Critical dynamics of isotropic antiferromagnets using renormalization-group methods: $T < T_N$ *

Gene F. Mazenko, Michael J. Nolan,† and R. Freedman‡
The James Franck Institute and The Department of Physics, The University of Chicago, Chicago, Illinois 60637
(Received 14 June 1976)

We analyze a simple model appropriate for the description of the critical dynamics of isotropic antiferromagnets in the ordered phase. We use renormalization-group methods and mode-coupling ideas to analyze the various correlation functions of interest below the Néel temperature. The transverse correlation functions are dominated by spin waves in agreement with the predictions of hydrodynamics in the appropriate $q\xi \ll 1$ limit . We find that the positions of the spin-wave peaks for the transverse magnetization and staggered magnetization correlation function differ significantly as $q\xi$ increases. The spin-wave peaks persist as $q\xi$ is increased in the transverse staggered magnetization, but become quickly overdamped as $q\xi$ increases for the transverse magnetization correlation function. These results are consistent with our previous calculations at $T = T_N$ where we found damped spin waves in the staggered magnetization correlation function, but not in the magnetization correlation function. Our most interesting result is that as a result of strong coupling of transverse spin waves into the longitudinal modes, hydrodynamics breaks down in treating the longitudinal magnetization correlation functions. This breakdown, first discovered by Villain using a mode-coupling approach, is manifested in the wave-number-dependent spin-diffusion coefficient going as $q^{-1/2}$ in three dimension for small q. The treatment of the longitudinal order-parameter correlation function is difficult within an ϵ expansion. We discuss these difficulties.

I. INTRODUCTION

In previous papers[1,2] we discussed a simple model which, we believe, describes the critical dynamics of isotropic antiferromagnets. We analyzed this model for temperatures greater than or equal to the Néel temperature, T_N, using dynamic renormalization-group methods. In this paper, we carry out the analysis of the dynamics in the ordered phase ($T < T_N$).

There has been a great deal of recent activity in the area of dynamic critical phenomena.[3] Almost all of this work has been focused on the disordered phase. For the most part, the systems that have been studied have a near-Lorentzian spectrum in the disordered phase. The key problems therefore have been to calculate the dynamic critical index z and evaluate the width characterizing the Lorentzian spectrum. In our previous work,[1,2] we showed that the order-parameter correlation-function spectrum for the antiferromagnet shows a rather strong non-Lorentzian shape for $q\xi \gtrsim 1$ (q is the wave number and ξ the correlation length) that is important for interpreting experiments. This calculation is in agreement with previous mode-coupling calculations,[4] experiments,[5] and subsequent renormalization-group calculations.[6,7] A key point one has to bear in mind in both the work for $T \geq T_N$ and the work discussed here is that the more structure one finds in a correlation function the more difficult it is to approximate using an ϵ expansion and the more one must supplement an ϵ expansion with other types of information.[8] We will see that the correlation functions below T_N show considerably more structure than those above and therefore one must work harder to interpret the results. The rewards, however, are that there is a great deal of new qualitative physics.

There have, to our knowledge, been only two previous renormalization-group calculations in the ordered phase for systems governed by equations of motions with mode-coupling terms. Ma and Mazenko[9,10] studied the dynamics of isotropic ferromagnets near six dimensions and found that the coupling of the transverse to longitudinal modes led to a breakdown of hydrodynamics for the longitudinal mode. In that case, however, the extrapolation to 3-dimensional systems is less clear since the special dimension for the dynamics ($d = 6$) is different from that for the statics ($d = 4$).

Recently Siggia,[11] and Hohenberg, Halperin and Siggia[12] have investigated the symmetric planar spin model for the λ transition in liquid helium for $T < T_c$. For $T \geq T_c$ the planar spin model is very similar to the model we study here. Below T_c however there are significant differences. In the antiferromagnet it is possible for the magnetization to couple into the product of the two different transverse components of the staggered magnetization. In the planar model there is only one transverse component. Thus one does not have the breakdown of hydrodynamics (see below) in the planar model found in the model studied here.

The new qualitative elements in the ordered phase mentioned above arise from the development of a spontaneous staggered magnetization below

T_N. Since the order parameter picks out a special direction, the response to perturbations parallel and transverse to the order parameter is very different. The transverse modes are dominated by spin waves while the longitudinal modes are relaxational in nature. Halperin and Hohenberg,[13] using hydrodynamical arguments, proposed forms for the various correlation functions for the isotropic antiferromagnet for $T < T_N$. Our calculations support their results for the spectrum of the transverse modes in the hydrodynamical limit $q\xi \ll 1$. We supplement their results with explicit calculations of the spin-wave velocity and damping correct to $O(\epsilon)$. We also show that the residues of the hydrodynamical poles are more complicated than in the simple hydrodynamical theory.

