Localized electromagnetic pulses in a collisional medium

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We investigate the effects of a velocity-dependent collision frequency on the nonlinear propagation of an intense electromagnetic wave. It is shown that a velocity-dependent collision frequency produces a nonlinear contribution to the current density and hence to the wave equation. Using a WKB-like approximation, we show that the evolution of the wave electric field is governed by the nonlinear Schrödinger equation. The effective potential for the photon is found to be attractive and thus leads to modulation instability. The threshold and growth rate are obtained. Possible final states of the instability are periodic or localized envelopes of waves. The criteria for the occurrence of the N soliton are presented.

I. INTRODUCTION

The filamentation instability of an intense electromagnetic radiation in nonrelativistic and relativistic media are well known.[1,2] In nonrelativistic media,[1] the radiation pressure of the wave introduces a nonlinearity by coupling with finite-amplitude ion fluctuations. As a result, unstable modulations occur which eventually lead to envelopes of localized light channels[3] in the medium. When the intensity of the light becomes fairly large, one[2] cannot exclude the possibility of other nonlinearities arising from relativistic effects. In particular, relativistic modulations of light have been an intriguing problem.[4,5] For this case, the nonlinearity arising from the relativistic mass variation competes with the wave group dispersion to produce one[4] and multidimensional[5] localized pulses. Furthermore, the nonlinearity originating from the nonparabolic momentum-energy relation[6,7] can also lead to the localized electromagnetic pulses in semiconductors. However, the above-mentioned investigations apply only to a *collisionless* medium.

In this paper, we consider the nonlinear propagation of an intense electromagnetic wave in a nonrelativistic, *collisional* medium. In particular, we shall be concerned with the effects of nonlinearity[8] arising due to the velocity dependence of the collision frequency.[9] Such type of collision model is appropriate to describe various phenomena in semiconductors[8] and Ramsauer gases.[9] It is of interest to mention that Stenflo and Yu[8] have shown that in semiconductors, the collision-induced nonlinear excitations may be more important than those due to other mechanisms.[7] On the other hand, the nonlinear phenomena such as echoes[9] and stimulated emissions[10] in a partially ionized gaseous plasma can successfully be explained by using the Harp model[9] for the collision frequency.

The plan of this paper is the following. In Sec. II, we present a brief description of the Lorentz collision operator[8,11] with velocity-dependent collision frequency.[9] Using this collision model, we then calculate the nonlinear response of the plasma to a finite-amplitude continuous electromagnetic wave train. Section III shows that, in the presence of the nonlinear current density,[8] the evolution of the wave electric field is governed by the nonlinear Schrödinger equation.[1] The modulational instability of a constant-amplitude wave packet is discussed in Sec. IV. The growth rate and threshold are obtained. Section V shows that possible final states of the unstable modulations may be localized wave packets. The analytical results for the latter are presented. A brief discussion of the results is contained in Sec. VI.

II. COLLISION MODEL AND THE NONLINEAR CURRENT DENSITY

The collision model which we use is the simple Lorentz collision operator,[8,11] with the velocity-dependent collision frequency represented by the Harp model.[9] According to the Lorentz model, the electrons collide elastically with neutral particles and produce a momentum transfer given by

$$C(F) = -\nu F + \frac{\nu}{4\pi}\int F\,d\Omega\,, \tag{2.1}$$

where $\nu(v^2)$ is the electron–neutral-particle collision frequency. The integration is over all solid angles Ω in velocity space. This model neglects the electron-electron scattering, and assumes that the collision frequency depends only on the magnitude of the electron velocity. The Harp model[8–10] for the velocity dependence of the collision frequency is

$$\nu(v^2) = \begin{cases} 0\,, & v^2 < v_0^2\,, \\ \infty\,, & v^2 > v_0^2\,, \end{cases} \tag{2.2}$$

where v_0 is chosen to be in the region where the actual collision frequency increases rapidly with velocity. Basically, the Harp model emphasizes the difference between the collisional characteristics of high- and low-energy electrons. In particular, it follows from Eqs. (2.1) and (2.2), that the electrons with speeds less than v_0 behave as a collisionless gas, while those with higher speeds behave as a collision-dominated gas. The widely used[8,10] Harp model is actually realistic for gases showing a strong Ramsauer effect, such as argon, which exhibits a sharp increase of the electron–neutral-particle collision frequency within a certain electron velocity range. For other plasmas, the use of the Harp model can be questioned. The reason is because the critical speed v_0 cannot be uniquely defined.

