Tricriticality and the failure of scaling in the many-component limit

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We analyze exactly in the limit $n \rightarrow \infty$, the n-component continuous-spin model with a cubic field term $\vec{H}_3 \cdot \vec{s}|\vec{s}|^2$. Full "four-field" tricritical behavior is exhibited for dimensionalities $d \geq 3$. However, orthodox tricritical scaling is shown to be impossible for $3 < d < 4$: to obtain the correct nonclassical spherical-model exponents on the λ line ($H = H_3 = 0$, $T > T_t$) it is essential to allow for a dangerous irrelevant scaling variable $p \propto 1/R_0^d$, where R_0 is the range of the pair interactions. The appropriate crossover exponent is $\phi_p = 3 - d$ so that p is marginal for $d = 3$: orthodox scaling is then possible but the scaling functions are nonuniversal. On the disordered symmetry plane ($H = H_3 = 0$) only the *corrections* to scaling survive and describe crossover to Gaussian tricritical behavior. For $T > T_t$ bicritical crossover from spherical to classical critical behavior occurs when H_3 varies but scaling is fully obeyed. Some inferences for systems with finite n are drawn.

I. INTRODUCTION

The scaling hypothesis for the vicinity of a tricritical point was first discussed by Riedel[1] and has since been developed further and extended into a general theory of behavior near a multicritical point.[2,3] In a system exhibiting *symmetric* tricritical behavior, such as a metamagnet[3] or fluid-helium three-four mixtures, a tricritical point may be identified as the point where a critical locus $T_c(g)$ in the thermodynamic symmetry plane (T, g) terminates and changes into a locus of first-order transitions (which, in reality, is frequently a line of triple points, i.e., of three-phase coexistence). Of course, the critical behavior of, say, the specific heat and ordering susceptibility is described by exponents at the tricritical point which differ from those on the critical line: the tricritical scaling hypothesis describes how the crossover from one sort of behavior to the other occurs. However, whereas an ordinary critical point is characterized by a single-ordering field, say h, which in a metamagnet is a staggered magnetic field, a full description of a tricritical point requires the identification of a further, independent, *odd field*, which we may call the *cubic field* h_3. The need for the *four* thermodynamic field variables g, T, h, and h_3, to describe tricritical behavior may be seen most directly from a Landau or classical phenomenological theory.[2,3] If m denotes the order parameter, the coefficients of the terms m^2, m^4, and m and m^3 in the free-energy expression must all vanish at tricriticality. (The coefficient of m^5 may be arranged to vanish identically, while that of m^6 must remain positive for stability.) The effects of h_3 on a tricritical phase diagram in the space (T, g, h) may be seen from Fig. 2 below. Although normally concealed in magnetic systems, because it is coupled to h, the cubic field plays an essential role in the description of tricriticality as observed in multicomponent fluid systems.[2] However, it has been hardly investigated theoretically. One of the aims of the work reported here was to help repair this omission.

The scaling hypothesis for ordinary critical points has been well verified by numerous theoretical studies, including, in recent years, by explicit renormalization-group calculations. Much experimental work on equations of state also demonstrates that scaling works well and confirms, for systems within the appropriate classes, the predicted universality of exponents *and* scaling functions. The same is far from true for tricritical points where, because of the relevance of four thermodynamic fields and the existence of various critical manifolds including lines of critical endpoints, many more complex and subtle features arise. However, it is known that the classical phenomenological theory verifies all the detailed predictions of tricritical scaling theory.[2,3] More recently, renormalization-group ϵ-expansion techniques have been applied by Nelson and Rudnick[4] to calculate the tricritical equation of state in the disordered region of the plane of symmetry (T, g), for a system with n-component, continuous vectorial spins, $\vec{s}_i$ (interacting, as usual, through a Landau-Ginzburg-Wilson-type Hamiltonian). At first sight, the results of this analysis are in good agreement with the general scaling theory. However, we will show that this impression is erroneous!

Indeed, some difficulties in a naive application of scaling have been noted previously in connection with certain exactly soluble models exhibiting tricritical behavior. The models are fairly special, but quite subtle analyses, sometimes entailing the recognition of particular nonlinear scaling fields, have been needed to cast the results in the anticipated scaling forms.[5,6]

The main goal of the present paper is to study tricritical behavior and test scaling in a general but exactly soluble limit that has already proved very fruitful in studying ordinary critical behavior. This is the infinite-component or $n\to\infty$ limit which is known to be essentially equivalent to the spherical model.[7,8] This limit has the advantage that the critical behavior still depends nontrivially on the dimensionality d and is nonclassical for $d<4$. Likewise the dependence of the critical exponents on the range and decay law of the interactions can be studied exactly in the spherical model.

Explicitly, on a regular d-dimensional lattice with sites $i=1,\ldots,N$ we consider the Hamiltonian

$$\mathcal{H}=\tfrac{1}{2}\sum_{(i,j)} J_{ij}\,|\vec{s}_i-\vec{s}_j|^2 - \sum_{i=1}^{N}[\vec{H}\cdot\vec{s}_i+(\vec{H}_3\cdot\vec{s}_i)|\vec{s}_i|^2/n] + \sum_{i=1}^{N}(\tfrac{1}{2}D|\vec{s}_i|^2+\tfrac{1}{4}U|\vec{s}_i|^4/n+\tfrac{1}{6}V|\vec{s}_i|^6/n^2)\,, \qquad (1.1)$$

where the spins $\vec{s}_i$ and the two fields

$$\vec{H}=(H,H,\ldots,H),\quad \vec{H}_3=(H_3,H_3,\ldots,H_3)\,, \qquad (1.2)$$

are classical n-component vectors. The first sum runs over all pairs (i,j) of sites, while

$$J_{ij}=J_{ji}=J(\vec{R}_i-\vec{R}_j)\geq 0,\quad J(\vec{0})=0\,, \qquad (1.3)$$

represents a ferromagnetic pair coupling with Fourier transform

$$\hat{J}(\vec{k})=\sum_{\vec{R}} e^{i\vec{k}\cdot\vec{R}}J(\vec{R})\,. \qquad (1.3a)$$

To obtain tricritical behavior we take U and V fixed with

$$U<0 \quad\text{and}\quad V>0\,, \qquad (1.4)$$

and regard the quadratic coupling D as a variable field. (This assignment of "variables" is largely dictated by simplicity and convenience: as will be evident below, one could choose D fixed and regard U as a variable, for example.) The factors $1/n$ and $1/n^2$ are inserted so as to yield sensible behavior in the limit $n\to\infty$.

As we show, this model is exactly soluble in the limit $n\to\infty$ for arbitrary H *and* H_3 (by an extension of the techniques used by Emery[8] and Sarbach and Schneider[9,10]). Furthermore, it exhibits a tricritical point for all dimensions $d\geq 3$ (including nonintegral d). It is, therefore, an excellent candidate to test the tricritical scaling hypothesis. The results, however, are somewhat disturbing[11]: in the range $3<d<4$, scaling does *not* hold in a direct form. A complete description of asymptotic tricriticality, including the nonclassical exponents on the critical line demands the recognition of a *dangerous irrelevant variable*[12] p, which can be associated with the (nonzero) inverse range of the pair interactions. Even in $d=3$ dimensions, where scaling is obeyed, we find that the scaling function is nonuniversal, depending explicitly on p.

The paper is set out as follows: in Sec. II the $n\to\infty$ limit is performed for a more general Hamiltonian with nonlinear odd couplings. In Sec. III, the phase diagram for the particular model embodied in (1.1) is discussed. Tricritical scaling is taken up in Sec. IV, where the need for the dangerous irrelevant variable is established. Section V summarizes the results and draws various conclusions. Some proofs and computational details are relegated to the Appendix.

II. GENERAL n-COMPONENT MODEL

In this section, we show how to compute the free energy in the thermodynamic limit of an n-component Hamiltonian more general than (1.1) and, we believe, more general than that covered by previously established theory. Specifically, in terms of n-component spins $\vec{s}_i=(s_i^\mu)$ with $\mu=1,2,\ldots,n$, located on the sites $i=1,2,\ldots,N$ of a regular d-dimensional lattice with periodic boundary conditions, consider

$$\mathcal{H}_{N,n}(\{\vec{s}_i\})=\mathcal{H}^I_{N,n}(\{\vec{s}_i\})+\sum_{i=1}^{N}[nW_1(|\vec{s}_i|^2/n)+\vec{1}\cdot\vec{s}_i W_2(|\vec{s}_i|^2/n)]\,, \qquad (2.1)$$

where

$$|\vec{s}_i|^2=\sum_{\mu=1}^{n}(s_i^\mu)^2 \quad\text{and}\quad \vec{1}\cdot\vec{s}_i=\sum_{\mu=1}^{n}s_i^\mu\,, \qquad (2.2)$$

while the pair interactions are given, as in (1.1), by

$$\mathcal{H}^I_{N,n}=\tfrac{1}{2}\sum_{(i,j)}J_{ij}|\vec{s}_i-\vec{s}_j|^2\,, \qquad (2.3)$$

where the J_{ij} again satisfy the ferromagnetic conditions (1.3). The weight functions $W_1(x^2)$ and $W_2(x^2)$ must have integrable Boltzmann factors and be sufficiently well behaved for large x to ensure the existence of the free energy in the thermodynamic limit.

