on T and H is wanted, we use here only the leading terms of $f(z)$ and discuss the asymptotic cases $|t| \gg h$ and $|t| \ll h$. The coefficients of the development of $f(z)$ in powers of z or $1/z$ are smaller than 1 (see below), and thus our results will be exact to order t/h or h/t throughout this paper. For simplicity we normalize here the functions $f(z)$ so that in the classical limit[36] (i.e., where $\beta = \frac{1}{2}$, $\delta = 3$, and $\gamma = 1$, γ being the critical exponent of the susceptibility) the $f(z)$ will yield the well-known $S = \frac{1}{2}$ equation of state.

For $|z|<1$, the leading term of $f(z)$ is z, and thus the equation of state is[19]

$$h/m^{\delta}=\tfrac{1}{3}+t/m^{1/\beta}\,, \tag{2.5}$$

while for the $|z|>1$ limit, the leading term of $f(z)$ yields[19]

$$h/m=t^{\gamma}/m_0\,. \tag{2.6}$$

For m_0 we take here its molecular-field value $m_0=(S+1)/S$, noting that within a factor, m_0 is the same as in the critical regime.[19] As is apparent, (2.5) is always applicable for $T<T_c$, while for $T>T_c$ it is applicable in the $h\gg t$ case. For $T>T_c$ in the $t\gg h$ case, Eq. (2.6) is to be used. We shall thus call the $1\gg h\gg|t|$ limit the "large field" case and the $1\gg|t|\gg h$ limit the "small field" case. (Note that $z=-\frac{1}{3}$ corresponds to the sgn(t) term used in Ref. 2 as the coefficient of the $|t|^{1-\alpha}$ term, see below.) The development of m to the first order in h/t or t/h is given, for the four cases under discussion, in Table I.

Correlation functions for the scaling regime have been calculated and are available in the literature[19,29] for the two extreme cases: $x\ll 1$ and $x\gg 1$, where $x=q/\kappa$ (see discussion in Sec. I). In the critical regime we restrict the discussion to results obtained for the Ising model[19] noting that for the present problem (see Sec. I) this is justified even for systems with more than one component of the order parameter. In the "small-q" $x\ll 1$ limit we can apply the result of Combescot *et al.*,[29] who showed that

$$\Gamma(x^2,t,z)=\chi(t,z)[1+x^2+\sigma_4(z)x^4+\sigma_6(z)x^6+\cdots]^{-1}\,. \tag{2.7}$$

$\chi(t,z)$ in (2.7) is the static susceptibility, and the correction terms are small since $\sigma_i(z)\ll 1$. In the "large-q" $x\gg 1$ limit we can use the result of Brezin *et al.*[19]:

$$\Gamma(q,t,z)=\frac{G_0}{q^{2-\eta}}\left[a-\frac{bt}{q^{1/\nu}}-\phi(z)\left(\frac{q}{m^{\nu/\beta}}\right)^{(\alpha-1)/\nu}\right], \tag{2.8}$$

where G_0, a, and b are constants. In the $m=0$ case[1,37] the last term is reduced to $ct^{1-\alpha}/q^{(1-\alpha)/\nu}$ where the constant c is defined by $c=z^{\alpha-1}\phi(z\to\infty)$. In this correlation function η is the critical-point exponent, ν is the correlation-length exponent, and α is the specific-heat exponent. The function $\phi(z)$ is given by[19]

$$\phi(z)=c_1\left(\int_1^{\infty}u^{\delta-1/\beta}[f'(0)-f'(z/u^{1/\beta})]\,du+\frac{f'(0)}{\delta-1/\beta+1}\right), \tag{2.9}$$

where c_1 is a constant and $f(\omega)$ is the renormalization-group Widom function.[19] For the present purpose it will be enough to develop $f(\omega)$ to first order in the dimension parameter $\epsilon=4-d$ and to consider the $\epsilon=n=1$ case. The normalized function $f(\omega)$ is thus approximated by

$$f(\omega)=1+\omega+\tfrac{1}{6}[(\omega+3)\ln(\omega+3)-3(\omega+1)\ln 3+2\omega\ln 2]\,. \tag{2.10}$$

However, the function (2.10) is not proper for $\omega\gg 1$, and in this case one has to use the expansion[19]

$$f(\omega)=\sum_{p=1}^{\infty}a_p\omega^{\gamma-2(p-1)\beta}\,. \tag{2.11}$$

The dominant term in (2.11) is the first term and one can approximate[19]

$$f(\omega)\approx 0.58\,\omega^{\gamma} \tag{2.12}$$

In the "large-field" case of $|z|<1$ we can thus substitute Eq. (2.10) into Eq. (2.9) to obtain $\phi(z)$. The result of this substitution is

$$\phi(z)=c_1\left(\frac{\beta f'(0)}{1-\alpha}+\frac{\beta}{6}\sum_{p=1}^{\infty}\frac{(-1)^p z^p}{3^p p(p-1+\alpha)}\right). \tag{2.13}$$

In the "small-field" case $|z|>1$ we have to consider both the region where Eq. (2.10) applies and the region where Eq. (2.12) applies. Such a consideration, as will be shown in Appendix A, yields

$$\phi(z)\approx c_1[(-\beta/18\alpha)z^{1-\alpha}+0.29\gamma z^{\gamma-1}]\,. \tag{2.14}$$

Comparison of Eq. (2.14) and Eq. (2.8) shows that $c=(-c_1\beta)/(18\alpha)$, and thus using the $m=0$ value[1,37] for c yields the value $c_1=36\alpha/(\beta\kappa_0^{(\alpha-1)/\nu})$, where κ_0 is the inverse of the spin-interaction range. Similarly, $b=3\kappa_0^{1/\nu}$ and $a=0.962$. We should note that the neglect of the higher-order terms in the development of $f(z)$, and thus of $\phi(z)$, does not allow one to attribute too much significance to

TABLE I. The leading terms of m in a ferromagnet.

Region	$T>T_c$	$T<T_c$
$\lvert t\rvert\gg h$	$m_0t^{-\gamma}h$	$(3\lvert t\rvert)^{\beta}+3\beta h(3\lvert t\rvert)^{\beta-\beta\delta}$
$\lvert t\rvert\ll h$	$(3h)^{1/\delta}-(3/\delta)t(3h)^{(\beta-1)/\beta\delta}$	$(3h)^{1/\delta}+(3/\delta)\lvert t\rvert(3h)^{(\beta-1)/\beta\delta}$

the calculated coefficients of the approximated terms (2.13) and (2.14). On the other hand, the asymptotic temperature and magnetic field dependences to order h/t or t/h are well established.

In the mean-field regime we have the Ornstein-Zernike function which is applicable only for $x \ll 1$. In the presence of magnetization this correlation function takes the form[38]

$$\Gamma(q, m, t) = G/(\kappa^2 + q^2), \tag{2.15}$$

where G is a constant and

$$\kappa^2 = \kappa_0^2(t + m^2)/(1 - m^2). \tag{2.16}$$

The main problem in this regime is that there is no available model-independent correlation function for the $x \gg 1$ limit (which is so important for critical resistivity[1]). For this reason we had to develop an approximate form that may account for the expectations from such a function. We have used[1,23] for this purpose the spin sum rule property and Eq. (2.15), yielding essentially the correlation function of the "non-small q's." The result obtained previously,[1,23]

$$\Gamma_q = \frac{(2\pi^2/\Omega)[S(S+1) - \langle\vec{S}\rangle^2]}{[\Lambda - \kappa \arctan(\Lambda/\kappa)](q^2 + \kappa^2)}, \tag{2.17}$$

can be easily generalized to the present case (for which the magnetization is considered) by using Eq. (2.16) rather than $\kappa = \kappa_0 t^{1/2}$. In (2.17), Ω is the volume per spin, S is the spin eigenvalue, and Λ is the effective radius of the Brillouin sphere. This function satisfies the sum rule, has the correct $q \to 0$ behavior, has the expected ($t^{1/2}$ for $m = 0$) temperature dependence of the energy[20,39] in the $q \gg \kappa$ limit, and is monotonically increasing with T for $T > T_c$ (as expected from the behavior of the magnetic energy[1]). In contrast with the first three properties, the latter property is not satisfied for $T < T_c$. Hence we had to modify our previous function[23] to find a mean-field "large-q" correlation function for $t < 0$.