The longitudinal modes are far more complicated than one would suspect from hydrodynamic arguments.[14] The complications arise due to a strong coupling between the longitudinal and transverse modes. It is well known[15] that this coupling causes the longitudinal static susceptibility $\chi_N^L(q)$ to diverge as $q^{-\epsilon}$ for small wave number q. We find that these same types of couplings cause drastic qualitative changes in the dynamic structure factor for the longitudinal modes. In particular, we find that hydrodynamics breaks down in treating the longitudinal magnetization (a noncritical conserved variable) dynamic autocorrelation function. We have calculated the dynamic-shape function for the longitudinal magnetization in the hydrodynamical regime $q\xi \ll 1$ and find that the shape is Lorentzian but that the characteristic frequency ω_c is proportional to $q^{2-\epsilon/2}$ rather than q^2 as predicted by hydrodynamics. In the hydrodynamical theory $\omega_c = D_L q^2$ as $q \to 0$ where D_L is the longitudinal-spin diffusion constant. For $T < T_N$ we find that $D_L \sim q^{-\epsilon/2}$ for small q and does not exist for $q \to 0$. This result is consistent with that of Villain[16] who found that ω_c is proportional to $q^{3/2}$ in three dimensions using a mode-coupling approach.

The effects of transverse modes on the longitudinal-order parameter correlation function are very complicated within an ϵ-expansion approach. We will discuss these complications and eventually conclude that the spectrum is essentially Lorentzian.

In Sec. II, we set up the problem, and discuss the equations of motions we study. We also summarize the known results for the static correlations for our system. We then show, in Sec. III, that the results for the various correlation functions are nontrivial even to zeroth order in ϵ. In Sec. IV, we discuss the method we used to calculate the dynamic correlation functions to $O(\epsilon)$ in the ordered phase. This section also points out how one can extract the spin-wave frequencies in terms of the exact static correlation functions. We use the results of Sec. IV in Sec. V to calculate the transverse correlation functions correct to $O(\epsilon)$ in the hydrodynamical regime. In Secs. VI and VII, we analyze the longitudinal-magnetization correlation function and the longitudinal-order parameter correlation function with emphasis on the coupling to the transverse modes. We conclude in Sec. VIII with a discussion of our results and their experimental implications.

II. FORMULATION OF PROBLEM FOR $T < T_N$

A. Equation of motion

The equations of motion we study are the same as those Freedman and Mazenko (FM)[1,2] and Halperin, Hohenberg, and Siggia[17] studied for $T \geq T_N$. In the scaling region the staggered magnetization density $\vec{N}(x,t)$ and the magnetization density $\vec{M}(x,t)$ satisfy the coupled equations

$$\frac{\partial \vec{N}}{\partial t} = \lambda \vec{N} \times \vec{H}_M - \Gamma_N \vec{H}_N + \vec{\eta}_N, \qquad (2.1a)$$

and

$$\frac{\partial \vec{M}}{\partial t} = \lambda \vec{N} \times \vec{H}_N + \Gamma_M \nabla^2 \vec{H}_M + \vec{\eta}_M, \qquad (2.1b)$$

where λ is a mode-coupling parameter, Γ_N is a bare kinetic coefficient associated with the nonconserved order parameter $\vec{N}(x,t)$, Γ_M is a bare transport coefficient associated with the conserved variable $\vec{M}(x,t)$, and $\vec{\eta}_\alpha(x,t)$ is a random Gaussian distributed noise source satisfying

$$\langle \eta_\alpha^i(\vec{x},t) \rangle = 0, \qquad (2.2a)$$

and

$$\langle \eta_\alpha^i(x,t) \eta_\beta^j(x',t') \rangle = 2\delta_{ij}\delta_{\alpha\beta}\hat{\Gamma}_\alpha \delta(x-x')\delta(t-t') \qquad (2.2b)$$

for $\alpha = M$ or N and $i = x, y, z$, where

$$\hat{\Gamma}_\alpha = \delta_{\alpha N}\Gamma_N - \delta_{\alpha M}\Gamma_M \nabla^2. \qquad (2.3)$$

In Eq. (2.1) $\vec{H}_\alpha(x,t)$ is an effective local field defined by

$$\vec{H}_\alpha = (\delta F / \delta \vec{\Psi}_\alpha), \qquad (2.4)$$

where $\vec{\Psi}_\alpha = \delta_{\alpha M}\vec{M} + \delta_{\alpha N}\vec{N}$, and F is the Ginzburg-Landau-Wilson free-energy functional

$$F[\vec{N},\vec{M}] = \frac{1}{2}\int d^d x \, [r_0 \vec{N}^2 + (\nabla\vec{N})^2 + \tfrac{1}{2}u(\vec{N}^2)^2 + r\vec{M}^2], \qquad (2.5)$$

where $r_0 = a'(T - T_N^0)$, a', r, and u are positive constants and T_N^0 is the mean-field transition temperature. When working in momentum space, all wavenumbers are restricted to be less than a cutoff Λ. Ma and Mazenko[10] have shown that the equilibrium correlation functions generated by equations of motion of the type (2.1) are the same as those generated by the static distribution function e^{-F}. This insures that the equilibrium properties calculated from (2.1) will agree with the usual[18] static ϵ-expansion results.