The partition function associated with (2.1) is

$$Z_{N,n}(T;J;W)=\int d\;[\text{illegible}]\;[-\mathcal{H}_{N,n}(\{\vec{s}_i\})/k_BT]\,, \qquad (2.4)$$

and the corresponding free-energy density (i.e., free energy per spin component) is

$$F_{N,n}(T;J;W) = -(k_BT/Nn)\ln Z_{N,n}(T;J;W)\,. \quad (2.5)$$

We are interested in the thermodynamic limit $N\to\infty$, and the infinite-component or spherical-model limit $n\to\infty$.

To calculate the free energy explicitly in these two limits, taken in *any* order, we will follow Kac and Thompson[13] and use standard results on the ordinary spherical model[14,15] to obtain a lower bound on $F_{N,n}(T;J;W)$; then, following Emery,[8b] we employ the thermodynamic variational principle (or Bogoliubov inequality)[8,16] to obtain a complementary upper bound. The detailed proofs are given in Appendix A. The result may be written

$$F(T;J;W) = \lim_{N,n\to\infty} F_{N,n}(T;J;W)$$
$$= \min_{\xi}\{\tfrac{1}{2}k_BT\mathfrak{F}_d(\zeta;J) + W_1(\xi^2) - \tfrac{1}{2}\zeta\xi^2 - \tfrac{1}{2}[W_2(\xi^2)]^2/\zeta\}\,, \quad (2.6)$$

where $\zeta=\zeta(\xi,T;J;W)$ is the solution of the constraint equation

$$\xi^2 = k_BTI_d(\zeta;J) + [W_2(\xi^2)]^2/\zeta^2\,, \quad (2.7)$$

while the underlying free-energy and correlation functions are given explicitly by

$$\mathfrak{F}_d(\zeta;J) = \int \frac{a^d d\vec{k}}{(2\pi)^d}\ln\{[\zeta + \hat{J}(\vec{0}) - \hat{J}(\vec{k})]/2\pi k_BT\}\,, \quad (2.8)$$

$$I_d(\zeta;J) = \frac{\partial \mathfrak{F}_d}{\partial\zeta} = \int \frac{a^d d\vec{k}}{(2\pi)^d}[\zeta + \hat{J}(\vec{0}) - \hat{J}(\vec{k})]^{-1}\,, \quad (2.9)$$

where $\hat{J}(\vec{k})$ is defined in (1.3) and a is the lattice spacing, and the integrals run over the appropriate Brillouin zone (which may be well approximated for many purposes by the sphere $|\vec{k}|<\pi/a$).

A few remarks are in order: for suitable $W_1(x^2)$ the model reduces to the usual fixed length spin model originally shown[7] to be equivalent to the Berlin-Kac spherical model.[13,14] If $W_2(x^2)$ is simply a constant, the result has been demonstrated before both in the n-component formulation[8] as well as in the spherical-model picture.[9,10] If one is satisfied to take the $n\to\infty$ limit *before* the thermodynamic limit, $N\to\infty$, Emery's integral representation[8a] is very convenient for computing the free energy.

It is not hard to see from (2.6) and (2.7) that a necessary and sufficient condition for the existence of the limiting free energy is

$$\lim_{x^2\to\infty} x^2/W_1(x^2) = \lim_{x^2\to\infty} xW_2(x^2)/W_1(x^2) = 0\,. \quad (2.10)$$

For future reference, we also quote the result

$$m = \langle s\rangle = \lim_{N,n\to\infty}\langle n^{-1}\sum_{\mu=1}^{n} s^{\mu}\rangle$$
$$= -W_2(\xi^2)/\zeta(\xi_0,T;J,W)\,, \quad (2.11)$$

for the magnetization (per spin component) in which $\xi_0(T;J,W)$ is the minimizing value of ξ. In addition, we have

$$m_2 = \langle s^2\rangle = \lim_{N,n\to\infty}\langle n^{-1}|\vec{s}|^2\rangle = \xi_0^2(T;J,W)\,. \quad (2.12)$$

III. THERMODYNAMICS AND THE PHASE DIAGRAM

It is instructive to consider the free energy (2.6) of the general model (2.1) in the case in which the terms $\mathfrak{F}_d$ and I_d do not contribute as, for example, when $T\to 0$. The constraint equation (2.7) together with (2.11) and (2.12) then reduces to

$$m_2 \equiv \langle s^2\rangle = \xi^2 = \langle s\rangle^2 \equiv m^2. \quad (3.1)$$

On using this, the free energy becomes

$$F(T;W) = \min_m[W_1(m^2) + mW_2(m^2)]\,, \quad (3.2)$$

which is evidently of the form of a classical, phenomenological free energy. One may thus anticipate that inclusion of the "fluctuation terms," $\mathfrak{F}_d$ and I_d, will yield a phase diagram qualitatively similar to that predicted by the corresponding phenomenological theory. Indeed, this observation guides the choice of the weight functions as

$$W_1(x^2) = \tfrac{1}{2}Dx^2 + \tfrac{1}{4}Ux^4 + \tfrac{1}{6}Vx^6\,, \quad (3.3)$$

with $U<0$ and $V>0$, and

$$W_2(x^2) = -H - H_3x^2\,, \quad (3.4)$$

which reduces the general Hamiltonian (2.1) to that presented in the Introduction. Note also that the criteria (2.10) are satisfied.

The integrals defining $\mathfrak{F}_d(\zeta;J)$ and $I_d(\zeta;J)$ can be studied for arbitrary J_{ij}, or $\hat{J}(\vec{k})$, but three cases of particular interest are the following.

Nearest-neighbor interactions

$$J_{ij} = \begin{cases} J>0, & \text{for } i,j \text{ adjacent sites,} \\ 0, & \text{otherwise,} \end{cases} \quad (3.5)$$

which are characteristic of all short or finite-range interactions.

Long-range interactions

$$J_{ij} = J_\sigma(a/|\vec{R}_{ij}|)^{d+\sigma}, \quad \text{with } J_\sigma,\ \sigma>0\,. \quad (3.6)$$

Kac potentials

$$J_{ij} = J_0(a/R_0)^d\varphi(|\vec{R}_{ij}|/R_0)\,, \quad (3.7)$$

in which R_0 represents the range of the interactions, while the shape function $\varphi(r)$ is bounded for all $r \geq 0$ and integrable on $(0, \infty)$. Notice that for fixed R_0, say, $R_0 = a$, *all* potentials can be written in the Kac form.

It will be relevant to note that the thermodynamics can be discussed exactly[17,18] for *all n* in the van der Waals or infinite-range limit, $R_0 \to \infty$. However, all critical and tricritical behavior is then entirely classical.

Since $W_1(\xi^2)$ and $W_2(\xi^2)$ are analytic, the minimization in (2.6) may be performed with the aid of differentiation. This leads to

$$F(T, D, H, H_3; J) = \tfrac{1}{2}k_B T \mathfrak{F}_d(\zeta; J) + \tfrac{1}{2}D\xi^2 + \tfrac{1}{4}U\xi^4 + \tfrac{1}{6}V\xi^6 - \tfrac{1}{2}\zeta\xi^2 - \tfrac{1}{2}(H + H_3\xi^2)^2/\zeta, \tag{3.8}$$

where ξ^2 and ζ are now given by the solution of the two coupled nonlinear equations

$$\zeta = D + U\xi^2 + V\xi^4 - 2H_3(H + H_3\xi^2)/\zeta, \tag{3.9}$$

$$\xi^2 = k_B T I_d(\zeta; J) + (H + H_3\xi^2)^2/\zeta^2. \tag{3.10}$$

The relation (3.9) is a necessary condition for the attainment of a minimum: positivity of the second ξ^2 derivative, namely,

$$U + 2V\xi^2 - 2H_3^2/\zeta - [1 - 2H_3(H + H_3\xi^2)/\zeta^2](\partial\zeta/\partial\xi^2) > 0, \tag{3.11}$$

is a sufficient condition. Solutions of (3.9) and (3.10) which satisfy (3.11) will be called *stable solutions*. However, if there are more than one pair of stable solutions, that pair, (ξ, ζ), giving the lowest free energy is to be chosen. Coexistence of different phases will thus be described by distinct solutions yielding the same free energy.

Note that, by using (2.11) and (2.12), which yield

$$\zeta = (H + H_3 m_2)/m, \quad \xi^2 = m_2, \tag{3.12}$$

the free energy may be expressed entirely in terms of the two densities $m \equiv \langle s \rangle$ and $m_2 \equiv \langle s^2 \rangle$. One finds

$$F = \tfrac{1}{2}k_B T[\mathfrak{F}_d(\zeta; J) - \zeta I_d(\zeta; J)] + \tfrac{1}{2}Dm_2 + \tfrac{1}{4}Um_2^2 + \tfrac{1}{6}Vm_2^3 - Hm - H_3 m m_2, \tag{3.13}$$

with $\zeta(H, H_3)$ given by (3.12), while m and m_2 and, in fact, the equation of state, are determined by the solution of the two equations

$$H = m(D + Um_2 + Vm_2^2) - H_3(2m^2 + m_2), \tag{3.14}$$

$$m_2 = k_B T I_d(\zeta; J) + m^2. \tag{3.15}$$

The stability condition (3.11) can be similarly reexpressed. Note the resemblance to the classical results which are, indeed, recaptured if the $\mathfrak{F}_d$ and I_d terms are dropped.