We propose for temperatures below T_c, to use the Ornstein-Zernike-like correlation function for $x \le \tilde{y}$ and the asymptotic classical correlation function $\Gamma_q \propto 1/q^2$ for $x \ge \tilde{y}$. The matching point $\tilde{y}$ between these two functions is adjusted in a way that will ensure the monotonicity requirement. Hence

$$\Gamma_q = A/(q^2 + \kappa^2) \quad \text{for } x \le \tilde{y} \tag{2.18}$$

and

$$\Gamma_q = A(\psi/q^2) \quad \text{for } x \ge \tilde{y}. \tag{2.19}$$

Here A is a constant that we determine by the spin sum rule.[1]

$$(\Omega/2\pi^2)\int_0^\Lambda \Gamma_q q^2\, dq = S(S+1) - \langle\vec{S}\rangle^2 \tag{2.20}$$

and ψ is a parameter that accounts for the matching of the functions (2.18) and (2.19) at $\tilde{y}$. Carrying out the integration (2.20) in the two regions $0 \le x \le \tilde{y}$ and $\tilde{y} \le x \le \Lambda$ yields

$$\Gamma_q = \frac{2\pi^2[S(S+1) - \langle\vec{S}\rangle^2]/\Omega}{\psi\Lambda + \kappa(\tilde{y} - \arctan\tilde{y} - \psi y)} \times \begin{cases} (\kappa^2 + q^2)^{-1} & \text{for } q < \tilde{y}\kappa \\ \psi q^{-2} & \text{for } q > \tilde{y}\kappa. \end{cases} \tag{2.21}$$

As can be seen from Eq. (2.21) the monotonicity requirement $d\Gamma_q/dT > 0$ yields $\psi < 1 - (\arctan\tilde{y})/\tilde{y}$, and thus $\psi < 1$. This is also apparent from the continuity requirement of the correlation function at $\tilde{y}$ which is given by $\psi/\tilde{y}^2 = 1/(1 + \tilde{y}^2)$. These two requirements approach compatibility the larger the value of $\tilde{y}$. However, since $1 - (\arctan\tilde{y})/\tilde{y} < \tilde{y}^2/(1 + \tilde{y}^2)$ for any $\tilde{y}$, there is always a mismatch between the functions (2.21). For example at $\tilde{y} = 30$, $\psi = 0.95$ and the mismatch is 5%. A similar matching between Eq. (2.7) and (2.8) in I has yielded the correlation function in the critical regime. Here as in the critical-regime case the details of the matching are not important for the determination of the critical behavior of the resistivity.

It should be pointed out that neither Eq. (2.17) nor Eq. (2.21) solves the problem of the "large-q" correlations function in the mean-field regime. These functions measure in fact an integral property, and thus only their asymptotic behavior at $x \gg 1$ or $x \ll 1$ should be taken literally. The confidence in the predictions at these asymptotic limits arises from the fulfillment of the mentioned expected properties of such correlation functions and their asymptotic behavior. Here we only suggest these functions as plausible forms that are correct in the asymptotic limits. The values of the parameters $\tilde{y}$ and ψ in Eq. (2.21) are of importance only if one wants to determine the value of κ at which a transition between the two asymptotic regions takes place, or to determine the correction to the one asymptotic behavior by the other. Since this is associated with the intermediate x region, where the functions are ill defined anyway, it is not expected that this determination will be meaningful. At best, one can find the order of magnitude of κ at which the transition is expected to occur (see Table I in I). The conclusions to be used in this paper are the Ornstein-Zernike-like behavior in the $x \ll 1$ limit and the $[S(S+1) - \langle\vec{S}\rangle^2][1 + \mathrm{sgn}(t)\kappa/\Lambda]$ behavior in the $x \gg 1$ limit. These conclusions are extensions of our previous results[1,23] to the $T < T_c$ region. The leading temperature and magnetic-field dependences of the two behaviors will be discussed in Sec. III A in connection with the critical resistivity in the mean-field regime.

B. The correlation functions of antiferromagnets

For antiferromagnets we consider the simple case of two sublattices. In this case the point of instability is the reciprocal magnetic-lattice vector $\vec{Q}$, and Γ_q in (1.1) is then to be replaced by $\Gamma_{|\vec{q}-\vec{Q}|}$. When no magnetic field is applied, $|\langle\vec{S}\rangle| = m(0)$, the field-free sublattice magnetization. The situation becomes more subtle when a magnetic field is present. Since this situation, as far as we know, has not been discussed in the literature, we elaborate on it somewhat.

For the small fields relevant to the present problem of critical behavior (i.e., $h \ll 1$) the average value of the spin is $|\langle\vec{S}\rangle| = \chi H/M(0,0)$ where χ is the antiferromagnetic susceptibility.[36] In the molecular-field approximation the susceptibility above T_c is given by $\chi = C/(T+\Theta)$, where $C = Ng^2\mu_B^2 S(S+1)/3k_B$ and Θ is the Curie-Weiss temperature of the antiferromagnet. One can then write

$$|\langle\vec{S}\rangle| = ah, \tag{2.22}$$

where $a = \tilde{a}\{1 - t[T_c/(T_c+\Theta)]\}$ and $\tilde{a} = (S+1)T_c/[3S(T_c+\Theta)]$. To calculate a in the critical regime let us recall that in the close vicinity of T_c the susceptibility of an antiferromagnet is proportional to the correlation function[40] Γ_Q. In this regime the function is given by[1,37]

$$\Gamma_Q = G_Q(2t^{1-\alpha} - 3t + 1), \tag{2.23}$$

where G_Q is a constant. (For simplicity we have assumed that the spin-spin interaction range $1/\kappa_0$ is of the order of $1/Q$). In view of the value of $\tilde{a}$ we can normalize $G_Q = 1$ and thus within a factor (which is of the order of the corresponding amplitude ratio[19]), a will be given in the critical regime by

$$a = \{(S+1)T_c/[3S(T_c+\Theta)]\}(2t^{1-\alpha} - 3t + 1). \tag{2.24}$$

Using the above average of $|\langle\vec{S}\rangle|$ and recalling that for $T > T_c$, $|\langle\vec{S}\rangle|$ is the same for both sublattices, the spin correlation function can be defined as

$$\Gamma_q^{AF} = (1/N)\sum_{\vec{R}}\sum_{\vec{R}'}\langle(\vec{S}_{\vec{R}} - a\vec{h})\cdot(\vec{S}_{\vec{R}'}e^{-i\vec{Q}\cdot(\vec{R}'-\vec{R})} - a\vec{h}) \times e^{i\vec{q}\cdot(\vec{R}'-\vec{R})}\rangle, \tag{2.25}$$

where we assumed that $\vec{S}_{\vec{R}}$ is parallel to the magnetic field $\vec{H}$. Hence the "antiferromagnetic" correlation function Γ_q^{AF} will be given by

$$\Gamma_q^{AF} = \Gamma_{|\vec{q}-\vec{Q}|} = \langle\vec{S}_{|\vec{q}-\vec{Q}|}\cdot\vec{S}_{-|\vec{q}-\vec{Q}|}\rangle - (ah)^2\delta(\vec{q}). \tag{2.26}$$