The distribution function $F(\vec{x},\vec{v},t)$ of the electrons in an unmagnetized, collisional plasma is governed by

$$\frac{\partial F}{\partial t}+\vec{v}\cdot\frac{\partial F}{\partial\vec{x}}-\frac{e}{m}\vec{E}\cdot\frac{\partial F}{\partial\vec{v}}=C(F)\,, \tag{2.3}$$

where $-e$ and m are the charge and constant mass of the electrons, and $\vec{E}(\vec{x},t)$ is the total electric field. The wave magnetic field is neglected. The choice of the constant electron mass allows us to exclude the nonlinearities arising from the relativistic mass variation, the nonparabolic momentum relation,[7] as well as energy dependence of the relaxation of the carrier.[12]

In general, the integration of Eq. (2.3) is complicated because of the fact that $\vec{E}$ depends on $\vec{x}$ and t in the third term. However, if we examine the phase in detail, we note that $\partial(\vec{k}\cdot\vec{r}-\omega t)/\partial t = -\omega[1-(v/c)\vec{k}\cdot\vec{v}]$. Hence, consistent with the neglect of the wave magnetic field we can neglect the factor v/c and treat x as though it were constant. This may also correspond to the assumption that the spatial variation of the wave electric field is slower than the fast time variation. We can then approximately write down the lowest order solution of (2.3) in the form

$$F(\vec{v},t)=\begin{cases}F_0\left(\vec{v}+\dfrac{e}{m}\displaystyle\int_{-\infty}^{t}\vec{E}\,dt\right), & v^2<v_0^2\,,\\ G(v^2,t)\,, & v^2>v_0^2\,,\end{cases} \tag{2.4}$$

where $G(v^2,-\infty)=F_0(\vec{v})$, and corresponds to the distribution of the electrons in the absence of the electric field, say at $t=-\infty$. Thus, we may take $F_0(\vec{v})$ to be isotropic in velocity space. The application of a time-dependent electric field causes the variation of the number of particles in each region. Consequently, the distribution function of the high-energy electrons turns out to be time dependent.

The x component of the current density is given by

$$J_x=-2\pi e\int_{-1}^{1}\int_{0}^{v_0}dv_x\,d\mu\,v^3\mu F_0 \times\left(v_x+\frac{e}{m}\int^{t}E_x\,dt\right), \tag{2.5}$$

where $\mu=v_x/v$, and the contributions of the electrons with velocity $v>v_0$ are ignored to the current density. The reason is that those electrons remain isotropic in velocity space due to the strong scattering. For sufficiently weak electric fields, Eq. (2.5) becomes[8]

$$J_x\approx\frac{N_0e^2}{m}\int_0^t E_x\,dt-\frac{8\pi e^4}{15m^3}\,v_0^5F_0''\left(\int_0^t E_x\,dt\right)^3, \tag{2.6}$$

where

$$F_0''=\left(\frac{\partial F_0(v)}{\partial(v^2)^2}\right)_{v^2=v_0^2}$$

and

$$N_0=-\frac{4\pi}{3}\int_0^{v_0}v^3\frac{\partial F_0}{\partial v}\,dv\,. \tag{2.7}$$

III. NONLINEAR WAVE EQUATION

Inserting (2.6) into Ampère's law, one obtains the one-dimensional equation for a linearly polarized electromagnetic wave train

$$\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}+\omega_{pe}^2\right)E=\frac{8\pi e^4}{15\epsilon_0m^3}\,\frac{v_0^5}{\omega^2}F''|E|^2E\,, \tag{3.1}$$

where $E\equiv E_x$, and $\omega_{pe}=(N_0e^2/m\epsilon_0)^{1/2}$ is the effective electron plasma frequency. Equation (3.1) governs the propagation of an electromagnetic wave in a weakly nonlinear dispersive medium.