B. Summary of results for $T \geq T_N$

FM performed a renormalization-group analysis of Eq. (2.1) correct to $O(\epsilon)$. It was shown that these equations have a stable dynamical fixed point. The stable fixed-point values of the dynamical parameters can be written in terms of the dimensionless variables f and a which are given by

$$f=\frac{\lambda^2\Lambda^{-\epsilon}}{\Gamma_M\Gamma_N}=\frac{\epsilon}{K_4}+O(\epsilon^2)\,, \tag{2.6a}$$

$$a=\frac{\Gamma_M r}{\Gamma_N}=\frac{1}{3}+O(\epsilon)\,, \tag{2.6b}$$

where $K_4=(8\pi^2)^{-1}$. It will turn out, when working with $T<T_N$, that in order to obtain numerical results valid to order ϵ we must have a to $O(\epsilon)$ while u and f must be known to $O(\epsilon^2)$. We have used Wilson's[19] matching technique and the memory-function formalism of Sec. IV to determine a and f to the next order in ϵ. The parameter u may be determined from the statics independently.[18] While we discuss the details of these calculations elsewhere, we present here the necessary numerical results. (See also Table I):

$$a=\tfrac{1}{3}(1+\epsilon a_1),\quad a_1=0.8063\,, \tag{2.7a}$$

$$f=\frac{\epsilon}{K_{4-\epsilon}}(1+\epsilon f_2),\quad f_2=-0.258\,, \tag{2.7b}$$

$$u=\frac{\epsilon}{11K_{4-\epsilon}}(1+\epsilon u_2),\quad u_2=0.5702\,. \tag{2.7c}$$

Our result for f_2 agrees with that obtained previously by Halperin *et al.*[17] However, our result for a does not agree with theirs. In our notation their result is $a_1=1.034$. The reason for this discrepancy is not understood. A dimensionless combination of these variables we will encounter often is

$$R=\frac{af}{u\Lambda^{-\epsilon}}=\tfrac{11}{3}\,[1-\epsilon(0.0219)]\,. \tag{2.8}$$

Freedman and Mazenko used the $O(\epsilon)$ values of the coupling parameters for $T\geq T_N$ to calculate the correlation functions

$$C^{ij}_{\alpha\beta}(q,\omega)=\langle\Psi_{i\alpha}(q,\omega)\Psi_{j\beta}(-q,-\omega)\rangle \tag{2.9}$$

in the scaling region to $O(\epsilon)$. It was shown that the correlation functions could be written in the dynamical scaling form[2]

TABLE I. Parameters calculated at lowest order and next order in ϵ.

Quantity	Def. eq.	Lowest order	Next order
f	$f=\dfrac{\lambda^2\Lambda^{-\epsilon}}{\Gamma_M\Gamma_N}$	$\dfrac{\epsilon}{8\pi^2}$	$\dfrac{\epsilon}{K_{4-\epsilon}}[1-\epsilon(0.258)]$
a	$a=\dfrac{\Gamma_M r}{\Gamma_N}$	$\frac{1}{3}$	$\frac{1}{3}[1+\epsilon(0.8063)]$
$u\Lambda^{-\epsilon}$		$\dfrac{\epsilon}{11K_4}$	$\dfrac{\epsilon}{11K_{4-\epsilon}}[1+\epsilon(0.5702)]$
R	$R=\dfrac{af}{u\Lambda^{-\epsilon}}$	$\frac{11}{3}$	$\frac{11}{3}[1-\epsilon(0.0219)]$
b_s	$b_s=\dfrac{c_s}{\Gamma_N\xi^{-1}(\Lambda\xi)^{\epsilon/2}}$	$\sqrt{\frac{11}{3}}$	$\sqrt{\frac{11}{3}}[1+\epsilon(0.0345)]$
d_s	$d_s=\dfrac{D_s}{\Gamma_N(\Lambda\xi)^{\epsilon/2}}$	$\frac{4}{3}$	$\frac{4}{3}[1+\epsilon(0.0682)]$
$\bar{D}$	$\bar{D}=\dfrac{D_s k^2}{c_s kx}$	$\dfrac{4}{\sqrt{33}}$	$\dfrac{4}{\sqrt{33}}[1+\epsilon(0.0337)]$
a_{MM}	$\tilde{\Gamma}^T_{MM}=\Gamma_M q^2(\Lambda\xi)^{\epsilon/2}(1-\epsilon a_{MM})$	0.4200	
a_{NN}	$\tilde{\Gamma}^T_{NN}=\Gamma_N(\Lambda\xi)^{\epsilon/2}(1-\epsilon a_{NN})$	0.3863	
a_{MN}	$\tilde{\Gamma}^T_{MN}=-(\Gamma_M\Gamma_N)^{1/2}qx\epsilon a_{MN}$	0.0622	
g_0	$g_0=\dfrac{f}{20\pi}\left(\dfrac{2}{b_s d_s}\right)^{1/2}$	$\frac{2}{5}\pi\sqrt{\frac{3}{2}}(\frac{3}{11})^{1/4}$	$\frac{1}{10}\pi\sqrt{\frac{3}{2}}(\frac{3}{11})^{1/4}[1-\epsilon(0.0513)]$
g_1	$g_1=\dfrac{5}{6}\left(\dfrac{d_s}{2b_s}\right)^{1/2}$	$\frac{5}{6}\sqrt{\frac{2}{3}}(\frac{3}{11})^{1/4}$	$\frac{5}{6}\sqrt{\frac{2}{3}}(\frac{3}{11})^{1/4}[1+\epsilon(0.0169)]$