Below T_c one has to consider the sublattice magnetization and to sum over the two sublattices. The correlation function in this case can be defined by

$$\Gamma_q^{AF} = \Gamma_{|\vec{q}-\vec{Q}|} = (1/N)\sum_{\vec{R}}\sum_{\vec{R}'}\langle[\vec{S}_{\vec{R}} - \vec{m}(0) - b\vec{h}]\cdot\{[\vec{S}_{\vec{R}'} - \vec{m}(0)]\,e^{-i\vec{Q}(\vec{R}-\vec{R}')} - b\vec{h}\}e^{i\vec{q}(\vec{R}'-\vec{R})}\rangle, \tag{2.27}$$

where $b = \chi_{\parallel}/M(0,0)$ and $\chi_{\parallel}$ is the parallel susceptibility.[36] For the small fields relevant to critical phenomena, and in the molecular-field approximation, b can be given (to order $|t|$) by[36]

$$b = \tilde{b}(1 - |t|/B), \tag{2.28}$$

where

$$\tilde{b} = \frac{(S+1)}{S} - \frac{6(2S^2+2S+1)}{5S^2(S+1)}\left(\frac{T}{T_c}\right)^2$$

and

$$B = 1 + \frac{3\Theta}{T_c}\left[1 - \frac{6(2S^2+2S+1)}{5(S+1)^2}\left(\frac{T_c}{T}\right)^2\right].$$

As for the $T > T_c$ case one can replace $(1 - |t|/B)$ in Eq. (2.28) by $\Gamma_Q = G_Q(-2|t|^{1-\alpha} - 3|t| + 1)$ and thus get, within a factor, the value of b for the critical regime. The correlation function for $T < T_c$ can then be written as

$$\Gamma_q^{AF} = \Gamma_{|\vec{q}-\vec{Q}|} = \langle\vec{S}_{|\vec{q}-\vec{Q}|}\cdot\vec{S}_{-|\vec{q}-\vec{Q}|}\rangle - m(0)^2\delta(\vec{q}-\vec{Q}) - (bh)^2\delta(\vec{q}). \tag{2.29}$$

It should be noted that m used in Eqs. (2.7) and (2.8) is the order parameter. This quantity is not well defined for antiferromagnets in the presence of a magnetic field. However, it can be shown[36] that the correction to (2.29) via the difference $m(H) - m(0)$ is of the order of $(bh)^2$. Hence (2.29) is adequate for the determination of the leading temperature and magnetic field dependences.

For the derivation of the explicit dependences one has to recall that in antiferromagnets the Néel temperature $T_c(H)$ is magnetic field dependent and its relation to the $H=0$ Néel temperature $T_c(0)$ is given by[36]

$$[T_c(H) - T_c(0)]/T_c(0) = -FH^2 = -fh^2, \tag{2.30}$$

where

$$F = \frac{3\chi_\perp^2[(S+1)^2 + S^2]}{20[(S+1)^2(Ng\mu_B S)^2]},$$

$\chi_\perp$ is the transverse susceptibility, $Ng\mu_B S$ is the saturated magnetic moment per unit volume, and

$$f = \frac{(S+1)^4}{60S^4[1+\Theta/T_c(0)]^2}\,.$$

The relation (2.30) is valid as long as

$$FH^2 \ll T_c(0)\,, \tag{2.31}$$

and it has to be modified when anisotropy is taken into account. For simplicity we consider here only the form (2.30), noting that it applies for both the critical regime and the mean-field regime.[41]

It is apparent that when critical effects are discussed, the distance between the (T,H) point and the critical curve $T_c(H)$ in the antiferromagnetic-phase plane is to be considered. For the small fields discussed here [Eq. (2.31)] one can approximate $dT_c(H)/dT \equiv \tan\Phi \simeq \Phi$, and thus the distance will be given by $[T - T_c(H)][1 - 2(FH)^2]$. The effective reduced temperature $t^* = [T - T_c(H)]/T_c(H)$ can then be expressed to order h^2 by

$$|t^*| = |t| + \mathrm{sgn}(t) f h^2 + t f h^2\,, \tag{2.32}$$

where $t = [T - T_c(0)]/T_c(0)$.

Since molecular-field calculations[17,36] were carried out for magnetic fields for which the condition (2.31) prevails, and since we are interested in critical effects, i.e., in the close vicinity of T_c, we assume the validity of this condition in our discussion on antiferromagnets. In view of this, we use for antiferromagnets, the correlation functions given in Sec. II A, but with m replaced by $m(0)$ and with t replaced by t^*

In the perturbation approximation that will be considered below, $\langle \vec{S}\rangle^2 \ll 1$. This implies that the leading temperature and magnetic-field-dependent terms of $\Gamma_{|\vec{q}-\vec{Q}|}$ will be proportional to $|t^*|^{1-\alpha}$ in the critical regime, and to $|t^*|^{1/2}$ in the mean-field regime. It is then the shift of the Néel temperature that will determine the critical behavior since the corrections introduced by the other terms in (2.29) are of higher order than th^2.

III. THE CRITICAL RESISTIVITY

In this section we derive the leading temperature and magnetic field dependences of the critical resistivity. This is done by the procedure used in I, i.e., by substituting the correlation functions that were given in Sec. II into the integral (1.1). The results obtained will be presented in their asymptotic ($|t| \gg h$ or $|t| \ll h$) limits for the various regions of κ or t. For comparison, the corresponding $h = 0$ results which were calculated in I, will be given for all regions. The qualitative features of the results will be explained on physical grounds.

A. Critical resistivity of ferromagnets

As is shown in I, the critical resistivity is given by an integral of the form (1.1). In the critical regime[20] the result of this integration is given by[1]

$$I_3 = D_1(y) t^{2\nu + \eta\nu} + D_2 - D_3 \Phi(z) m^{(1-\alpha)/\beta} - D_4 t\,, \tag{3.1}$$

where $D_1(y)$ is a function of the matching point y between the "small-q" correlation function (2.7) and the "large-q" correlation function (2.8), and D_2, D_3, and D_4 are constants that depend on the effective radius of the Fermi surface Σ [i.e., the cutoff of the integral (1.1)]. Here we have added the magnetization-dependent term which in the $m = 0$ case, calculated in I, reduces to $D_3|t|^{1-\alpha}$. Equation (3.1) is dominated by the $x \gg 1$ correlation function[1] as long as $y < \Sigma/\kappa$. Then, the term of the dominant temperature dependence and the only term which depends on the magnetization is

$$\Delta I_3 \propto -\Phi(z) m^{(1-\alpha)/\beta}\,. \tag{3.2}$$

Hence the temperature and magnetic field dependences of the critical resistivity will be determined by ΔI_3, the dominant part of I_3. When (3.2) is written explicitly in terms of the functions $\Phi(z)$ (given in Sec. II) and the proper m values (given in Table I), one obtains the explicit t and h dependences of the critical resistivity in the critical regime.[2] This is done here for the two limits $|t| \gg h$ and $|t| \ll h$. The results for these limits are presented for $T > T_c$ in the first row of Table II and for $T < T_c$ in the first row of Table III. The results account for the increase in the number of large-q spin fluctuations upon the increase of temperature through T_c. Hence these results describe the situation for metals[1,2] in the critical regime even for relatively large values of $|t|$ ($\leq 10^{-2}$). For semiconductors, on the other hand, this will be true only in the very close vicinity of T_c (see below). The magnetoresistance is found to be always negative, as expected, from the suppression of the critical fluctuations by the magnetic field. For better realization of the predicted behaviors of the resistance and magnetoresistance (given by the first rows of Tables II and III) we illustrate in Fig. 1 the corresponding leading temperature and magnetic field dependences. The significant dependences to be noted in Fig. 1 are the energy-like behavior of the resistivity $|t|^{1-\alpha}$ [Fig. 1(a)], the $h^2 t^{-\gamma-1}$ dependence of the magnetoresistance above T_c [Fig. 1(b)], and the $h|t|^{1-\alpha-\beta\delta}$ dependence of the magnetoresistance below T_c [Fig. 1(c)]. In the $|t| \ll h$ limit the $h^{(1-\alpha)/\beta\delta}$ dependence is found both above and below T_c. The results indicate that for small enough fields ($|t| \gg h$) the magnetoresistance has a diverginglike behavior, but when T_c is finally approached (and $h \gg |t|$), the mag-

TABLE II. Critical resistivity of a ferromagnet for $T > T_c$.