We now use a modulational representation[1] and accordingly express $E=A(x,\tau)e^{-i\omega t}+\text{c.c.}$, where $A(x,\tau)$ is the slowly varying complex amplitude. Then, from Eq. (3.1), we obtain the following equation for the evolution of the wave electric field

$$i\frac{\partial A}{\partial\tau}+\beta\frac{\partial^2A}{\partial x^2}+\alpha|A|^2A=0\,, \tag{3.2}$$

where $\beta=c^2/2\omega$, $\omega^2=\omega_{pe}^2+c^2k^2$, $x=x-v_g\tau$, $v_g=\partial\omega/\partial k$, and $\alpha=4\pi e^4v_0^5F''/15m^3\epsilon_0\omega^3$.

IV. MODULATIONAL INSTABILITY

Let us study the stability[1] of a constant envelope wave packet with respect to low-frequency long-wavelength phonons (Ω,K; $\Omega\ll\omega$, and $K\ll k$). For this case, the electric field A has a finite amplitude A_0, frequency $\omega_0\approx\omega_{pe}$, and wave vector $k_0\approx k$ as $x\to\pm\infty$. Accordingly, the term involving $|A|^2$

in Eq. (3.2) should be replaced by $\alpha(|A|^2-|A_0|^2)A$. On separating the real and imaginary parts of A as

$$A=[\rho(x,\tau)]^{1/2}\exp[i\theta(x,\tau)]\,, \tag{4.1}$$

we obtain from (3.2),

$$\frac{\partial\rho}{\partial\tau}+\frac{c^2}{\omega}\frac{\partial}{\partial x}\left(\rho\frac{\partial\theta}{\partial x}\right)=0 \tag{4.2}$$

and

$$\alpha(\rho-\rho_0)-\frac{\partial\theta}{\partial\tau}+\frac{c^2}{4\omega\rho}\frac{\partial^2\rho}{\partial x^2}-\frac{c^2}{\omega}\left(\frac{\partial\theta}{\partial x}\right)^2-\frac{c^2}{8\omega\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2=0\,. \tag{4.3}$$

Linearizing (4.2) and (4.3) in the form,

$$\rho=\rho_0+\tilde\rho e^{iKx-i\Omega\tau}\,,\quad \tilde\rho\ll\rho_0\,, \tag{4.4}$$

$$\theta=\tilde\theta e^{iKx-i\Omega\tau}\,, \tag{4.5}$$

and combining the resulting equations we find

$$\begin{vmatrix} -i\Omega & -\rho_0c^2K^2/\omega \\ \alpha-c^2K^2/4\omega\rho_0 & i\Omega \end{vmatrix}=0\,. \tag{4.6}$$

From (4.6), one obtains the following dispersion relation:

$$-\Omega^2+c^4K^4/4\omega^2-(\alpha c^2K^2/\omega)|A_0|^2=0\,. \tag{4.7}$$

It follows from (4.7), that modulations are unstable if

$$|A_0|^2\geq c^2K^2/4\omega\alpha\,. \tag{4.8}$$

Since $v_t<v_0$, the distribution function, $F_0\sim\exp(-v^2/v_t^2)$, can be assumed to be Maxwellian. Equation (4.8) thus becomes

$$\frac{e^2|A_0|^2}{m^2\omega^2c^2}\geq\frac{15}{4\pi^{1/2}}\left(\frac{v_t}{v_0}\right)^5\frac{K^2v_t^2}{\omega_{pe}^2}\exp\left(\frac{v_0^2}{v_t^2}\right), \tag{4.9}$$

where we used $\omega\approx\omega_{pe}$, and the relation

$$\alpha=\frac{4\pi^{1/2}}{15}\frac{\omega_{pe}^2}{\omega^3}\frac{e^2}{m^2}\left(\frac{v_0}{v_t}\right)^5\exp\left(\frac{-v_0^2}{v_t^2}\right). \tag{4.10}$$

the maximum growth rate is given by

$$\gamma=\frac{8\pi^{1/2}}{15}\frac{\omega_{pe}^4}{\omega^3}\left(\frac{|A_0|^2}{8\pi n_0T_e}\right)\left(\frac{v_0}{v_t}\right)^5\exp\left(\frac{-v_0^2}{v_t^2}\right)\,, \tag{4.11}$$

and is attained for $K_c=(2\alpha\omega/c^2)^{1/2}$. Thus, a perturbation whose wavelength λ is longer than $2\pi/K_c$ becomes unstable.