$$C^{ij}_{\alpha\beta}(q,\omega)=\delta_{ij}\delta_{\alpha\beta}\chi_\alpha(q)f^{\alpha}_x(\nu)/\omega(q,\xi)\,, \tag{2.10}$$

where $\chi_\alpha(q)$ is the static susceptibility for the variable $\alpha=M$ or N, $\omega(q,\xi)$ is the characteristic frequency, $f^\alpha_x(\nu)$ is the shape function, $x=q\xi$ and $\nu=\omega/\omega(q,\xi)$. The characteristic frequency has the dynamical scaling form

$$\omega(q,\xi)=q^z\Omega(x)\,, \tag{2.11}$$

where the dynamical scaling index is $z=\frac{1}{2}d$. FM calculated the scaling function $\Omega(x)$ (see Fig. 7 of Ref. 2 for a plot of this function) and found the same qualitative behavior as that calculated previously by Joukoff-Piette and Resibois[20] and Huber and Krueger.[21] This form for the scaling function compares favorably with the neutron scattering data of Nathans[22] and Lau *et al.*[23] Analytical expressions were obtained by FM for the shape functions $f^\alpha_x(\nu)$ correct to $O(\epsilon)$ for all values of x and ν. Here, we briefly summarize these results for $\epsilon=1$. The shape function $f^M_x(\nu)$ for the magnetization is essentially Lorentzian for all values x showing the characteristic hydrodynamic pole at $\nu=0$ for $x\to 0$. This behavior is in qualitative agreement with the neutron scattering experiment of Tucciarone *et al.*[24] The shape function $f^N_x(\nu)$ for the order parameter has the following behavior:

(i) In the relaxational regime, $x\ll 1, f^N_x(\nu)$ is a Lorentzian centered about $\nu=0$.

(ii) As x increases and the critical regime $x\gg 1$ is approached, two fluctuation-induced peaks appear in the spectrum displaced symmetrically about the origin. For $\epsilon=1$ these two peaks first appear for $x\sim 2$.

(iii) The position of the peaks moves continuously outward from the origin until $x\sim 10$ where the position of the peaks attains a limiting value $\nu_M=\pm 0.65$. The non-Lorentzian shape function $f^N_x(\nu)$ found by FM is similar to that observed in the isotropic antiferromagnet $RbMnF_2$ by inelastic neutron scattering.[5] Similar peaks in $f^x_N(\nu)$ were found previously by Wegner[4] at $T=T_N$ (i.e., $x=\infty$) by numerically solving a set of coupled equations similar to Eqs. (2.1). Wegner derived these equations using a mode-coupling approach. Recent calculations by Janson[6] for a generalized isotropic spin model confirmed our earlier calculations.

C. Static results for $T<T_N$

The major new feature for $T<T_N$ is that some component of the staggered magnetization has a finite equilibrium average in zero external conjugate field. That is, we can write

$$\langle N_z(x,t)\rangle = N\,, \tag{2.12}$$

where N is the spontaneous staggered magnetization.

In our model, the static properties of the magnetization and the staggered magnetization are uncoupled. Therefore, the statics of the magnetization field remain Gaussian as above T_N. The spontaneous staggered magnetization, however, leads to qualitative changes in the statics for the order parameter. The calculation of the static order-parameter properties for $T<T_N$ has been discussed within the context of the ϵ expansion by Brézin *et al.*,[25] Nelson,[26] and by Mazenko.[27] We summarize the results here. By expanding the quantity

$$\left\langle \frac{\delta F}{\delta \vec{N}(x)}\right\rangle = 0 \tag{2.13}$$

in a perturbation series expansion we obtain the equation of state (in zero external field) relating the spontaneous magnetization to the temperature. To lowest order in ϵ one obtains

$$r_0+uN^2=0\,. \tag{2.14}$$

One easily sees that since $u\sim\epsilon$ that $N\sim\epsilon^{-1/2}$. This makes the perturbation theory considerably more complicated than for $T\geq T_N$. We find going to first order in ϵ that the spontaneous staggered magnetization is related to the temperature by