Region	$h = 0$	$h \ll t$	$h \gg t$
Critical $x \gg 1$	$t^{1-\alpha}$	$t^{1-\alpha} - (6\alpha\gamma/\beta)t^{-\gamma-1}(m_0h)^2$	$-(3h)^{(1-\alpha)/(\beta\delta)} + (1-\alpha)\left(\frac{1}{\beta\delta} + \frac{1}{18\alpha f'(0)}\right) t(3h)^{-\alpha/(\beta\delta)}$
Critical $x \ll 1$	$t^{2\nu} \ln t$	$t^{2\nu}(1 + \{6a_2/[a_1(2-\eta)]\}\, t^{-2(\gamma+\beta)}(m_0h)^2) \ln t$	$h^{2\nu/(\beta\delta)} \ln h$
Mean-field $x \gg 1$	$t^{1/2}$	$\frac{\pi\kappa_0}{2\Lambda}\left(\frac{t+(m_0h/t)^2}{1-(m_0h/t)^2}\right)^{1/2} - [1+(\kappa_0/\Lambda)^2](m_0h/t)^2$	$\frac{\pi\kappa_0}{2\Lambda}\left(\frac{(3h)^{2/3}-t}{1-(3h)^{2/3}-2t}\right)^{1/2} - [1+(\kappa_0/\Lambda)^2][(3h)^{2/3}-t]$
Mean-field $x \ll 1$	$t \ln t$	$-[t+(m_0h/t)^2] \ln \{1+(\Sigma/\kappa_0)^2[t+(m_0h/t)^2]^{-1}\}$	$[(3h)^{2/3}-t] \ln [(3h)^{2/3}-t]$

netoresistance saturates and depends only on the magnetic field. As will be discussed in Sec. IV all the magnetic field dependences mentioned here were found experimentally.[14,18]

For a semiconductor one can apply (3.2) only in the very close vicinity of T_c where the condition $y<\Sigma/\kappa$ is still satisfied. (This is for $|t| \lesssim 10^{-5}$ as can be seen from Table I of I.) Since the critical regime may extend to larger t's we have to consider[1] the small-x correlation function (2.7). This yields a de Gennes–Friedel-type dependence of the resistivity:[1]

$$\rho \propto \kappa^2 \ln\kappa \; . \tag{3.3}$$

While κ is well known[38] in the $h=0$ and the $|t| \ll h$ cases, we have not found an explicit expression for it in the literature in the $|t| \gg h > 0$ case. In view of our interest in this limit we have derived this expression by using the proper m in the susceptibility relation $\kappa = (\partial m/\partial h)^{1/(\eta-2)}$. For $T>T_c$ we have used the relation $h/m^{\delta} = f(z)$ up to the second term of Eq. (2.11) and found that

$$\kappa^2 = t^{2\nu}(1+\{6a_2/[a_1(2-\eta)]\}t^{-2(\gamma+\beta)}m_0^2 h^2) \; .$$

For $T<T_c$ we have considered the equation of state (2.5) with the corresponding m of Table I and found that

$$\kappa^2 = (3|t|)^{2\nu} + [2/(2-\eta)](\beta\delta - 1/\beta - 1)h(3|t|)^{-\beta} .$$

The results obtained for the critical resistivity in this $x \ll 1$ region are given in the second rows of Tables II and III. It is thus expected that in semi-

TABLE III. Critical resistivity of a ferromagnet for $T < T_c$.

Region	$h = 0$	$h \ll \lvert t\rvert$	$h \gg \lvert t\rvert$
Critical $x \gg 1$	$-\lvert t\rvert^{1-\alpha}$	$-(3\lvert t\rvert)^{1-\alpha}\{(1-\alpha)-(1-\alpha)/[18f'(0)]\}$ $\times\{1-(1-\alpha)/[18\alpha f'(0)]\}^{-1}h(3\lvert t\rvert)^{1-\alpha-\beta\delta}$	$-(3h)^{(1-\alpha)/(\beta\delta)} - (1-\alpha)\{(1/\beta\delta)+1/[18\alpha f'(0)]\}$ $\times\lvert t\rvert(3h)^{-\alpha/(\beta\delta)}$
Critical $x \ll 1$	$\lvert t\rvert^{2\nu} \ln\lvert t\rvert$	$\{(3\lvert t\rvert)^{2\nu}+[2/(2-\eta)](\beta\delta-1/\beta-1)$ $\times h(3\lvert t\rvert)^{-\beta}\} \ln\lvert t\rvert$	$h^{2\nu/(\beta\delta)} \ln h$
Mean-field $\{x \gg 1$ $(\tilde{y}\kappa < \Sigma)$	$-\lvert t\rvert^{1/2}$	$-[3h/(3\lvert t\rvert)^{1/2}][1+(\kappa_0/\Lambda)^2]-(\zeta\kappa_0/\Lambda)$ $\times\{[2\lvert t\rvert+3h/(3\lvert t\rvert)^{1/2}]/[1-3\lvert t\rvert)^{1/2}]\}^{1/2}$	$-(3h)^{2/3}[1+(\kappa_0/\Lambda)^2]-(\zeta\kappa_0/\Lambda)$ $\times\{[(3h)^{2/3}+\lvert t\rvert]/[1-(3h)^{2/3}-2\lvert t\rvert]\}^{1/2}$
Mean-field $\{x \ll 1$ $(\tilde{y}\kappa > \Sigma)$	$\lvert t\rvert \ln\lvert t\rvert$	$-[2\lvert t\rvert+3h/(3\lvert t\rvert)^{1/2}]$ $\times \ln\{1+(\Sigma/\kappa_0)^2[2\lvert t\rvert+3h/(3\lvert t\rvert)^{1/2}]^{-1}\}$	$-[\lvert t\rvert+(3h)^{2/3}]$ $\times \ell n\{1+(\Sigma/\kappa_0)^2[\lvert t\rvert+(3h)^{2/3}]^{-1}\}$

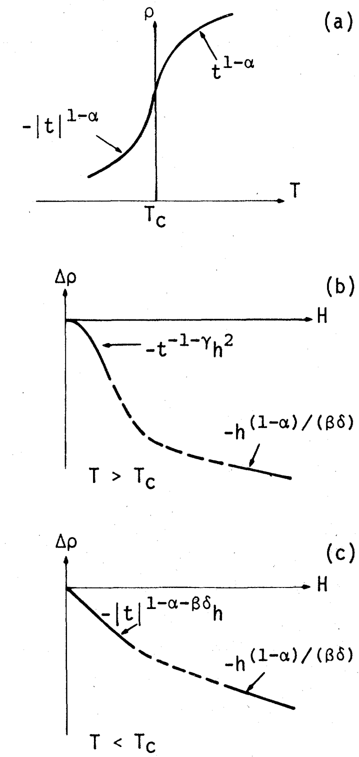

FIG. 1. Temperature dependence of the critical resistivity of a ferromagnetic metal (a). Magnetic field dependence of the magnetoresistivity of a ferromagnetic metal above the critical temperature (b) and below the critical temperature (c). The dashed curves represent the transition region between small magnetic field behavior and large magnetic field behavior. The results apply in both the critical and the mean-field regimes.