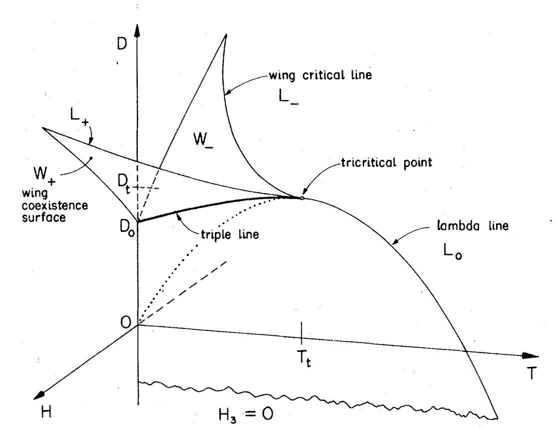

FIG. 1. Phase diagram of the model Hamiltonian (1.1) in the space T-D-H for $H_3 = 0$, showing the tricritical point, the parabolic λ line L_0 (and its continuation, dotted), and the wing critical lines, L_+ and L_-.

The densities conjugate to the three fields $H \equiv H_1$, $H_2 \equiv -\frac{1}{2}D$, and H_3 are defined by

$$m_l(T, H_1, H_2, H_3) = -\frac{\partial}{\partial H_l} F(T, H_1, H_2, H_3), \tag{3.16}$$

for $l = 1, 2, 3$. By differentiating (3.13) and taking (3.14) and (3.15) into account, we find, simply,

$$m_1 \equiv \langle s \rangle = m, \quad m_2 \equiv \langle s^2 \rangle = \xi^2, \quad m_3 \equiv \langle s^3 \rangle = m m_2. \tag{3.17}$$

The nature of the phase diagram resulting from the free energy (3.8) has already been studied[9] in some detail for the case $H_3 \equiv 0$. The results are shown in Fig. 1. In the disordered region of the (T, D) plane one has, by (3.12), $\zeta = 1/\chi$ $(H = H_3 = 0)$ where

$$\chi = \partial m/\partial H, \tag{3.18}$$

is the ordering susceptibility. A critical or λ line L_0 thus occurs when $\zeta = 0$ which yields the conditions

$$\xi_c^2 = k_B T_c I_d(0; J) \equiv k_B T_c I_d^0, \quad D + U\xi_c^2 + V\xi_c^4 = 0. \tag{3.19}$$

The first equation yields a nonzero critical temperature when $d > 2$ for nearest neighbor coupling or for $d > \sigma$ for long-range interactions. The second equation has two solutions when $D \leq \frac{1}{4}U^2/V$ but only the larger one satisfies the stability condition. The critical line is thus parabolic and given explicitly by

$$k_B T_c(D) = [|U| + (U^2 - 4VD)^{1/2}]/2VI_d^0, \tag{3.20}$$

as shown in Fig. 1 (where the other branch of the parabola is shown as a dotted curve). On the λ line the susceptibility associated with m_2 varies as

$$\bar{\chi} = \frac{\partial m_2}{\partial H_2} \sim (2V\xi_c^2 + U)^{-1}, \qquad (3.21)$$

and is finite until the tricritical point is reached. Combining the condition $1/\bar{\chi} = 0$ with (3.20) locates the tricritical point at the vertex of the parabola, and the endpoint of the critical line, namely,

$$k_B T_t = \xi_t^2 / I_d^0, \quad D_t = \tfrac{1}{4} U^2/V, \quad \xi_t^2 = m_{2,t} = \tfrac{1}{2}|U|/V. \qquad (3.22)$$

For $H \neq 0$ (but $H_3 = 0$), Eqs. (3.14) and (3.15) may yield two pairs of stable solutions. Equating the corresponding free energies determines two symmetrically placed surfaces of coexistence—the "wings," W_+ and W_-. The wings are bounded by two lines of critical points, L_+ and L_-, and meet along a triple line in the (T, D) plane which intersects the D axis at $T = 0$, $D_0 = \frac{3}{4} D_t$ (see Fig. 1). For $d > 3$, for finite-range interactions (or $d > \frac{3}{2}\sigma$ for long-range interactions) the wing critical lines meet the λ line at the tricritical point (3.22) as indicated in Fig. 1.[9] This also occurs for $d = 3$ (or $d = \frac{3}{2}\sigma$) provided $|U|$ is sufficiently small [explicitly $|U| < 2V^{1/2}/p(J)$, where $p(J)$ is defined in terms of $I_d(\zeta; J)$ below].[9] Otherwise the wing critical lines meet one another smoothly in a disordered region of the (T, D) plane above T_t, and the tricritical point disappears, being replaced by a critical λ endpoint on the wing surface.[9]

Now the vanishing of ζ on the λ *line* L_0 in the (T, D) plane implies that the singular point of the functions $\mathfrak{F}_d(\zeta)$ and $I_d(\zeta)$ is attained. As in the ordinary spherical model, therefore, the λ line is characterized by dimensionally dependent exponents given by[15]

$$\dot{\alpha} = 1 - \dot{\gamma}, \quad \dot{\beta} = \tfrac{1}{2}, \quad \dot{\Delta} = \dot{\beta}\dot{\delta} = \tfrac{1}{2} + \dot{\gamma}, \qquad (3.23)$$

where the susceptibility exponent for finite-range forces is

$$\dot{\gamma} = \begin{cases} 2/(d-2), & \text{for } 2 < d \leq 4, \\ 1, & \text{for } d \geq 4, \end{cases} \qquad (3.24)$$

while for long-range forces one has

$$\dot{\gamma} = \begin{cases} \sigma/(d-\sigma), & \text{for } \sigma < d \leq 2\sigma \\ 1, & \text{for } d \geq 2\sigma; \end{cases} \qquad (3.25)$$

finally for infinite range forces ($R_0 \to \infty$) $\dot{\gamma} = 1$ holds always. Here and below the fluxion dot denotes exponents *on* the $H = H_3 = 0$ or λ line L_0.

On the other hand, at a *wing critical point*, $\zeta = H/m$ ($H_3 = 0$) exceeds χ^{-1} and so does not vanish Consequently the wing critical points cannot exhibit the spherical-model dimensionality dependence; in fact they are completely classical, that is

$$\ddot{\alpha} = 0, \quad \ddot{\beta} = \tfrac{1}{2}, \quad \ddot{\gamma} = 1, \quad \ddot{\Delta} = \ddot{\beta}\ddot{\delta} = 1\tfrac{1}{2}. \qquad (3.26)$$

Although at first surprising, a difference in critical exponents for nonzero H, or H_3, should be anticipated for the following reason.[11] The characteristic spherical-model values (3.23)–(3.25), describe an isotropic or rotationally symmetric n-component spin model in the limit $n \to \infty$. On the other hand, the presence of a field H or H_3 selects a particular axis and so breaks the rotational symmetry, and hence changes the exponents. For finite n one expects the new exponents to be Ising-like (corresponding to $n_{\text{eff}} = 1$) as indeed found explicitly in renormalization-group calculations.[19] Evidently, the crossover for the limit $n = \infty$ is to classical behavior: the physical mechanism responsible for this merits further study.

Turning now to the case $H_3 \neq 0$, one observes that symmetry with respect to H is lost. In particular the line of critical points above T_t no longer lies in the $H = 0$ plane. Rather, as illustrated in Fig. 2, if, say, H_3 is negative the critical line lies in the half space $H < 0$ and continues smoothly and, indeed, analytically into the former wing critical line L_-, while L_+ terminates on the associated coexistence surface at a critical endpoint. This critical endpoint is also the new terminus of the triple line. Of course, for $H_3 > 0$, the roles of L_+ and L_-, and positive and negative H are reversed. In fact, the topology of the phase diagram is identical to that predicted by classical theory.[2,3]

Furthermore, one finds, in confirmation of the reasoning presented above, that the exponents are

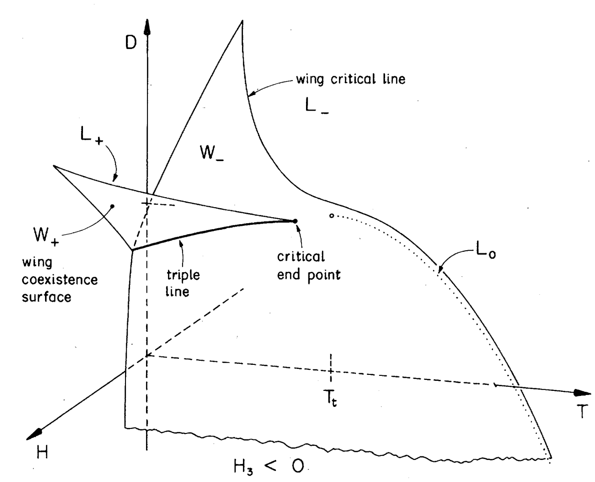

FIG. 2. Schematic phase diagram corresponding to Fig. 1 when H_3 is negative, showing a critical endpoint. The line L_0, present when $H_3 = 0$, is shown dotted for the sake of comparison.

classical on *all* critical lines for $H_3 \neq 0$. (Mathematically this is again because ζ remains positive when χ^{-1} vanishes for $H_3 \neq 0$.) It follows that as one varies H_3 at fixed $D < D_t$ one will observe *bicritical behavior*[3,19] as H_3 passes through zero. The critical line L_0 at $H_3 = H = 0$ appears, from this perspective, as a line of bicritical points: it is for this reason that we retain the term "λ line" for L_0 in contradistinction to the other critical lines. This feature of bicriticality does not, of course, show up in the classical phenomenological theory. Nor is it to be expected in the tricritical region of a real multicomponent fluid system since, in that case, *all* critical exponents are expected to be Ising-like ($n_{\text{eff}} = 1$).