$$\tau=\frac{\Lambda^2}{2}\left(\frac{2uN^2}{\Lambda^2}\right)^{1/2\beta}, \tag{2.15}$$

where

$$\beta=\tfrac{1}{2}(1-\tfrac{3}{11}\epsilon)\,, \tag{2.16}$$

$$\tau\equiv |r_0-r_0^c|\sim a'|T-T_N|\,, \tag{2.17}$$

and $r_0^c=-5\epsilon\Lambda^2/22$ is the critical value of r_0. If we define the correlation length as

$$\xi^{-2}=\Lambda^2(uN^2/\Lambda^2)^{\beta/\nu} \tag{2.18}$$

then we obtain the usual relation between temperature and correlation length

$$\tau=\tfrac{1}{2}\Lambda^2 2^{1/2\beta}(\xi\Lambda)^{-1/\nu}\,, \tag{2.19}$$

since to $O(\epsilon)$, $\beta/\nu=1+\frac{1}{2}\epsilon$. Using these definitions one can show that the longitudinal and transverse static correlation functions are given, correct to $O(\epsilon)$,[28] by

$$[\chi^L_N(q)]^{-1}=2\xi^{-2}\left(f_L(x)+\frac{11+7\epsilon/2}{9+2x^{-\epsilon}}\right), \tag{2.20}$$

where we use (only in this section) the convention $x=q\xi/\sqrt{2}$ and,

$$f_L(x)=x^2-\frac{18\epsilon}{11}\left(1+\frac{(x^2+4)^{1/2}}{x}\ln\tfrac{1}{2}[(x^2+4)^{1/2}-x]\right), \tag{2.21}$$

while

$$[\chi_N^T(q)]^{-1} = q^2 f_T(x)\,, \tag{2.22}$$

where

$$f_T(x) = 1 + \frac{2\epsilon}{11}\left[1 + (1+x^{-2})(x^{-2}\ln(1+x^2) - 1)\right]. \tag{2.23}$$

In the small x limit we find

$$f_L(x) = x^2\left(1 + \frac{9\epsilon}{22}\right) + O(x^3) \tag{2.24}$$

while

$$f_T(0) = 1 + \epsilon/11\,. \tag{2.25}$$

The main points we wish to make about χ_N^L and χ_N^T is that both diverge as $q \to 0$. $\chi_N^T \sim q^{-2}$ which is the expected Nambu-Goldstone mode. The divergence of the longitudinal susceptibility. $\chi_N^L(q) \sim q^{-\epsilon}$, is related to the so-called coexistence curve singularities discussed by Brézin *et al.*[25] and is discussed in some detail in Ref. 27. The divergence of these static quantities will have important consequences in treating the dynamics for $T < T_N$.

III. ZEROTH-ORDER SOLUTION

Even to zeroth order in ϵ, we obtain interesting results for the dynamics of our model. Remembering that $N \sim \epsilon^{-1/2}$ we can easily show that to zeroth order in ϵ the equations of motion become,

$$\frac{\partial}{\partial t}\delta N_L(q,t) - \Gamma_N[\chi_N^{L,0}(q)]^{-1}(q)\delta N_L(q,t) = \eta_N^L(q,t)\,, \tag{3.1}$$

$$\frac{\partial}{\partial t}M_L(q,t) - \Gamma_M q^2 r M_L(q,t) = \eta_M^L(q,t)\,, \tag{3.2}$$

$$\frac{\partial}{\partial t}\vec{N}_T(q,t) - \Gamma_N q^2 \vec{N}_T(q,t) - \lambda N r \hat{z}\times\vec{M}_T(q,t) = \vec{\eta}_N^T(q,t) \tag{3.3}$$

$$\frac{\partial}{\partial t}\vec{M}_T(q,t) - \Gamma_M q^2 r \vec{M}_T(q,t) - \lambda N q^2 \hat{z}\times\vec{N}_T(q,t) = \vec{\eta}_M^T(q,t) \tag{3.4}$$

where we have

$$\delta N_L(q,t) \equiv N_z(q,t) - \langle N_z(q,t)\rangle\,, \tag{3.5}$$

$$[\chi_N^{L,0}(q)]^{-1} = q^2 + 2uN^2 \tag{3.6}$$

and $\vec{N}_T$ and $\vec{M}_T$ are transverse to the z component of $\vec{N}$. We note that the various spin and "isospin" components are coupled in the transverse directions. We can obtain an equation which is diagonal in the spin indices by making a unitary transformation. That is we introduce the standard "helicity" basis by writing

$$\Psi_{\alpha i} = \sum_s t_{is}^* \Psi_{\alpha s}\,, \tag{3.7}$$

$$\Psi_{\alpha s} = \sum_j t_{js}\Psi_{\alpha j}\,, \tag{3.8}$$

where $i = x, y, z$ and $s = 0, +, -$. The transformation coefficients are given explicitly by

$$\begin{aligned} &t_{j0} = \delta_{jz}\,,\; t_{j+} = \frac{1}{\sqrt{2}}(\delta_{jx} + i\delta_{jy})\,,\\ &t_{j-} = \frac{1}{\sqrt{2}}(\delta_{jx} - i\delta_{jy})\,,\\ &t_{zs} = \delta_{s,0} t_{xs} = \frac{1}{\sqrt{2}}(\delta_{s,+} + \delta_{s,-})\,,\\ &t_{ys} = \frac{i}{\sqrt{2}}(\delta_{s,+} - \delta_{s,-})\,. \end{aligned} \tag{3.9}$$