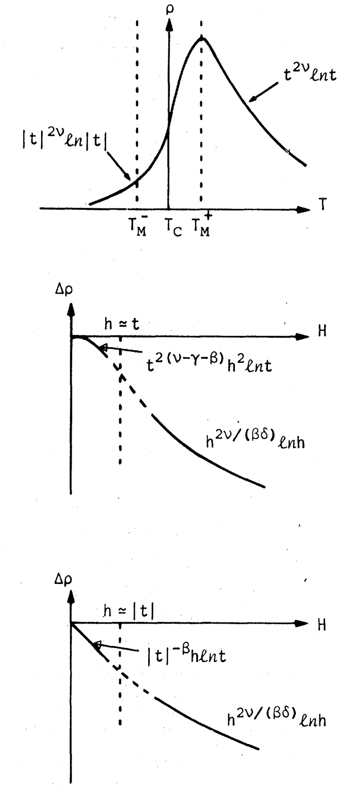

FIG. 2. Temperature dependence of the critical resistivity of a ferromagnetic semiconductor, (a). Magnetic field dependence of the magnetoresisitivity of a ferromagnetic semiconductor above T_c, (b), and below T_c, (c). The dashed curves represent the transition region between small magnetic field behavior and large magnetic field behavior. The results apply in both the critical and the mean-field regimes.

conductors a transition from the behavior given in the first rows to the behavior given in the second rows will take place. This transition occurs when κ, which depends on $|t|$ and h, increases through the vicinity of the point $\kappa = \Sigma/y$. For $T > T_c$ the transition is associated with the point (T_M^+, H_m) above which the dominant fluctuations have q's larger than the electronic cutoff Σ. For semiconductors, Σ is quite small ($\Sigma/\kappa_0 \ll 1$) and the transition manifests itself by a maxima of the resistivity at the above point. A similar situation, but without a peak at T_M^-, is expected for $T < T_c$.

The predicted behavior of the critical resistivity of a ferromagnetic semiconductor is shown in Fig. 2. In the relatively narrow region, where the $x \gg 1$ asymptotic correlation function applies ($y < \Sigma/\kappa$), the behavior is exactly the same as for metals. Hence, the region between T_M^- and T_M^+ in Fig. 2(a) depicts the behavior shown in Fig. 1(a) but on a reduced temperature scale. The behavior of the resistivity outside this region is characterized by the $|t|^{2\nu}\ln|t|$ decrease of the resistivity (note that $\ln|t| < 0$ since $|t| < 1$). The magnetic-field dependence of the magnetoresistance in this $y > \Sigma/\kappa$ region is the same as in the $T_M^+ > T > T_M^-$ region for the $|t| \ll h$ case but changes to $h^{2\nu/(\beta\delta)}\ln h$ in the $|t| \ll h$ case. The temperature dependence of the magnetoresistance is $t^{2(\nu-\gamma-\beta)}$ above T_c[Fig. 2(b)] and $|t|^{-\beta}$ below T_c [Fig. 2(c)]. It is worth noting that these temperature dependences coincide with the dependences in the inner ($T_M^- < T < T_M^+$) region when the classical values of the exponents ($\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = 1$, $\delta = 3$, and $\nu = \frac{1}{2}$) are used.

The qualitative features of the behavior of a semiconductor are expected on physical grounds. As the temperature increases through T_c the dominant fluctuations have larger and larger q's. Very close to T_c these q's are small enough to match the wave vector of the carriers, Σ, and the effective large-angle scattering[1,2] takes place. As the temperature increases through T_M^+ the dominant fluctuations have q's which are larger than Σ and the less-effective small-angle scattering process takes over. Correspondingly, the resistivity decreases with temperature. As in the $x \gg 1$ region the effect of magnetic field is to inhibit the fluctuations and thus to yield negative magnetoresistance ($\ln h < 0$ since $h < 1$). Below T_c the increase of temperature is accompanied by a monotonic increase in the number of the fluctuations. Hence the resistivity increases monotonically with increasing temperature and decreases with increasing magnetic field.

In the mean-field regime, i.e., when $|t| > t_G$, the latter being the Ginzburg reduced temperature,[1,20] we have to use the correlation functions (2.17) and (2.21) with the classical κ(2.16). Carrying out the integration (1.1) for the $T > T_c$ case yields then:

$$I_3 = (1 - m^2)\left\{\tfrac{1}{2}\Sigma^2 - \tfrac{1}{2}\kappa_0^2 \frac{t+m^2}{1-m^2}\ln\left[1 + \left(\frac{\Sigma}{\kappa_0}\right)^2 \frac{1-m^2}{t+m^2}\right]\right\} \times\left\{\Lambda\left[1 - \frac{\kappa_0}{\Lambda}\left(\frac{t+m^2}{1-m^2}\right)^{1/2}\right]\arctan\left(\frac{[\Lambda(1-m^2)^{1/2}]}{\kappa_0(t+m^2)^{1/2}}\right)\right\}^{-1}. \tag{3.4}$$

As discussed in Sec. II A, the functions (2.17) and (2.21) yield the exact temperature and magnetic-field dependences only in the asymptotic limits $\kappa/\Sigma \ll 1$ and $\kappa/\Sigma \gg 1$. Hence, we should consider (3.4) in these limits. The explicit temperature and magentic-field dependences are found by substituting the mean-field values of κ [Eq. (2.16)] and m (the values given in Table I but with classical exponents) into

$$\Delta I_3 \propto -m^2[1 + (\kappa_0/\Lambda)^2] + (\pi/2)(\kappa/\Lambda) \tag{3.5}$$

for the $\kappa/\Sigma \ll 1$ case, and into Eq. (3.3) for the $\kappa/\Sigma \gg 1$ case. The results of this substitution in the first case are given in the third row of Table II while those of the second case are given in the fourth row of this table. As can be expected in view of (3.4), the exact behavior of the critical resistivity as well as the transition point from the third-row behavior to the fourth-row behavior, will depend on the parameters of the magnetic (κ_0 and Λ) and electronic (Σ and Λ) systems.[1] The latter transition can be looked upon as a transition from a mean-field behavior [Eq. (3.4)] to a molecular-field behavior. This is because the results obtained in the molecular-field approximation[16,17] coincide with the results given here in the fourth row of Table II. The physical picture associated with the behavior described by the third and fourth rows of Table II is similar to the picture suggested for the behavior associated with the first and second rows of this table, respectively. Further, comparison of the predicted power-law behaviors in the critical and mean-field regimes shows that they are the same, provided the classical values of the critical exponents are used.

The following sequence of behaviors is predicted for different electronic systems: In metals, a transition from the first-row behavior to that of the third-row behavior is expected around t_G. In semiconductors, a transition from the first-row behavior to that of the second-row behavior is expected at $\kappa \approx \Sigma/y$ and then, with increasing κ, a transition to the fourth-row behavior, in the vicinity of t_G. In intermediate cases, such as of "semimetals," ($0.1 \leq \Sigma/\Lambda \leq 0.5$) we may expect[1] the behavior of a metal, but finally (large t or h), the resistivity will assume the fourth-row behavior.

The situation below T_c is quite similar but we have to use Eqs. (2.21) in the corresponding κ regions of the mean-field regime. For $\tilde{y}\kappa < \Sigma$ we get

$$I_3 = (1 - m^2)\{\psi\Sigma^2 + \kappa^2[\tilde{y}^2(1-\psi) - \ln(1+\tilde{y}^2)]\} \times\{2[\psi\Lambda + \kappa(\tilde{y} - \arctan\tilde{y} - \psi\tilde{y})]\}^{-1}, \tag{3.6}$$

while for $\tilde{y}\kappa > \Sigma$

$$I_3 = (1 - m^2)\{\tfrac{1}{2}\Sigma^2 - \tfrac{1}{2}\kappa^2\ln[(\Sigma^2+\kappa^2)/\kappa^2]\} \times[\psi\Lambda + \kappa(\tilde{y} - \arctan\tilde{y} - \psi\tilde{y})]^{-1}. \tag{3.7}$$

Following the same procedure as above and writing the dominant terms $-m^2[1+(\kappa_0/\Lambda)^2] - \zeta\kappa/\Lambda$, where $\zeta = (\tilde{y} - \arctan\tilde{y} - \psi\tilde{y})/\psi$ is a constant of the order 1, we get, in correspondence with (3.6) and (3.7), the results of the third and fourth rows of Table III. As in the $T > T_c$ case the dependences of the magnetoresistance, $ht^{1-\alpha-\beta\delta}$ and $h^{(1-\alpha)/\beta\delta}$, apply also to the mean-field regime when the classical values of the critical exponents are considered. Hence the entire behavior illustrated in Figs. 1 and 2 represents also the mean-field behavior provided that $1-\alpha$ is replaced by $\frac{1}{2}$ and the critical exponents are replaced by their classical values. Again the results of the fourth row coincide with molecular-field calculations[16,17] and the sequence of transitions in the behavior of metals, semiconductors, and "semimetals" is similar to that found for $T > T_c$.