Let us now estimate the magnitude of the growth rates for both gaseous and semiconductor plasmas. First, for an argon plasma, the use of relevant parameters, namely, $N_0=10^{10}$ cm^{-3}, $(\omega_{pe}/\omega)^2\approx1$, $T_e=10^3$ °K, $|A_0|^2/8\pi n_0T_e\approx10^{-2}$ yields a maximum growth rate $\gamma\approx0.01\omega_{pe}$ for $v_0\simeq1.6v_t$. Next, we apply our results to the semiconductor and take a pure sample of the InSb. For a peak field strength of $A_0=121$ V/cm, we obtain $\omega=1.25\times10^{13}$ rad/sec. Choosing a typical value of $\omega_{pe}=1.25\times10^{13}$ rad/sec, and $T_e=77$ °K, we find $\gamma\approx0.02\omega_{pe}$ for $v_0\approx1.6\,v_t$. Our conclusion is that because the typical growth rates are fairly large, the instability discussed here should be observable in gaseous and semiconductor plasmas.

V. LOCALIZED ELECTROMAGNETIC PULSES

A. Stationary solution

Possible final states of the modulationally unstable electromagnetic wave is given by the solutions of the cubic nonlinear Schrödinger equation. The most general stationary solution is the periodic wave train

$$A(x,\tau)=R^{1/2}e^{i\phi(x,\tau)}\,, \tag{5.1}$$

where $Rd\phi/d\tau=M$, and

$$R(x)=C_2+(C_1-C_2)C_n^2([|\alpha/6\beta|(C_1-C_3)]^{1/2}x,\kappa)\,. \tag{5.2}$$

Here $\kappa^2=(C_1-C_2)/(C_1-C_3)$, and $C_1\geq C_2\geq C_3$ are the three solutions of the equation

$$(2\alpha/\beta)C^3-(2\alpha/\beta)C^2+PC+4M^2=0\,, \tag{5.3}$$

A particular simple solution,[1] which follows from Eq. (5.1) is

$$A(x,\tau)=A_M\,\mathrm{sech}[k_0(x-v_gt)]\exp(-iWt)\,, \tag{5.4}$$

where $k_0=(\alpha/2\beta)^{1/2}A_M$, $W=\frac{1}{2}\alpha A_M^2$, and A_M is the maximum amplitude of the soliton. We note that the collision-induced nonlinearity competes with the group dispersion of the electromagnetic wave train, allowing for localized solitonlike solutions.

We now discuss under what conditions the envelope solutions presented above persist. Because we are chiefly concerned with the continuous excitations of electromagnetic wave trains, the stable quasistationary localized pulses will occur in a nonlinear dispersive medium. The reason that this occurs is because the energy dissipated in the perpendicular direction (perhaps due to some instability) is counterbalanced by the energy fed into the quasistationary solitons.

B. N-soliton solutions

Zakharov and Shabat[13] applied inverse scattering method to solve the cubic nonlinear Schrödinger equation and obtained N-soliton solutions. In the following, we use the results of Zakharov and Shabat[13] to obtain the conditions necessary for the occurrence of localized electromagnetic pulses. Introducing the variables $t'=\tau/\beta$, $\eta=x/\beta$, in Eq. (3.2) leads to

$$i\frac{\partial A}{\partial t'}+\frac{\partial^2 A}{\partial \eta^2}+p|A|^2A=0\,, \tag{5.5}$$

where $p=\alpha\beta$. Let us consider the particular simple excitation at $t'=0$, viz.,

$$A(0,\eta)=A_M\,\mathrm{sech}(\eta/\Delta\eta)\,, \tag{5.6}$$

and study the soliton production by this excitation. To attain this goal, it is necessary to solve for the discrete eigenvalue ξ of the equations[13]

$$v_1'+i\xi v_1=qv_2\,, \tag{5.7}$$

$$v_1'-i\xi v_2=-q^*v_1\,, \tag{5.8}$$

where $q=i(\frac{1}{2}p)A(t'=0,\eta)$, and the prime denote differentiation with respect to η.