It is easy to demonstrate that these transformation coefficients satisfy the completeness relations

$$\sum_j t_{js'} t_{js}^* = \delta_{s,s'} \tag{3.10a}$$

and

$$\sum_s t_{is} t_{js}^* = \delta_{ij}\,. \tag{3.10b}$$

After applying this transformation we can write the zeroth-order equation of motion as a vector equation for $\varphi = \Psi - \langle\Psi\rangle$:

$$\frac{\partial}{\partial t}\varphi_\alpha^{s,0}(q,t) - \hat{\Gamma}_\alpha(q)[\chi_\alpha^{s,0}(q)]^{-1}\varphi_\alpha^{s,0}(q,t) + is\lambda N r_{-\alpha}(q)\varphi_{-\alpha}^{s,0}(q,t) = \eta_\alpha^s(q,t)\,, \tag{3.11}$$

where

$$[\chi_\alpha^{s,0}(q)]^{-1} = \delta_{\alpha,M} r + \delta_{\alpha,N}[q^2 + r_0 + uN^2(1 + 2\delta_{s,0})]\,, \tag{3.12}$$

$$[\tilde{\chi}_{-\alpha}(q)]^{-1} = \delta_{\alpha,M} q^2 + \delta_{\alpha,N} r \equiv r_{-\alpha}(q)\,, \tag{3.13a}$$

$$r_\alpha(q) = \delta_{\alpha,N} q^2 + \delta_{\alpha,M} r\,, \tag{3.13b}$$

and

$$\eta_\alpha^s(q,t) = \sum_j t_{js}\eta_\alpha^j(q,t)\,. \tag{3.14}$$

The zeroth-order correlation functions $C_{\alpha\beta}^{ss',0}(q,\omega)$ are defined by

$$\langle\varphi_\alpha^{s,0}(q,\omega)\varphi_\beta^{s,0}(q',\omega')\rangle = (2\pi)^{d+1}\delta^{(d)}(q+q') \times \delta(\omega+\omega')C_{\alpha\beta}^{ss',0}(q,\omega) \tag{3.15}$$

and can be easily calculated by Fourier transforming the equation of motion over time, solving the resulting set of two coupled linear equations giving φ^0 in terms of η and then using the statistical properties of the noise.

The zeroth-order correlation functions may be summarized as follows. The longitudinal correlation functions are given by the simple Lorentzians

$$C_{MM}^{00,0}(q,\omega)=\frac{2\Gamma_M q^2}{\omega^2+(\Gamma_M r q^2)^2} \tag{3.16}$$

and

$$C_{NN}^{00,0}(q,\omega)=\frac{2\Gamma_N}{\omega^2+\{\Gamma_N[\chi_N^{L,0}(q)]^{-1}\}^2} \tag{3.17}$$

which, if we replace $[\chi_N^{L,0}(q)]^{-1}$ by $[\chi_N^{0}(q)]^{-1}$, are identical to the zeroth-order correlation functions for $T \geq T_N$. We further note that

$$C_{MN}^{00,0}(q,\omega)=C_{NM}^{00,0}(q,\omega)=0\,, \tag{3.18}$$

and

$$C_{\alpha\beta}^{++,0}(q,\omega)=C_{\alpha\beta}^{--,0}(q,\omega)=C_{\alpha\beta}^{\pm 0,0}(q,\omega)=0\,. \tag{3.19}$$

The remaining nonzero correlation functions are those between transverse fields, $C_{\alpha\beta}^{+-,0}(q,\omega) = C_{\alpha\beta}^{-+,0}(-q,-\omega)$. These correlation functions can be written in the dynamical scaling form

$$C_{NN}^{+-,0}(q,\omega)=\frac{\chi_N^T(q)}{\omega_c(q)}f_{NN}^{T,0}(\nu,x)\,, \tag{3.20}$$

$$C_{MM}^{+-,0}(q,\omega)=\frac{\chi_M(q)}{\omega_c(q)}f_{MM}^{T,0}(\nu,x)\,, \tag{3.21}$$

$$C_{MN}^{+-,0}(q,\omega)=C_{NM}^{+-,0}(q,\omega)=\frac{(\chi_N^T(q)\chi_M(q))^{1/2}}{\omega_c(q)}f_{MN}^{T,0}(\nu,x)\,, \tag{3.22}$$

where $\omega_c=c_s q$ is the characteristic frequency, $c_s^2=R\Gamma_N^2\xi^{-2}$ is the spin-wave velocity squared evaluated to zeroth order in ϵ, $uN^2=\xi^{-2}$, $x=q\xi$ and $\nu=\omega/\omega_c$. We then find that