B. Critical resistivity of antiferromagnets

As was shown in I for electrons scattering in an antiferromagnet, one has to consider the momentum transfer with respect to the point of instability $\vec{Q}$. The resistivity integral (over the Brillouin zone[1]) (1.1) takes then the form

$$\rho \propto \int [\Gamma_{|\vec{q}-\vec{Q}|} + \langle\vec{S}(t,h,q)\rangle^2] q\, d^3q\ . \tag{3.8}$$

in view of the results obtained in Sec. II B, we can use the relations

$$\langle\vec{S}\rangle^2 = (ah)^2\delta(\vec{q}) \quad \text{for } T > T_c$$

and

$$\langle\vec{S}\rangle^2 = m^2(0)\,\delta(\vec{q}-\vec{Q}) + (bh)^2\delta(\vec{q}) \quad \text{for } T < T_c \,.$$

The case of a semiconductor, i.e., when $\Sigma \ll Q$, is straightforward and Eq. (3.8) reduces to

$$\rho \propto \Gamma_Q(|t^*|) \,. \tag{3.9}$$

Hence using the result (2.32) of Sec. II B one can determine the temperature and magnetic-field dependences of ρ in the cases $|t| \gg h$ and $|t| \ll h$. In the first case

$$\rho \propto t^{1-\alpha} + (1-\alpha)ft^{-\alpha}h^2 \quad \text{for } T > T_c \tag{3.10}$$

and

$$\rho \propto -|t|^{1-\alpha} + (1-\alpha)f|t|^{-\alpha}h^2 \quad \text{for } T < T_c. \tag{3.11}$$

In the second case

$$\rho \propto (fh^2)^{1-\alpha} + (1-\alpha)t(fh^2)^{-\alpha} \quad \text{for } T > T_c \tag{3.12}$$

and

$$\rho \propto (fh^2)^{1-\alpha} + (1-\alpha)|t|(fh^2)^{-\alpha} \quad \text{for } T < T_c \,. \tag{3.13}$$

The behavior described by these formula is summarized in Fig. 3. The temperature dependence of the resistivity of an antiferromagnetic semiconductor when $H=0$ is shown[1] in Fig. 3(a) and the temperature dependence of the magnetoresistivity is shown in Fig. 3(b). Qualitatively, the results shown in Fig. 3(b) can be gathered from Fig. 3(a) since the shift of the $\rho(T, H=0)$ curve, parallel to itself towards lower temperatures, will yield an increase of the resistivity at each T. If this shift is induced by a magnetic field, as in the present case, positive magnetoresistance will result.

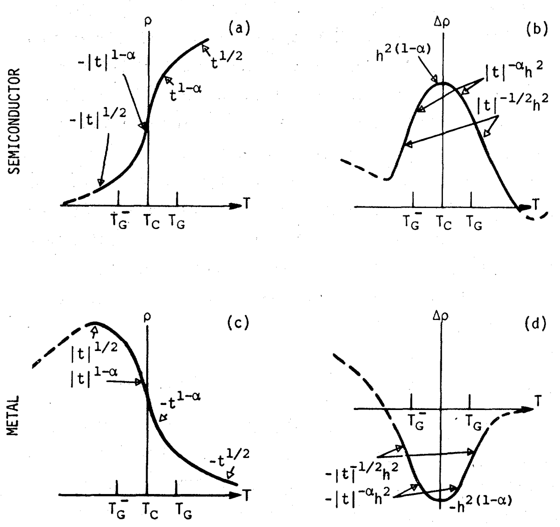

FIG. 3. Temperature dependence of the critical resistivity (a) and the critical magnetoresistivity (b) of an antiferromagnetic semiconductor. The temperature dependence of the critical resistivity (c) and the magnetoresistivity (d) of an antiferromagnetic metal. The dashed curves represent the regions where the molecular-field approximation applies.

The physical meaning of the above predictions is quite apparent when one recalls (see Sec. III A) that for effective scattering of the carriers one has to have $\Sigma \approx Q$ in the vicinity of T_c. If this condition is not fulfilled, as in the present case, the effectiveness of the scattering will increase with increasing temperature. This will be the case as long as the dominant fluctuations will have q's for which $Q \approx q > \Sigma$. Hence, the resistivity will assume the dependence shown in Fig. 3(a). The effect of an applied magnetic field, at a given temperature, is to enhance the shift to the more effective (in the present case) lower-q fluctuations and thus to yield positive magnetoresistance. This shift is responsible for the inhibition of the antiferromagnetic ordering and the lowering of T_c. The results of the magnetoresistivity shown in Fig. 3(b) indicate that for the fields considered by the present theory [Eq. (2.31)] a universal quadratic (small field, $|t| \gg h$ case) or a nearly quadratic (large field, $|t| \ll h$ case) magnetic-field dependences are predicted. As for ferromagnets, the diverginglike behavior of the magnetoresistance is obtained up to some small, but finite, $|t|$. Then the magnetoresistance saturates with temperature and the behavior of Eqs. (3.12) and (3.13) takes over.

For the mean-field regime [see Eqs. (3.5)–(3.7)] we can repeat the same procedure, i.e., take the leading $\rho(T, H=0)$ term $\kappa \propto \text{sgn}(t)|t|^{1/2}$ and replace t by t^*. Developing then in the limits $|t| \gg h$ and $|t| \ll h$ yields the results given in Eqs. (3.10)–(3.13) except that $1-\alpha$ is to be replaced by $\frac{1}{2}$ and $-\alpha$ by $-\frac{1}{2}$.

From (2.21) we have concluded that the leading terms of the mean-field correlation function will be $\Gamma_Q \propto -\langle\vec{S}\rangle^2 + \text{sgn}(t)\kappa/\Lambda$. So far we have considered only the second term, but in the large field ($t \ll h$) limit the first term may become dominant and a negative magnetoresistance results. This negative magnetoresistance reflects the effect of the reduction in the total number of fluctuations by the applied field. This result is expected from the molecular-field theory[17] for which the correlation function is $-m^2 \propto -h^2/[(T-\Theta)/T_N]$. One should note that when this behavior takes over the sign of the magnetoresistance changes and the magnetoresistance diminishes with increasing temperature.

Using the same approach far below T_c and recalling that for this case $\langle\vec{S}\rangle^2 = m^2(0)$ (but $|t|$ is replaced by $|t^*|$), we get again the molecular-field result[16] $\rho \propto -|t^*| = -|t| + fh^2 + |t|fh^2$. The resistivity in this region is decreasing with decreasing temperature due to the decrease in the number of fluctuations. The positive magnetoresistance results in this region from the inhibition of the antiferromagnetic ordering by an applied magnetic field. The regions where the molecular-field approximation is valid are represented in Fig. 3 by the dashed curves. This is to stress the fact that the behavior predicted for this region should be considered more as a qualitative estimate of the expected behavior since other scattering processes as well as band-structure effects can yield variations in the above temperature and magnetic-field dependences.