We turn now to a study of the asymptotic equation of state in the tricritical region.

IV. SCALING IN THE TRICRITICAL REGION

We may restrict attention to the cases $3 \leq d < 4$ for short-range coupling, and $3 \leq 2d/\sigma < 4$ for long-range interactions, in which a tricritical point is present but nonclassical exponents occur on the λ line. It is convenient to put

$$\epsilon = 4 - (2/\sigma)d, \tag{4.1}$$

with $\sigma \Rightarrow 2$ for short-range forces.[20] In fact, all dimensional-dependent exponents which enter can be written in terms of the λ-line susceptibility exponent

$$\dot{\gamma} = (1 - \tfrac{1}{2}\epsilon)^{-1} \quad \text{with } 1 \geq \epsilon > 0. \tag{4.2}$$

Note that *on* the *borderline* of tricriticality one has $\epsilon = 1$ and $\dot{\gamma} = 2$.

Asymptotic equation of state

To study the free energy and the equation of state near tricriticality on the basis of (3.13), (3.14), and (3.15), we need the singular, small ζ behavior of the basic correlation function integral $I_d(\zeta; J)$, defined in (2.9); this may be written rather generally[15]

$$I_d(\zeta; J) = I_d(0; J)(1 - p\zeta^{1/\dot{\gamma}} + \cdots), \tag{4.3}$$

where the positive coefficient p, which will play an important role in what follows, may be written for Kac potentials as

$$p = b_{d,\varphi}(a/R_0)^d J_0^{-1/\dot{\gamma}}, \tag{4.4}$$

where $b_{d,\varphi}$ is a pure number depending on the shape function $\varphi(r)$. Recall, however, as mentioned after (3.7), that for fixed R_0 ($=a$, say), both the nearest-neighbor coupling (3.5) and the long-range interactions (3.6) may be written in Kac form. In the long-range limit, $R_0 \to \infty$, the parameter p approaches zero but one has the finite result

$$\lim_{R_0 \to \infty} I_d(0; J) = 1/\hat{J}(0) = 1/J_0\hat{\varphi}_0, \tag{4.5}$$

where $\hat{\varphi}_0 = \int \varphi(|\vec{r}|)d\vec{r}$. Via (3.22) this sets the tricritical temperature T_t, in terms of which we define the reduced temperature variable

$$t = (T - T_t)/T_t. \tag{4.6}$$

We may notice from (3.19) that the λ line L_0 is then specified by

$$\xi_c^2 = m_{2,c}(T) = m_{2,t}(1 + t), \quad D_c(T) = D_t(1 - t^2). \tag{4.7}$$

On introducing (4.3) into the (m_2, ζ, m) relation (3.15), one finds, for $\zeta, t \to 0$,

$$\tilde{m}_0 \equiv m_2 - m_{2,t} \approx m_{2,t}\, t - m_{2,t}p\zeta^{1/\dot{\gamma}} + m^2. \tag{4.8}$$

In terms of $\tilde{m}_0$ and the auxiliary field variables,

$$h_0 = H + m_{2,t}H_3, \quad h_3 = H_3, \tag{4.9}$$

the (H, m, m_2, H_3) relation (3.14) becomes

$$h_0/m = D - D_t + V\tilde{m}_0^2 - h_3(2m + \tilde{m}_0/m). \tag{4.10}$$

If one now substitutes for $\tilde{m}_0$ and uses $\zeta = (h_0 + h_3\tilde{m}_0)/m$, which follows from (3.12), one can rewrite this in the form

$$\dot{c}h/m + m_{2,t}p\zeta^{1/\dot{\gamma}}(|U|t + 2Vm^2 - \dot{c}h_3/m) \approx g - (2 + \dot{c})h_3m + |U|tm^2 + Vm^4, \tag{4.11}$$

where we have introduced the *nonlinear fields*

$$h = H + m_{2,t}(1 + t)H_3 = H + m_{2,c}(T)H_3, \tag{4.12}$$

$$g = D - D_t(1 - t^2) = D - D_c(T), \tag{4.13}$$

which will play the role of appropriate scaling variables. (Note that g and h vary quite analytically with T.) The coefficient is given by

$$\dot{c} = \begin{cases} 1, & \text{for } \dot{\gamma} < 2 \text{ or } \epsilon < 1, \\ 1 - D_t p^2, & \text{for } \dot{\gamma} = 2 \text{ or } \epsilon = 1, \end{cases} \tag{4.14}$$

where the discontinuous behavior on the tricritical borderline arises from the term $V\tilde{m}_0^2$ in (4.10) which, in turn, leads to a $\zeta^{2/\dot{\gamma}}$ term which is linear, and hence cannot be neglected, when $\dot{\gamma} = 2$. (Note that if $D_t p^2 > 1$ then $\dot{c}$ is negative and tricritical behavior is destroyed; this corresponds to the criterion $|U| > 2V^{1/2}/p$ mentioned in Sec. III.)

Finally, we may rewrite (4.8) as

$$\tilde{m} \equiv m_2 - m_{2,t}(1 + t) \equiv m_2 - m_{2,c}(T) = m^2 - m_{2,t}p\zeta^{1/\dot{\gamma}}, \tag{4.15}$$

with $\zeta = (h + h_3\tilde{m})/m$. Note that we may think of the variable $\tilde{m} = \langle s^2 \rangle - \langle s^2 \rangle_c$ as the density conjugate to the secondary, nonordering field g.

The free energy (3.13) may readily be written in terms of the new variables. It is convenient to subtract off an analytic background term

$$F_0(T)=\tfrac{1}{2}Dm_{2,t}+\tfrac{1}{4}Um_{2,t}^2+\tfrac{1}{6}Vm_{2,t}^3+\tfrac{1}{2}m_{2,t}gt-\tfrac{1}{3}Vm_{2,t}^3t^3+\tfrac{1}{2}k_BT\mathfrak{F}_d(0;J), \quad (4.16)$$

and to set $-U=V=1$ or, equivalently, to rescale the variables by setting

$$t\Rightarrow t/2m_{2,t}V^{1/3}$$

and

$$(g,h,h_3,m,\tilde{m})\Rightarrow(V^{1/3}g,\ V^{1/6}h,\ V^{1/2}h_3,\ V^{-1/6}m,\ V^{-1/3}\tilde{m}). \quad (4.17)$$

The remaining part of the free energy may then be written asymptotically as

$$\Delta F(t,g,h,h_3)\approx\tfrac{1}{2}g\tilde{m}+\tfrac{1}{4}t\tilde{m}^2+\tfrac{1}{6}\tilde{m}^3-hm-h_3m\tilde{m}+\tfrac{1}{2}(1+\dot{\gamma})^{-1}\dot{p}\,[(h+h_3\tilde{m})/m]^{1+(1/\dot{\gamma})}, \quad (4.18)$$

where, in rescaled terms

$$\dot{p}=\frac{p\,|U|}{2V^{(2+\epsilon)/6}}=\tfrac{1}{2}b_{d,\varphi}\left(\frac{a}{R_0}\right)^d\frac{|U|}{J_0^{1/\dot{\gamma}}V^{(2+\epsilon)/6}}, \quad (4.19)$$

while m and $\tilde{m}$ are to be found by solving

$$\dot{c}h/m+\dot{p}\zeta^{1/\dot{\gamma}}(t+2m^2-\dot{c}h_3/m)=g-(2+\dot{c})h_3m+tm^2+m^4 \quad (4.20)$$

together with

$$\zeta=(h+h_3\tilde{m})/m\,,\quad\text{and}\quad\tilde{m}=m^2-\dot{p}\zeta^{1/\dot{\gamma}}. \quad (4.21)$$

The equations now have a universal form apart from the coefficient $\dot{p}$ (and $\dot{c}$ when $\epsilon=1$). Note, furthermore, that if $\dot{p}=0$, as in the long-range limit, one has $\tilde{m}\equiv m^2$ and the equations reduce to those of the classical phenomenological theory.[2c,3]

Plane of symmetry: disordered region

We are now in a position to discuss the scaling properties of the asymptotic free energy and equation of state. Consider first the plane of symmetry defined by $H=H_3=0$ or, equivalently, $h=h_3=0$. In the disordered region [$D>D_c(T)$ and its continuation below T_t], with $h_3=0$ one has $m\to0$ and, from (4.21),

$$\zeta=h/m\to1/\chi=1/(\partial m/\partial h)_{h=h_3=0}, \quad (4.22)$$

as $h\to0$, where χ is the differential susceptibility. The relation (4.20) then yields

$$\dot{c}\chi^{-1}+\dot{p}t\chi^{-1/\dot{\gamma}}=g. \quad (4.23)$$

On dividing through by t^{ϕ_0} and choosing

$$\phi_0=\gamma(\phi_0-1)=\gamma/(\gamma-1)=2/\epsilon, \quad (4.24)$$

one sees directly that (4.23) has a scaling solution of the form

$$\chi(t,g)=|t|^{-\gamma_0}X_0(g/|t|^{\phi_0}), \quad (4.25)$$

in which $\gamma_0=\phi_0=2/\epsilon$. Furthermore, the scaling function $X_0(x_0)$ is the solution of the equation

$$\dot{c}X_0^{-1}\pm\dot{p}X_0^{-1/\dot{\gamma}}=x_0, \quad (4.26)$$

where the + and − refer to $t\geq0$ or $t\leq0$, respectively. The exponents γ_0 and ϕ_0 are, in fact, just two of the set of *Gaussian tricritical exponents*,[12,20]

$$\alpha_0=(-4/\epsilon)+3,\quad\beta_0=(1/\epsilon)-\tfrac{1}{2},$$
$$\gamma_0=\phi_0=2/\epsilon,\quad\Delta_0=\beta_0\delta_0=(3/\epsilon)-\tfrac{1}{2}. \quad (4.27)$$

We remark that the borderline of tricriticality always corresponds to $\epsilon=1$.