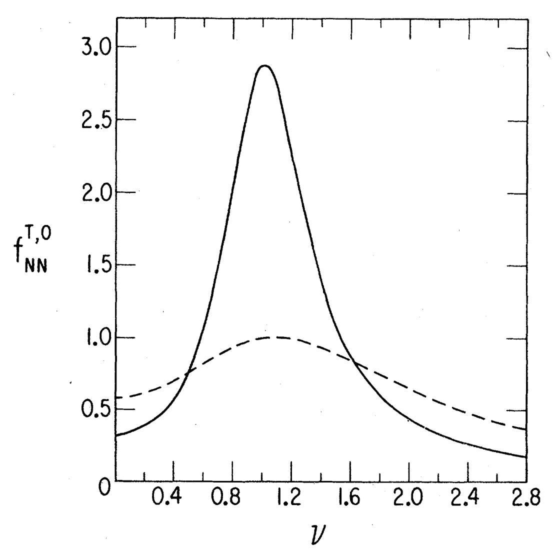

FIG. 1. Zero order in ϵ transverse order-parameter shape function, $f_{NN}^{T,0}(x,\nu)$ as a function of ν for $x=1$ (solid line) and $x=3$ (dashed line).

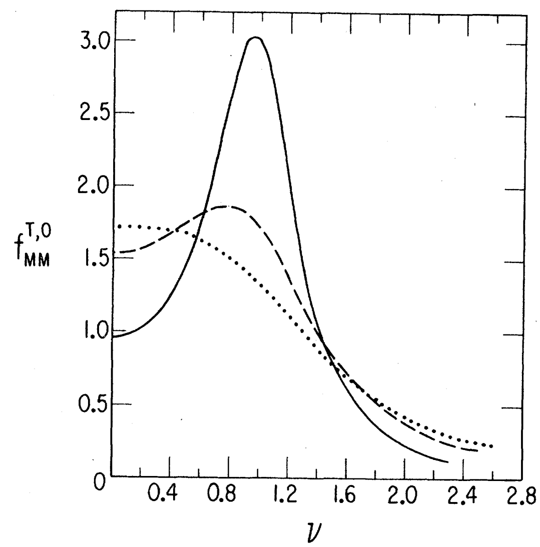

FIG. 2. Zero order in ϵ transverse magnetization shape functions, $f_{MM}^{T,0}(x,\nu)$ as a function of ν for $x=1$ (solid line), $x=2$ (dashed line) and $x=3$ (dotted line).

$$f_{NN}^{T,0}(\nu,x)=\frac{2x}{33}\left(\frac{3}{11}\right)^{1/2}\frac{(33\nu^2+x^2+11)}{D_0(x,\nu)}\,, \tag{3.23}$$

$$f_{MM}^{T,0}(\nu,x)=\frac{2x}{33}\left(\frac{3}{11}\right)^{1/2}\frac{(11\nu^2+3x^2+33)}{D_0(x,\nu)}\,, \tag{3.24}$$

$$f_{MN}^{T,0}(\nu,x)=-\frac{8}{3}\,\frac{x\nu}{D_0(x,\nu)}\,, \tag{3.25}$$

where

$$D_0(x,\nu)=\nu^4+2\nu^2\left(\frac{5x^2}{33}-1\right)+\left(1+\frac{x^2}{11}\right)^2. \tag{3.26}$$

We have plotted the results for these f's for various values of x in Figs. 1–3. The zeros of $D_0(x,\nu)$ occur at

$$\nu_0=\pm\left(1-\frac{x^2}{33}\right)^{1/2}\pm\left(\frac{3}{11}\right)^{1/2}\frac{2ix}{3}. \tag{3.27}$$

Then for small x the spin-wave poles are at $\nu_0 = \pm 1$ (or $\omega=\pm c_s q$) with a damping proportional to x. As x increases the damping increases and the effective spin-wave velocity decreases until $x^2=33$ where the real part of the pole vanishes and the mode becomes diffusive.