For an antiferromagnetic metal the momentum-transfer cutoff may be larger than Q and one has to carry out the integration (3.8) with the correlation functions (2.26) and (2.29). Sufficiently close to T_c and for small enough fields, $\langle \vec{S} \rangle^2$ is small compared with $\langle \vec{S}_{\vec{R}} \cdot \vec{S}_{\vec{R}'} \rangle$ and thus perturbation treatment of the potential $\langle \vec{S} \rangle^2$ is possible. In this paper we discuss only the close vicinity of T_c where this treatment is allowed. When this approximation is not allowed one has to consider the changes in band structure that take place below T_c. As will be discussed in Sec. IV the experimental evidence[25] suggests that this is a valid approximation up to $|t^*| \lesssim 0.1$, which is certainly satisfactory for our discussion on critical effects. Using this perturbation approach one can separate the two parts of the integral in (3.8). Then, by the δ-function integration one eliminates the $(ah)^2$ or the $(bh)^2$ terms. Finally, by the use of the transformation $\vec{q} \to \vec{q} + \vec{Q}$, the integral (3.8) takes the form

$$\rho \propto \int [\Gamma_q + m^2(0)\delta(\vec{q})] |\vec{q} + \vec{Q}| \, d^3q \ . \tag{3.14}$$

Addition and subtraction of $Q \int \Gamma_q d^3q$ then yields

$$\rho \propto Q \int [\Gamma_q + m^2(0)] d^3q + \int \Gamma_q (|\vec{q} + \vec{Q}| - Q) \, d^3q \ . \tag{3.15}$$

Using the spin sum rule (2.14) and the correlation functions (2.26) and (2.29), and assuming $Q \approx \Lambda = \Sigma$ (as in a proper metal), one finds that the first integral is simply

$$(2Q\pi^2/\Omega)[S(S+1) - (ah)^2] \quad \text{for } T > T_c$$

and

$$(2Q\pi^2/\Omega)[S(S+1) - (bh)^2] \quad \text{for } T < T_c \ .$$

The second integral, when $\vec{Q}$ is on the Brillouin-zone boundary (the normal case), can be easily shown to be[1]

$$-\tfrac{3}{2}\pi \int_0^Q \Gamma_q q^3 \, dq \ . \tag{3.16}$$

This term has the leading temperature and magnetic-field dependences: $-(t^*)^{1-\alpha}$ above T_c and $|t^*|^{1-\alpha}$ below T_c. The leading terms are then the same as those obtained for semiconductors [Eqs. (3.10)–(3.13)] but with the opposite sign.

The physical reason for the decrease of the resistivity upon temperature increase through T_c follows the decrease in the number of the large-q fluctuations with which the carriers interact effectively ($\Sigma = k_F \approx Q$). The magnetic field in the vicinity of T_c inhibites the large-q fluctuations and enhances the less effective (in metals) small-q fluctuations and thus negative magnetoresistance results. It is interesting to note that in the present perturbation approximation the effect of the antiferromagnetic periodic potential does not contribute to the critical resistivity up to order $m^2(0)$. This comes about due to the cancellation (to the above order) of the coherent scattering term [proportional to $m^2(0)$] by the modification of the free-electron-like eigenfunctions of the carries[42] [used for the derivation of (1.1)]. Overlooking the effect of the coherent scattering[26] thus yields an erroneous $m^2(0)$ dependence of the critical resistivity below T_c.

"Far" below T_c ($|t^*| \gtrsim 0.1$), the situation is not necessarily consistent with our perturbation approximation. However, the decrease in the number of fluctuations as low temperatures are approached must lead, finally, to the decrease of the resistivity with decreasing temperature in this range. Hence the behavior of metals is qualitatively similar to that of semiconductors and it can be described by the molecular-field approximation.[16] "Far" above T_c ($t \gtrsim 0.1$) the shift of the Néel temperature with magnetic field is not important and, again, the dominant contribution to the magnetoresistance can be treated by this approximation.[16,17] The above results for the resistivity of antiferromagnetic metals are summarized in Fig. 3(c) while those of the magnetoresistivity are summarized in Fig. 3(d). In the corresponding temperature regions, where only the qualitative features of the behavior are understood, the expected behavior is illustrated by a dashed curve. Again, as for semiconductors, the qualitative features of the magnetoresistance can be gathered from a shift of the $\rho(t)$ curve [of Fig. 3(c)] parallel to itself in the direction of low temperatures.

The results given above are for a proper metal ($k_F \gtrsim Q$). If the Fermi surface is not spherical the qualitative behavior may vary between that of a metal and that of a semiconductor. In particular if $Q_1 > k_{F1}$ in one direction and $Q_2 < k_{F2}$ in another direction, the tendency of the temperature dependence and the sign of the magnetoresistance can become complicated and direction dependent. However, the correlation between these two properties will always be conserved, and one can characterize the critical resistivity of antiferromagnets as semiconductor-like [Fig. 3(a) and Fig. 3(b)] or as metallic-like [Fig. 3(c) and Fig. 3(d)].

IV. COMPARISON WITH EXPERIMENTAL DATA

Most of the available data on the resistivity of magnetic materials around T_c was reviewed in I. Qualitatively, the temperature dependence of the resistivity that is shown in Figs. 1–3 have been confirmed.[1] The important consequence derived from the data examination was that the proportionality $d\rho/dT \propto C_p$ is well established at least semiquantitatively. Very recently, renewed analyses of available resistivity data have shown that the expectations, $\alpha = \alpha'$ as well as reasonable values for α, α', and A/A' confirm this expectation quantitatively. This was shown to be the case for the ferromagnetic metal iron[33] (see Sec. I) as well as for the antiferromagnetic metal dysprosium.[25] In the latter case, the data were good enough to find exactly t_G and to show that in its vicinity a transition from critical to mean-field behavior does take place. Hence, as far as existing data and analyses are concerned there is no indication that the quasistatic approximation as well as the use of the Ising correlation functions are not justified (no experimental deviation from the relation $d\rho/dT \propto C_p$ has been reported). Moreover, the quantitative verification of a specific-heat-like behavior in antiferromagnetic metals shows that the perturbation approximation used in Sec. III B is justified and that there is no need to involve critical band shifts in order to explain the resistivity of these materials in the critical regime.

For semiconductors the situation is more difficult since the critical change in the number of carriers[10,34] overshadows the critical change in the mobility. For comparison with the present predictions one has to examine the resisitivity of degenerate semiconductors or metals that are expected to have a small effective Fermi surface.[1] In the cases known[1,43] (the ferromagnet $GdNi_2$ and the antiferromagnets MnTe, GdSb, HoSb, and PrB_6) the qualitative behavior confirms the expectations [Figs. 2(a) and 3(a)]. However, no quantitative results such as the power-law dependences of the critical resistivity were reported for such systems in the critical regime.

In the study of the magnetoresistance most studies were of qualitative nature. Hence, the negative magnetoresistance of ferromagnetic metals[4] and ferromagnetic semiconductors[7] was confirmed. The predicted sign change of the magnetoresistance in antiferromagnetic metals as shown in Fig. 3(d) was found in the rare-earth metals.[8,9] On the other hand, we do not know about magnetoresistance measurements in an antiferromagnetic degenerate semiconductor, and the behavior shown in Fig. 3(b) is still unconfirmed experimentally. It should be mentioned that, as for the resistivity, the existing data concerning the magnetoresistance in semiconductors[10,15] are associated with the change in the number of carriers.[6,33,34]

The more quantitative studies of the critical magnetoresistance were concerned mainly with the magnetic-field dependences of the magnetoresistance. Almost all dependences shown in Figs. 1 and 3 were confirmed. In ferromagnetic nickel the linear magnetic-field dependence below T_c and the quadratic magnetic-field dependence above T_c were established.[14] In ferromagnetic gadolinium the $h^{(1-\alpha)/\beta\delta}$ dependence was shown to exist.[18] This finding, although semiqualitative (no number for $\alpha/\beta\delta$ was deduced), is the only result for magnetoresistance that can be associated with the critical regime. Gadolinium is also the only material for which the temperature dependence of the magnetoresistance was determined.[44] This was for $T < T_c$ where the $|t|^{-1/2}h$ dependence was found. In the antiferromagnetic metal holmium[13] the quadratic magnetic-field dependence was found below and above T_c. The dependence that was reported for a ferromagnetic semiconductor[12] seems to be associated with a change in the number of carriers, and thus quantitative comparison of experimental data with the predictions of Fig. 2 is still a hard task.