From Eqs. (5.6), (5.7), and (5.8) one obtains

$$(1-\zeta^2)\frac{d^2v_1}{d\zeta^2}-\zeta\frac{dv_1}{d\zeta}+\left(\frac{p}{2}(A_M\Delta\eta)^2+\frac{(\xi\Delta\eta)^2+i\xi\zeta\Delta\eta}{1-\zeta^2}\right)v_1=0\,, \tag{5.9}$$

where $\zeta=\tanh(\eta/\Delta\eta)$, and $\Delta\eta$ corresponds to the width of the soliton.

Equation (5.9) admits a solution which is finite at $\zeta=1$ (i.e., for $\eta=+\infty$). We have

$$v_1=\left(\frac{1-\zeta}{2}\right)^{s_0}\left(\frac{1+\zeta}{2}\right)^{t_0}F\left(a,\ b,\ d,\ \frac{1-\zeta}{2}\right), \tag{5.10}$$

where F is the hypergeometric function,[15] and

$$s_0=\tfrac{1}{2}(1-i\xi\Delta\eta)\,,\quad t_0=-\tfrac{1}{2}i\xi\Delta\eta\,,$$
$$a=s-(\tfrac{1}{2}p)^{1/2}A_M\Delta\eta\,,\quad s=0.5-i\xi\Delta\eta\,, \tag{5.11}$$
$$b=s+(\tfrac{1}{2}p)^{1/2}A_M\Delta\eta\,,\quad d=1.5-i\xi\Delta\eta\,.$$

In order that v_1 be finite at $\zeta=-1$ (i.e., for $\eta=-\infty$), it is necessary[14] that $a=-n$, where $n=0,\ 1,\ 2\ldots$ Thus, Eq. (5.10) leads to the eigenvalues

$$\xi=i[-n-0.5+(\tfrac{1}{2}p)^{1/2}A_M\Delta\eta]/\Delta\eta\,. \tag{5.12}$$

Since solitons are formed only if discrete eigenvalues exist. This happens for

$$(\tfrac{1}{2}p)^{1/2}A_M\Delta\eta>0.5\,. \tag{5.13}$$

Using $\Delta\eta=\Delta x/\beta$ and $p=\alpha\beta$, Eq. (5.11) reduces to

$$(2\alpha/\beta)^{1/2}A_M\Delta x>1\,. \tag{5.14}$$

It should be noted that one can also obtain eigenvalues for various other functional forms of the excitation. It is expected that condition (5.14) should not change dramatically.

VI. SUMMARY

In this paper, we have considered the nonlinear propagation of an electromagnetic wave in a partially ionized collisional gas. The nonlinearity, originating from the velocity dependences of the electron collision frequency, gives rise to nonlinear modification to the wave dispersion relation. The wave is found to be modulationally unstable against the low-frequency perturbations. It is shown that unstable modulations may evolve to localized pulses. Our study of the N-soliton production should be useful to the understanding of nonlinear wave phenomena occurring in a collisional medium.

To the best of the author's knowledge, envelope solitons are observed in collisionless laboratory plasmas.[15] However, we believe that the results of this paper should apply to collision dominated gaseous, as well as semiconductor plasmas. Detailed comparisons with available experimental observations are, however, needed in the future to verify the theoretical results presented here.

Finally, we mention some of the assumptions involved in our investigation. In particular, we have ignored the heating of the electrons in the low-speed domain due to the electromagnetic radiation. Usually, after a time of the order of the energy relaxation time, the thermal speed of the conduction electrons might rise and consequently the distribution function may flatten. A detailed study accounting for this particular effect is beyond the scope of this paper. On the other hand, to have a better understanding of nonlinear wave phenomena in collision dominated plasmas, one must also refine the widely used[8-10] Harp model for the collision frequency.

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