Note that (4.25) can be rewritten in the alternative, but quite equivalent, form

$$\chi(t,g)=g^{-\gamma_{0,t}}\tilde{X}_0(t/g^{\phi_{0,t}}), \quad (4.28)$$

where $\gamma_{0,t}=1$ and $\phi_{0,t}=1/\phi_0$. The subscript t corresponds to Griffiths' notation[2]: the various tricritical exponents are related through scaling by

$$2-\alpha=\phi(2-\alpha_t),\quad\beta=\phi\beta_t,\quad\gamma=\phi\gamma_t,$$
$$\delta=\delta_t,\quad\Delta=\phi\Delta_t, \quad (4.29)$$

where the subscript zero has been dropped since these relations are general.

Now the exponents γ_0 and ϕ_0 as given by (4.27) agree with those found by Amit and De Dominicis[21] in their renormalization-group treatment of the large-n limit of a model equivalent to (1.1) with $H_3=0$. More recently Rudnick and Nelson[4b] developed a method of constructing the equation of state of the n-component model, again with $H_3=0$, to leading order in ϵ (for short-range forces). Their results for the exponents *and* for the scaling function $X_0(x)$ agree precisely with ours in the limit $n\to\infty$. Some care is needed, however, in translating their variables $t\approx\Delta r$, u, and χ, to ours if the correct dependence of $\dot{p}$ on the range R_0 and on other parameters is to be verified: neglecting purely numerical factors, the appropriate substitutions are found (see, e.g., Ref. 20) to be

$$t \Rightarrow gV^{1/3}/J_0R_0^2, \quad u \Rightarrow t|U|\epsilon a^d/V^{1/3}J_0 nR_0^4,$$
$$\chi \Rightarrow \chi/V^{1/3}J_0R_0^2. \qquad (4.30)$$

Note, in particular, the correspondence $(t,u) \Rightarrow (g,t)$ and the various powers of R_0.

Having found ζ through (4.22), (4.25), and (4.26), one may return to (4.18) to obtain the free energy for $\epsilon < 1$ as

$$\Delta F(t,g) \approx \tfrac{1}{2}\dot{p}\,|t|^{2-\alpha_0}X_0^{-1/\dot{\gamma}}[(1+\dot{\gamma})^{-1}X_0^{-1} - x_0 \pm \tfrac{1}{2}\dot{p}X_0^{-1/\dot{\gamma}}], \qquad (4.31)$$

where $X_0(x_0)$, with $x_0 = g/t^{\phi_0}$, is determined by (4.26) while the specific heat exponent α_0 takes the Gaussian value given in (4.27). (The ± sign again refers to $T \geq T_t$ or $T \leq T_t$.) On the borderline of tricriticality at $\epsilon = 1$ (or $d = \frac{3}{2}\sigma$), one finds an extra term, namely, $\frac{1}{3}\dot{p}^2X_0^{-2/\dot{\gamma}}$, in the bracketed factor. We see that the free energy $\Delta F(t,g)$ has a scaling form with Gaussian exponents in accord with the conclusions of Nelson and Rudnick.[4(a)] We will now show, however, that this comforting conclusion no longer stands when we examine the ordered region.

Ordered region on the plane of symmetry

Let us retain the condition $h_3 = 0$ but allow $h \to 0+$ in the ordered region; then the magnetization m will be nonzero and hence (4.21) implies $\zeta \equiv 0$. Then (4.20) yields

$$m^4 + tm^2 + g = 0, \qquad (4.32)$$

as an equation for the spontaneous magnetization. On dividing by t^2 one sees that this has a scaling solution

$$m(t,g) = |t|^{\beta} Z(g/|t|^{\phi}), \qquad (4.33)$$

where the scaling function $Z(x)$ is the largest solution of

$$Z^4 \pm Z^2 + x = 0. \qquad (4.34)$$

However, the exponents β and ϕ now belong to the set of *classical tricritical exponents*[2,3,22-24]

$$\alpha = -1, \quad \beta = \tfrac{1}{2}, \quad \gamma = 2, \quad \Delta = \beta\delta = \tfrac{5}{2}, \quad \phi = 2, \qquad (4.35)$$

which, *except* on the tricritical borderline $\epsilon = 1$, are quite distinct for the Gaussian exponents (4.27). Note that one may rewrite (4.33) in the equivalent form

$$m(t,g) = g^{\beta_t}\tilde{Z}(t/g^{\phi_t}), \qquad (4.36)$$

with $\beta_t = \frac{1}{4}$ and $\phi_t = \frac{1}{2}$, in concordance with the relations (4.29).

Since $\zeta \equiv 0$ in the ordered region one has $\tilde{m} = m^2$ by (4.21), and hence the free energy is given by the scaling form

$$\Delta F \approx \tfrac{1}{2}|t|^{2-\alpha}Z^2(x)[x \pm \tfrac{1}{2}Z^2(x) + \tfrac{1}{3}Z^4(x)], \qquad (4.37)$$

with $x = g/|t^{\phi}|$ and $Z(x)$ given by (4.34) while α takes the classical value -1 as stated in (4.35). Furthermore, it is evident that the parameter $\dot{p}$ does not enter these results at all so that *all* the results are, in fact, completely classical in the ordered region!

Full scaling formulation

We have just demonstrated that although the free energy, magnetization, and susceptibility scale on the symmetry plane $h = h_3 = 0$, they do so *separately* and *inconsistently* in the ordered and disordered regions. This inconsistency becomes particularly evident if one attempts to scale (4.18) to (4.21) for nonzero h (even with $h_3 = 0$). If one tries to utilize the Gaussian exponent $\phi_0 = 2/\epsilon$ one finds it impossible to scale for general h *except* on the tricritical borderline when $\epsilon = 1$. Conversely, one can scale asymptotically with the classical exponents (4.35) but one discovers that the $\dot{p}\zeta^{1/\dot{\gamma}}$ factors must be neglected asymptotically. The results are then wholly classical and, in particular, *the critical exponents on the λ line are not given correctly by the tricritical scaling functions.*

The way out of this dilemma, in accord with renormalization-group concepts, is to recognize that the parameter $\dot{p}$ should be identified as a *dangerous irrelevant variable*[12] with its own characteristic scaling exponent ϕ_p.

To this end, let us introduce the basic tricritical scaling variables, x, y, y_3, and z, by writing

$$g = x|t|^{\phi}, \quad h = y|t|^{\Delta}, \quad h_3 = y_3|t|^{\Delta_3}, \quad \dot{p} = z|t|^{\phi_p} \qquad (4.38)$$

and the auxiliary scaling functions Z and Q via

$$m = |t|^{\beta}Z(x,y,y_3,z), \quad \tilde{m} = |t|^{\tilde{\beta}}Q(x,y,y_3,z). \qquad (4.39)$$

On substituting in (4.18) to (4.21) and requiring that the t dependence drop out, the exponents are forced to adopt the classical values (4.35) with, in addition,

$$\Delta_3 = \tfrac{3}{2}, \quad \tilde{\beta} = 2 - \alpha - \phi = 1, \qquad (4.40)$$

and, for the dangerous irrelevant variable,

$$\phi_p = -1 + \epsilon = -(2/\sigma)d + 3. \qquad (4.41)$$

Evidently ϕ_p is negative for $\epsilon < 1$ so that, as anticipated, $\dot{p}$ is technically irrelevant[12,20] with $z = \dot{p}/|t|^{\phi_p}$ vanishing as $t \to 0$. However, on the tricritical borderline one has $\phi_p = 0$ so that $\dot{p} \equiv z$ becomes *marginal*; the coefficient $\dot{c}$ may then be written

$$\mathring{c}=1-\mathring{p}^2=1-z^2 \quad (\epsilon=1). \tag{4.42}$$

The scaling functions Q and Z are determined by

$$Q=Z^2-zZ^{-1/\mathring{\gamma}}(y+y_3Q)^{1/\mathring{\gamma}}, \tag{4.43}$$

and, after a substitution, by[11]

$$\mathring{c}(y+y_3Q)\pm z(y+y_3Q)^{1/\mathring{\gamma}}Z^{2/\epsilon}(1\pm 2Z^2) = Z(x-2y_3Z\pm Z^2+Z^4), \tag{4.44}$$

where, as before, the plus sign is to be chosen for $T \geq T_t$ and the minus for $T \leq T_t$. Finally the scaling form for the free energy is

$$\Delta F/|t|^{2-\alpha} \approx \tfrac{1}{2}xQ \pm \tfrac{1}{4}Q^2+\tfrac{1}{6}Q^3-yZ-y_3ZQ + \tfrac{1}{2}(1+\mathring{\gamma})^{-1}z[(y+y_3Q)/Z]^{1+(1/\mathring{\gamma})}. \tag{4.45}$$

Note the universal forms for the scaling functions when $\epsilon < 1$.