In summary, both the qualitative, and the presently available, quantitative results on the temperature dependence of the critical resistivity are in accord with the present predictions. The experimental semiquantitative studies of the magnetoresistance are also in agreement with the results of this work. On the other hand, there are almost no quantitative experimental determinations of the temperature dependence of the magnetoresistance.

V. SUMMARY AND CONCLUSIONS

The temperature and magnetic-field dependences of the critical resistivity were determined for ferromagnets and antiferromagnets, metals and semiconductors. It was shown that the temperature dependence of the critical resistivity below T_c is similar to that presented in I for $T > T_c$. A transition from a $-|t|^{1-\alpha}$ temperature dependence to a $-|t|^{1/2}$ temperature dependence is expected for ferromagnetic metals when the temperature is decreased below T_c. For a semiconductor the corresponding transition will be from a $-|t|^{1-\alpha}$ behavior to a $|t|^{2\nu}\ln|t|$ behavior and then to a $|t|\ln|t|$ behavior. In the intermediate cases of a relatively small Fermi surface the sequence of transitions will be $-|t|^{1-\alpha}$ to $-|t|^{1/2}$ to $|t|\ln|t|$ behavior. Similar temperature dependences are predicted for antiferromagnets except that no $|t|^{2\nu}\ln|t|$ behavior is expected and that the signs of $|t|^{1-\alpha}$ and $|t|^{1/2}$ are reversed for antiferromagnetic metals. In both magnetic systems when the Fermi surface is not spherical there may be three different cutoffs in the resistivity integral, and a more complicated behavior then described above may be found. However, even in this case, close enough to T_c a $|t|^{1-\alpha}$ behavior should always be observed and far enough (but $|t|\ll 1$) from T_c a $|t|^{1/2}$ or a $|t|\ln|t|$-type behavior is expected. The experimental results of studies in which the temperature dependence was determined are in agreement with the above predictions.

The magnetoresistance for small field ($h \ll |t|$) is expected to be always proportional to h^2 above T_c. Below T_c, a linear dependence of the magnetoresistance is expected for ferromagnets and a quadratic dependence for antiferromagnets. These dependences are also in agreement with the available experimental data. In the $h \gg |t|$ limit a rather universal behavior of the magnetic-field dependence of the resistivity, $h^{(1-\alpha)/\beta\delta}$ for ferromagnets and $h^{2(1-\alpha)}$ for antiferromagnets, is predicted. We do not know of a direct experimental proof for this prediction, but it is in accord with results of specific-heat measurements. The temperature dependence of the magnetoresistance is predicted to be peaked but nondivergent for ferromagnets and antiferromagnets both above and below T_c.

The present results yield new qualitative explanations and predictions for the critical resistivity of antiferromagnets. The "strange oscillatory" nature of the temperature dependence of the magnetoresistivity in the rare-earth metals (such as holmium) has not been explained before. Now, the negative magnetoresistance at the ferromagnetic-antiferromagnetic and the antiferromagnetic-paramagnetic transitions temperatures, as well as the positive magnetoresistance between them, are understood. The explanation is given here in terms of the magnetic-field effect on the relevant momentum transfers that take place in the scattering process at each temperature region. A qualitative prediction that calls for experimental verification is that of positive critical magnetoresistance in antiferromagnets which exhibit a semiconductor-like behavior. Hence in antiferromagnetic degenerate semiconductors such as MnTe or antiferromagnetic metals such as PrB_6 we expect positive magnetoresistivity in the vicinity of Néel temperature.

We feel, however, that the important predictions of the present paper are the power-law dependences of the critical magnetoresistance. This is because the predicted power laws are simple combinations of the critical exponents α, β, γ, δ, and ν. It seems, then, possible that the relatively simple measurements of critical resistivity and critical magnetoresistivity can yield many critical parameters. We believe that the main obstacles to full utilization of this method, the data analysis and the determination of the asymptotic regions, can be removed. Recent success in applying a new method of analysis to the critical-resistivity data of iron and dysprosium indicates that such data can even yield the temperature dependence of the critical parameters. The form of the temperature and magnetic-field dependences suggests that the association of the temperature and magnetic-field regions with the asymptotic limits $|t| \gg h$ or $|t| \ll h$ will not be too difficult. For example, by ensuring a linear magnetic-field dependence of the magnetoresistivity of a ferromagnet below T_c the $|t| \gg h$ region can be identified. In this region a measurement of the temperature dependence of the magnetoresistivity can yield the corresponding power law. From the combination of the power laws found in different regions one can then deduce the critical exponents. In this manner the exponents α, γ, and $(\beta\delta)$ can be found for ferromagnetic metals, and the exponents ν, β, γ, and δ can be found for ferromagnets that have a semiconductor-like behavior.

From the above it is seen that the present results account for the available experimental data. It can be concluded that comparison of sufficiently precise experimental data with our results can yield critical exponents and critical ratios. Such a comparison may even detect crossovers from one critical behavior to another.

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APPENDIX

To calculate $\phi(z)$ for $z>1$ we divide the integration interval in (2.3) into two regions, $1 \leq u \leq z^{\beta}$ and $z^{\beta} \leq u \leq \infty$. In the first region we have to use Eq. (2.5) [or rather Eq. (2.6)], and in the second region we have to use Eq. (2.4). Hence

$$\phi(z)=c_1\left(\int_1^{z^\beta} du\, u^{\delta-1/\beta}[f'(0)-0.58\gamma(z/u^{1/\beta})^{\gamma-1}] - \frac{1}{6}\int_{z^\beta}^{\infty} du\, u^{\delta-1/\beta}\ln(1+z/3u^{1/\beta}) + f'(0)/(\delta-1/\beta+1)\right). \quad \text{(A1)}$$

The first integral yields

$$f'(0)\frac{z^{\delta\beta+\beta-1}-1}{\delta-1/\beta+1} - 0.58\gamma z^{\gamma-1}\frac{z^{\delta\beta+\beta-\gamma}-1}{\delta-\gamma/\beta+1}.$$

The second integral can be written as

$$-\frac{\beta}{6}\left(\frac{z}{3}\right)^{1-\alpha}\int_0^{1/3} dv\, v^{-2+\alpha}\ln(1+v),$$

where the substitution $v=z/3u^{1/\beta}$, and the scaling relations $\delta\beta+\beta-1=1-\alpha$ and $\beta\delta+\beta-\gamma=2\beta$ were used. Carrying out this integration by using the approximation $\ln(1+v)\approx v$ yields the value $1/3^{\alpha}\alpha$ for this integral. Hence (A1) can be approximated by

$$\phi(z)\approx c_1\left[\left(\frac{\beta}{1-\alpha}-0.29\gamma-\frac{\beta}{18\alpha}\right)z^{1-\alpha}+\left(\frac{0.58\gamma\beta}{2\beta}\right)z^{\gamma-1}\right] \quad \text{(A2)}$$

The first two coefficients of $z^{1-\alpha}$ are almost equal and thus

$$\phi(z)\approx c_1[(-\beta/18\alpha)z^{1-\alpha}+0.29\gamma z^{\gamma-1}]. \quad \text{(A3)}$$

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