When the cubic field h_3 vanishes, these expressions simplify since Q may be eliminated. In particular, the equation of state is then given by

$$\mathring{c}y \pm zy^{1/\mathring{\gamma}}Z^{\epsilon/2}(1\pm 2Z^2)=Z(x\pm Z^2+Z^4). \tag{4.46}$$

The standard argument[20] for an irrelevant variable, such as z, is that, as regards the asymptotic behavior, it may be set equal to zero. One sees from these results, however, that z is "dangerous" in the sense that setting $z=0$ eliminates all terms involving the nonclassical exponent $\mathring{\gamma}$ and hence precludes correct exponents on the λ line. In the following, we explore the role of this variable in more detail.

Dangerous irrelevant variable in the disordered region

It is instructive to use the full scaling formulation to study the symmetry plane $(h_3=0, h\to 0)$ again in the disordered region. In order to allow $y, y_3 \to 0$ in (4.43) and (4.44) we must put

$$Z(x,y,y_3,z)\approx y/Y(x,z) \quad \text{with} \quad Y=1/t^{\gamma}\chi, \tag{4.47}$$

Then by (4.43) we have $Q=-zY^{1/\mathring{\gamma}}$ and hence (4.44) yields

$$cY \pm zY^{1/\mathring{\gamma}}=x, \tag{4.48}$$

which determines $Y(x,z)$. Finally, the free energy is given by

$$\Delta F(t,g;\mathring{p}) \approx \tfrac{1}{2}|t|^{2-\alpha}zY^{1/\mathring{\gamma}}[(1+\mathring{\gamma})^{-1}Y-x\pm\tfrac{1}{2}zY^{1/\mathring{\gamma}}+O(z^2)]. \tag{4.49}$$

Evidently, the *total* free energy is now proportional to the irrelevant variable $z \propto p$: essentially the scaling part of the free energy has vanished and only a "correction-to-scaling" term remains. Of course, this expression for ΔF should reduce to the Gaussian scaling form (4.31) found previously. To check this make the substitution

$$Y=|t|^{\gamma_0-\gamma}/X_0, \tag{4.50}$$

with $\gamma_0=2/\epsilon$ and $\gamma=2$ (as before). One then finds that (4.48), which determines $Y(x,z)$, reduces precisely to (4.26), which determines $X_0(x_0;\mathring{p})$ with $x_0=g/|t|^{\phi_0}$ and $\phi_0=2/\epsilon$. Of course, the explicit values of the exponents $\gamma, \phi, \gamma_0, \phi_0, \mathring{\gamma}$ and ϕ_p play a crucial part in this reduction. Lastly, on substituting (4.50) in (4.49) and likewise putting $x=x_0|t|^{\phi_0-\phi}$ and $z=\mathring{p}/|t|^{\phi_p}$ one exactly recaptures the Gaussian form (4.31). Of course, when $\epsilon=1$, so that $z\equiv\mathring{p}$ is only a marginal variable, one must retain the $O(z^2)$ term in (4.49) to obtain the full answer. However, in that case the Gaussian and classical tricritical exponents agree precisely in any case!

It is also interesting to verify more explicitly that the full scaling formulation, taking account of z, gives the expected nonclassical behavior on the λ line. To this end, recall from (4.13) that $g\equiv 0$ on the λ line, while $t>0$. Hence, the approach to the λ line is characterized by $g,x\to 0$. Since $1/\mathring{\gamma}<1$, the relation (4.48) may be solved as

$$Y(x,z)\approx (x/z)^{\mathring{\gamma}} \quad \text{for} \quad x\to 0. \tag{4.51}$$

From (4.47), one hence finds that the susceptibility varies as

$$\chi \approx \mathring{C}(T)/g^{\mathring{\gamma}} \quad \text{with} \quad \mathring{C}(T)\approx(\mathring{p}t)^{\mathring{\gamma}}. \tag{4.52}$$

Precisely the same result, exhibiting the expected spherical-model singularity, is of couse, found from (4.25) and (4.26). Note that the amplitude is (asymptotically) proportional to $\mathring{p}$. The leading singularities of the specific heat and of the nonordering susceptibility, $\tilde{\chi}=\partial m_2/\partial g$, on the λ line are likewise proportional to $\mathring{p}$.

On the critical isotherm, $t=0$, one finds, by solving (4.48) for large x, that

$$\chi \approx (1/g)[\mathring{c}+\mathring{p}t(g/\mathring{c})^{\epsilon/2}+\cdots]. \tag{4.53}$$

However, in the long range limit, $R_0\to\infty$ (or $\mathring{p}\equiv 0$), the classical form $\chi\approx\mathring{c}/g$ holds for all t and so describes a mean-field divergence on the λ line $(\mathring{\gamma}\Rightarrow 1)$.

Leading corrections to scaling

As soon as one leaves the plane of symmetry $H=H_3=0$ (or $h=h_3=0$) the variable z is no longer dangerous (provided $\epsilon < 1$): its role is only to determine the leading corrections to scaling. Indeed by supposing z small we may then solve (4.43) for Q to first order and substitute in (4.44). After some rearrangement, the scaled equation of state to first order in z, may be written

$$y/Z = x - 3y_3 Z \pm Z^2 + Z^4 - z[2Z^2 \pm 1 - (y_3/Z)] \times [(y/Z) + y_3 Z]^{1/\dot{\gamma}}. \qquad (4.54)$$

(Note that $\hat{c} = 1$ since we are now restricted to $\epsilon < 1$.)

Perhaps the first point to be made is that when the symmetry plane is approached in the ordered region (i.e., $y, y_3 \to 0$ with $Z^2 > 0$), the term involving z drops out completely. Thus, as found before, whatever the value of p the ordered behavior on the symmetry plane is fully classical. Note, however, that this does *not* imply that the three-phase coexistence figure[25] in the density space m_1, m_2, m_3 is not completely without corrections: the reason is that to attain the critical endpoints, which lie on this figure, one must go to nonzero H and H_3 (see Figs. 1 and 2). We hope to investigate this in detail in the future.

It is not hard to analyze (4.54) explicitly on various loci. Of interest if the *tricritical isotherm* ($t = g = 0$) with $h_3 = 0$. By considering $Z \to \infty$ we find[11]

$$h \approx m^\delta (1 - 2\dot{p} m^{|\phi_p|/\beta} + \cdots), \qquad (4.55)$$

where $\delta = 5$, in accord with (4.35). The correction exponent $|\phi_p|/\beta = 2(1-\epsilon)$ has just the value which is to be expected on the basis of scaling.[20]

Likewise one may study the tricritical isotherm as $h_3 = H_3$ varies with $h = 0$; but note by (4.12) this means $H = m_{2,t} H_3$. Setting $y = 0$ in (4.54) and solving for y_3 with $Z \to \infty$ yields

$$h_3 \approx \tfrac{1}{3} m^{\Delta_3/\beta} (1 - e_3 \dot{p} m^{|\phi_p|/\beta} + \cdots). \qquad (4.56)$$

where $\Delta_3/\beta = 3$ and $e_3 = 5/3^{1+(1/\dot{\gamma})}$. Again the correction to scaling term has the expected power law.

Finally we may indicate the behavior of, say, the nonordering susceptibility, on the tricritical isotherm in small fields, h and h_3 in the disordered region: as $g \to 0$ with $t = 0$ one obtains

$$\tilde{\chi} \propto \frac{\partial \tilde{m}}{\partial g} \approx g^{-\alpha_t} (A + B\dot{p} g^{|\phi_p|/\phi} + \cdots), \qquad (4.57)$$

where $\alpha_t = 2 - (2-\alpha)/\phi = \frac{1}{2}$ is the classical nonordering exponent, while $|\phi_p|/\phi = \frac{1}{2}(1-\epsilon)$ is the appropriate correction exponent. The amplitudes A and B may be evaluated explicitly and themselves take scaling forms: correct to second order in h and h_3 one finds

$$A(h, h_3) \approx h^2/g^{5/2}, \quad B(h, h_3) \approx B_0 - B_1 h h_3/g^2, \qquad (4.58)$$

where $B_0 = \frac{1}{2}(1 - \frac{1}{2}\epsilon)$ and $B_1 = \frac{1}{2}\epsilon(1 + \frac{1}{2}\epsilon)$. Now when $h \to 0$ the leading amplitude A vanishes and one is left only with

$$\tilde{\chi} \approx B_0 \dot{p}/g^{\alpha_t + (\phi_p/\phi)} = B_0 \dot{p}/g^{\alpha_{0,t}}. \qquad (4.59)$$

However, one discovers that $\alpha_{0,t} = \frac{1}{2}\epsilon$ which is recognized as precisely the correct Gaussian exponent![4] This illustrates how the "accidental" vanishing of the leading, classical scaling terms when $h, h_3 \to 0$, leaves higher-order terms which scale with the Gaussian exponents originally found for the symmetry plane.

Bicritical crossover on the λ line

It was mentioned at the end of Sec. III, that the critical behavior on the wings is always classical but that, since the exponents are nonclassical on the λ line, a *crossover* in critical behavior must occur above the tricritical point when h_3 deviates from zero. In fact at fixed $T > T_t$ the phase diagram projected onto the (g, h_3) plane will have a *bicritical* aspect.[3(a),19] It is of interest to elucidate this behavior which is, of course, contained in the equation of state as given by (4.20) and (4.21) or, equally, in fully scaled form by (4.44) with (4.43). We will work with the former equations which may be rewritten

$$\zeta + \dot{p} t \zeta^{1/\dot{\gamma}} = g - 2h_3 m + t m^2 + m^4, \qquad (4.60)$$

where, on the left, $2m^2$ has been neglected relative to t, while eliminating $\tilde{m}$ yields

$$m\zeta = h + h_3 m^2 - h_3 \dot{p} \zeta^{1/\dot{\gamma}}. \qquad (4.61)$$

In the region of interest, t is positive and fixed, while g and h_3 become small. Differentiating the equations with respect to m (at fixed t, g, and h_3) yields

$$\frac{\partial \zeta}{\partial m}[1 + \dot{p} t (1 - \tfrac{1}{2}\epsilon) \zeta^{\epsilon/2}] = -2h_3 + 2tm + 4m^3, \qquad (4.62)$$

and, with inverse susceptibility $\chi^{-1} = \partial h/\partial m$,

$$\zeta = \chi^{-1} + 2h_3 m - [m + \dot{p} h_3 (1 - \tfrac{1}{2}\epsilon) \zeta^{-\epsilon/2}] \frac{\partial \zeta}{\partial m}, \qquad (4.63)$$

From the first of these, we see that $\partial \zeta/\partial m$ vanishes on the locus

$$m = m_c = (h_3/t)[1 + O(h_3^2)]. \qquad (4.64)$$

Furthermore (4.63) then yields a simple explicit relation between χ^{-1} and ζ. Lastly, by differentiating (4.63) again, one finds that $\partial^2 h/\partial m^2 \equiv \partial\chi^{-1}/\partial m$ also vanishes on the locus which may thus be identified as the *critical isomomental*.

Since χ^{-1} vanishes at the critical point, one thus obtains the critical-point values to leading orders in h_3, namely,

$$\zeta_c = 2h_3^2/t, \quad h_c = h_3^3/t^2 + 2^{1/\dot{\gamma}}\dot{p}h_3^{3-\epsilon}/t^{1/\dot{\gamma}} \qquad (4.65)$$

and, for the critical locus in the (g, h_3) plane,

$$g_c(h_3) = 3h_3^2/t + 2^{1/\dot{\gamma}}\dot{p}t^{\epsilon/2}h_3^{2/\dot{\psi}}. \qquad (4.66)$$

Here $\dot{\psi} = \dot{\gamma}$ is the "shift exponent"[3,26] for the variable h_3^2 which, by symmetry, is really the most appropriate variable to consider.[3,19(b)] Since $1/\dot{\psi} = 1 - \frac{1}{2}\epsilon < 1$, the critical line projected into the (g, h_3^2) plane displays a characteristic bicritical cusp.[19]

On the critical isomomental (4.64), we may use (4.63) to substitute for ζ in (4.60); this yields an asymptotic equation for the susceptibility χ, namely,

$$\chi^{-1} + \dot{p}t[\chi^{-1} + 2(h_3^2/t)]^{1/\dot{\gamma}} = \tilde{g} \equiv g - g_c^0, \qquad (4.67)$$

where $g_c^0 = 3h_3^2/t$ is the critical line in the mean-field limit, $\dot{p} = 0$. This equation describes the crossover from the λ line to the wings *and* to the mean-field limit $\dot{p} \to 0$. If we are not interested in the latter, we may, asymptotically, drop the first term and solve explicitly to obtain

$$\chi(g, h_3) \approx \frac{(\dot{p}t)^{\dot{\gamma}}}{\tilde{g}^{\dot{\gamma}} - 2\dot{p}^{\dot{\gamma}}t^{\dot{\gamma}-1}h_3^2}, \quad (\dot{p} > 0). \qquad (4.68)$$

On the λ line $h_3 = 0$ this clearly yields the expected spherical-model behavior with susceptibility exponent $\dot{\gamma}$; conversely on the wings when $h_3 \neq 0$ the denominator vanishes for positive $\tilde{g}$ and one obtains classical behavior, i.e., $\chi \sim (g - g_c)^{-1}$.

This result is readily cast in scaling form: one has

$$\chi \approx C\tilde{g}^{-\dot{\gamma}}\dot{X}(z), \quad \text{with} \quad z = h_3^2/\tilde{g}^{\dot{\phi}}, \qquad (4.69)$$

where $C = (\dot{p}t)^{\dot{\gamma}}$. The nonlinear field $\tilde{g} \approx g$ is defined in (4.67) while the bicritical crossover exponent is given by

$$\dot{\phi} = \dot{\psi} = \dot{\gamma} = 1/(1 - \tfrac{1}{2}\epsilon), \qquad (4.70)$$

as expected for spherical models.[26,27] The scaling function itself is simply

$$\dot{X}(z) = [1 - (z/\ddot{z})]^{-1} \text{ with } \ddot{z} = \tfrac{1}{2}\dot{p}^{-\dot{\gamma}}t^{1-\dot{\gamma}}. \qquad (4.71)$$

Knowing the behavior of χ and hence ζ on the critical locus it is straightforward to find the free energy on the locus and hence study the crossover in the specific heat.

V. CONCLUSIONS

We have exhibited and solved exactly a nontrivial model displaying a full tricritical phase diagram including critical endpoints induced by a cubic field H_3, and exhibiting for $3 \leq d < 4$, *nonclassical* critical exponents on the λ line. The wing critical exponents, arising when $H \neq H_3 \neq 0$, are classical; thus, a bicritical crossover occurs when H_3 deviates from zero above the tricritical point. Furthermore, we have demonstrated that for dimensionalities $3 < d < 4$, it is *impossible* to scale the full asymptotic free energy in the tricritical region in an orthodox way which reproduces correctly the nonclassical critical exponents on the λ line. To achieve a full description of the asymptotic singularities it is essential to recognize a dangerous irrelevant variable p and to scale this variable with its own special critical exponent $\phi_p = -(1-\epsilon)$. It is found that p is proportional to $1/R_0^d$ where R_0 is the range of the interactions. Only in the van der Waals limit $R_0 \to \infty$ does orthodox scaling apply and then all behavior is quite classical.

Two comments should be made about the analysis leading to these conclusions. First, it is worth emphasizing that the explicit temperature dependence of the "spin weighting terms" in the model [as implied by the definitions (2.1) and (2.4)] has no influence on the scaling properties. Only nonuniversal features, such as the global shape and disposition of the phase diagram, are affected by the way T is introduced. Second, the choice of the linear or nonlinear combinations (4.12), (4.13), and (4.15) for the definitions of h, g, and $\tilde{m}$ is not essential for the scaling analysis: the free-energy scales in a qualitatively similar way in terms of $h_0 = H + m_{2,t}H_3$, and the simple deviations $g_0 = D - D_t$, $\tilde{m}_0 = m_2 - m_{2,t}$.[28] Introduction of h, g, and $\tilde{m}$ merely allows one to cast the results in a form which is closer to the simple phenomenological Landau theory.[2,3]

In view of these observations it is appropriate to compare our results with those of renormalization-group calculations based on Landau-Ginzburg-Wilson Hamiltonians. We have, in fact, shown that in the limit $n \to \infty$, which is what our work describes (see Appendix A), the various expressions obtained by Amit and De Dominicis[21] and by Nelson and Rudnick[4] for exponents, scaling functions, etc. in the disordered region $H = H_3 = 0$, are reproduced precisely. However, the scaling properties which appear in Nelson and Rudnick's $\epsilon = 4 - d$ expansions as describing crossover from ordinary (n-component) critical behavior on the critical line, to *Gaussian* tricritical behavior, must be regarded purely as manifestations of the correction-to-scaling terms when one views the full tricritical region (with $H, H_3 \neq 0$).[28] Furthermore, there seems no reason to doubt that this state of affairs persists for all n. In other words, the breakdown of orthodox tricritical scaling for $d>3$, which we have demonstrated explicitly for $n=\infty$, will also be seen in appropriate calculations for $n<\infty$. Indeed the basic conclusion[22] that the tricritical exponents are classical for $d>3$ with, in particular $\phi=2$, is in direct conflict with the Gaussian exponents, including $\phi=2/\epsilon$, found in the symmetry plane ($H=H_3=0$) by Nelson and Rudnick for all n; this necessarily implies some sort of breakdown of orthodox scaling. The dangerous irrelevant variable $p \propto 1/R_0^d$, which we have identified, should thus play a similar crucial role for all n.

Of course, for finite n the exponents on the λ line will be those of an ordinary n-vector critical point. Similarly, classical exponents should no longer be found at the wing critical points. Rather, as explained in Sec. III, these exponents should become Ising-like ($n_{\rm eff}=1$) since the fields $\vec{H}$ and $\vec{H}_3$ break the $O(n)$ symmetry and single out a unique axis. Therefore, bicritical crossover induced by H_3 should be observed on the lambda line above T_t for all $n>1$.

The exponent ϕ_p of the dangerous irrelevant variable vanishes identically in three dimensions. Indeed, as shown, it is then possible to obtain a full scaling description using only the standard classical tricritical exponents. However, the variable p is *marginal* at $d=3$ and, as one sees by putting $z \equiv p$ in Eqs. (4.43)–(4.45), the scaling functions are *no longer universal*. This nonuniversal dependence on p shows up in the phase diagram and in various other ways which we hope to explore in more detail. For finite n, the situation in $d=3$ dimensions must be more complicated because of the presence of logarithmic factors, $\ln|g|$, $\ln|t|$, etc., raised to fractional powers.[22,29] (Note that renormalization-group calculations for $d=3$ indicate that the logarithmic factors disappear in the limit $n\to\infty$ yielding results in accord with our analysis.[29]) However, it seems likely that the logarithmic corrections enter in an essentially nonuniversal way and correspond, in fact, simply to the effects of the same marginal variable p which reflects the finite range of the interactions. A full analysis of asymptotic tricritical behavior, which allows for the logarithmic corrections, is clearly an important task. One might hope, however, that the nonuniversal effects already arising in the $n\to\infty$ limit will be a reasonable, perhaps even roughly quantitative guide to the true behavior.

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APPENDIX: LIMITING FREE ENERGY

Lower bound

To obtain a lower bound to the free energy defined through (2.5), (2.4), and (2.1) we rewrite the Nn-fold spin-component integrals defining the partition function by separating the radial and angular parts, that is we write

$$\int d^N\vec{s} = \int_0^\infty d^N\rho\, \rho^{N'} \prod_{i=1}^{N} \int_{|\vec{s}_i|=\rho_i} d\sigma_i\,, \tag{A1}$$

where $d\sigma_i$ denotes the surface element of an n-dimensional sphere of radius ρ_i. Now make the change of variables

$$\xi = \rho_i/n^{1/2}, \quad \vec{x}_i = \vec{s}_i/\xi_i\,, \tag{A2}$$

for all i. On setting $k_BT\equiv 1$ for brevity, the result may be written

$$Z_{N,n}(J,W) = n^{N/2}\int_0^\infty d^N\xi \exp\Big(-nN\Phi_{N,n}(\{\xi_i\}) - n\sum_{i=1}^{N}\left[W_1(\xi_i^2) - (1-n^{-1})\ln \xi_i\right]\Big)\,, \tag{A3}$$

where $\Phi_{N,n}$ is seen to be the free energy of a system with n-component spins, $\vec{x}_i$, of fixed length $|\vec{x}_i| = n$ with "fluctuating" pair interactions $J_{ij}\xi_i\xi_j$ and an inhomogeneous, but otherwise ordinary magnetic field, $\vec{H}_i = \xi_i W_2(\xi_i)$. Now by adding and subtracting terms $-\frac{1}{2}\pi\xi_i^2$ in the exponent, factorizing, and using $\int_0^\infty e^{-\pi\xi^2/2}\,d\xi = 1$, we obtain

$$Z_{N,n} \leq n^{N/2} \exp\left[-n \min_{\xi_i \geq 0}\left(N\Phi_{N,n}(\{\xi_i\}) + \sum_{i=1}^{N} [W_1(\xi_i^2) - (1-n^{-1})\ln\xi_i - \tfrac{1}{2}\pi\xi_i^2/n]\right)\right] . \tag{A4}$$

But since the J_{ij} are positive and spatially uniform the minimum can be shown to occur when all the ξ_k are equal.[13] On taking logarithms, dividing by Nn, and letting N and n approach infinity in any way we thus obtain

$$F(J, W) = \lim_{N,n\to\infty} F_{N,n}(J, W) \geq \min_{\xi}[\Phi(\xi) + W_1(\xi^2) - \ln\xi] , \tag{A5}$$

where

$$\Phi(\xi) = \lim_{N,n\to\infty} \Phi_{N,n}(\{\xi_i \equiv \xi\}) , \tag{A6}$$

is the limiting free energy of a standard spherical model[14] with pair interactions $\xi^2 J_{ij}$ and a magnetic field $\xi W_2(\xi^2)$. By well-known results[14,15] we have

$$\Phi(\xi) = \tfrac{1}{2}\mathfrak{F}_d(\zeta;J) + \ln\xi - \tfrac{1}{2}\zeta\xi^2 - \tfrac{1}{2}[W_2(\xi^2)]^2/\zeta , \tag{A7}$$

where $\mathfrak{F}_d(\zeta;J)$ was defined in (2.8) while ζ is the solution of the self-consistency, or saddle point, or constraint equation

$$\xi^2 = I_d(\zeta;J) + [W_2(\xi^2)]^2/\zeta , \tag{A8}$$

with $I_d(\zeta;J)$ defined in (2.9). On substituting these results in (A5) and restoring the factors $k_B T$, we obtain a lower bound for $F(T;J,W)$ of precisely the required form (2.6).

Upper bound

An upper bound may be found, following Emery,[8(b)] by a variational method. To this end write

$$\mathcal{H}_{N,n}(\{\vec{s}_i\}) = \mathcal{H}^0(\{\vec{s}_i\}) + \mathcal{H}^1(\{\vec{s}_i\}) \tag{A9}$$

where, with positive ζ and real η,

$$\mathcal{H}^0(\{\vec{s}_i\}) = \mathcal{H}^{\rm I}_{N,n}(\{\vec{s}_i\}) + \sum_{i=1}^{N} (\tfrac{1}{2}\zeta|\vec{s}_i|^2 + \eta\,\vec{1}\cdot\vec{s}_i) , \tag{A10}$$

and

$$\mathcal{H}^1(\{\vec{s}_i\}) = \sum_{i=1}^{N} nW(|\vec{s}_i|^2/n,\ \vec{1}\cdot\vec{s}_i/n) , \tag{A11}$$

where for convenience we have introduced

$$W(x^2, y) = W_1(x^2) - \tfrac{1}{2}\zeta x^2 + y[W_2(x^2) - \eta] . \tag{A12}$$

The thermodynamic variational principle[8(b),16] then yields

$$F_{N,n} \leq F^0_{N,n} + \langle W(|\vec{s}_i|^2/n,\ \vec{1}\cdot\vec{s}_i/n)\rangle_0 , \tag{A13}$$

where $F^0_{N,n}$ and $\langle\ \rangle_0$ denote the free energy and expectation calculated with $\mathcal{H}^0$, while translational invariance has been used in writing the second term. Evidently $F^0_{N,n}$ is the free energy of a Gaussian model in a uniform magnetic field, so one has simply

$$\lim_{N,n\to\infty} F^0_{N,n} = \tfrac{1}{2}\mathfrak{F}_d(\zeta:J) - \tfrac{1}{2}\eta^2/\zeta . \tag{A14}$$

To evaluate the Gaussian expectation in (A13) when $n\to\infty$, consider the second term in (A12) which contributes

$$\langle n^{-1}\vec{1}\cdot\vec{s}\,\tilde{W}_2(|\vec{s}|^2/n)\rangle_0 = \langle n^{-1}\vec{1}\cdot\vec{s}\int_{-\infty}^{\infty} dt\, W_2(t)\ \delta(t - |\vec{s}|^2/n)\rangle_0 \tag{A15}$$

where we have put $W_2(x^2) - \eta = \tilde{W}_2$ and dropped the irrelevant subscript i. Introducing a Gaussian representation for the δ function yields

$$\lim_{N,n\to\infty}\langle n^{-1}\vec{1}\cdot\vec{s}\,\tilde{W}_2(|s|^2/n)\rangle_0 = \lim_{\lambda\to\infty}(\lambda/\pi)^{1/2}\int_{-\infty}^{\infty} dt\, W_2(t) \lim_{N,n\to\infty}\langle n^{-1}\vec{1}\cdot\vec{s}e^{-\lambda(t-|\vec{s}|^2/n)^2}\rangle_0 \tag{A16}$$

where the interchange of the $\lambda\to\infty$ limit is justifiable. Now by (A10) the spin components s^μ are independent random variables. Thus on expanding the exponential expectations, $\langle s^\mu(s^\nu)^2 \ldots (s^\kappa)^2\rangle$ may be factorized. On resumming the leading terms, one finds

$$\langle n^{-1}\vec{1}\cdot\vec{s}e^{-\lambda(t-|\vec{s}|^2/n)^2}\rangle_0 = \langle s_0\rangle e^{-\lambda(t-\langle s^2\rangle_0)^2}[1 + O(n^{-1})] . \tag{A17}$$

where the basic Gaussian expectations are

$$\langle s\rangle_0 = m = -\eta/\zeta\,, \tag{A18}$$

and

$$\langle s^2\rangle_0 = \xi^2 = I_d(\zeta;J) + \eta^2/\zeta^2\,. \tag{A19}$$

On substituting (A17) in (A16), one finally obtains

$$\lim_{N,n\to\infty} \langle n^{-1}\vec{1}\cdot\vec{s}\,\tilde{W}_2(|\vec{s}\,|^2/n)\rangle_0 = \langle s\rangle_0\,\tilde{W}_2(\langle s^2\rangle_0)\,. \tag{A20}$$

The same argument applied to the first pair of terms in (A1) yields, by (A13),

$$F_{N,n} \le F^0_{N,n} + W_1(\xi^2) - \tfrac{1}{2}\zeta\,\xi^2 + m[W_2(\xi^2) - \eta]\,. \tag{A21}$$

Now to this point ξ and η are arbitrary parameters: they may thus be varied to optimize, i.e., minimize, the bound. Since η and m are monotonically related, varying η is equivalent to varying m regarding η and ζ as functions of ξ^2 and m. Taking into account (A18) and (A19) then yields the relation

$$\eta = W_2(\xi^2)\,, \tag{A22}$$

so that the last term in (A21) cancels from the bound. Likewise η can be eliminated from (A19) in favor of $W_2(\xi^2)$ which yields (2.7) and enables all reference to m and η to be dropped. However, the relation (A18) yields the useful relation (2.11) for the magnetization, m. Finally, one is left with a minimization on ξ which, on using (A14), has the same form as the lower bound (A5) and the desired result (2.6).

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