It is a simple matter to compute the spin-wave peak positions as a function of x from $f_{NN}^{T,0}$ and $f_{MM}^{T,0}$. In the case of $f_{NN}^{T,0}$ we obtain

$$\nu_{\text{peak}}^{N}=\left[-\frac{1}{3}\left(\frac{x^2}{11}+1\right)+\frac{4}{3}\left(\frac{x^2}{11}+1\right)^{1/2}\right]^{1/2} \tag{3.28}$$

while for $f_{MM}^{T,0}$ we obtain

$$\nu_{\text{peak}}^{M}=\left[-3\left(\frac{x^2}{11}+1\right)+4\left(\frac{x^2}{11}+1\right)^{1/2}\right]^{1/2}. \tag{3.29}$$

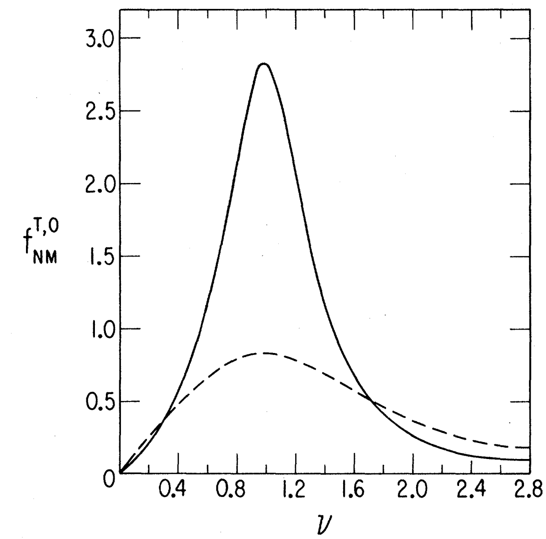

FIG. 3. Zero order in ϵ transverse off-diagonal shape function, $f_{MN}^{T,0}(x,\nu)$ as a function of ν for $x=1$ (solid line) and $x=3$ (dashed line).

We have plotted ν_{peak}^{N} and the height of the spin-wave peak for $f_{NN}^{T,0}$ in Figs. 4 and 5. It is particularly interesting that the position of the spin-wave first moves out from its hydrodynamical value for $x \lesssim 5$ and then moves rapidly to small values as x increases with the peak disappearing at $x \sim 12.85$. The spin-wave peaks persist to a higher value of x than one would obtain from an analysis of the zeros of $D_0(x,\nu)$. The peak height for $f_{NN}^{T,0}$ is a rapidly decreasing function of x. The spin-wave position for the transverse magnetization is plotted in Fig. 4. Unlike ν_{peak}^{N}, it is a monatomically decreasing function of x, with the mode disappearing at $x=2.92$. We see that the x where the real part of zero of $D_0(x,\nu)$ disappears ($x=\sqrt{33}$) is roughly midway between where the spin waves in the magnetization and in the staggered magnetization disappear. This analysis shows us the importance of finite x corrections to hydrodynamical results even when the "hydrodynamical" mode is relatively well defined.

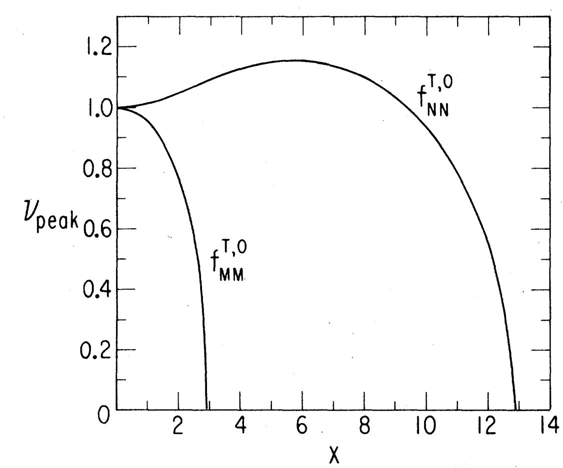

FIG. 4. Spin-wave location, ν_{peak}, as a function of x for the zero-order transverse shape functions $f_{MM}^{T,0}$ and $f_{NN}^{T,0}$.

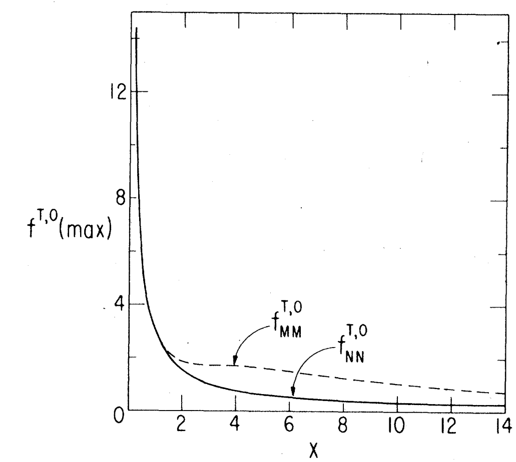

FIG. 5. Maximum value of the zero-order transverse change functions $f_{MM}^{T,0}$ and $f_{NN}^{T,0}$ as a function of x.

IV. MEMORY-FUNCTION APPROACH TO PERTURBATION THEORY

A. Motivation

We are now prepared to investigate the $O(\epsilon)$ corrections to the correlation functions for $T<T_N$. In principle, this involves a straightforward perturbation theory expansion in $\lambda \sim O(\epsilon^{1/2})$, and $u \sim O(\epsilon)$ remembering that $N \sim O(\epsilon^{-1/2})$. Ma and Mazenko carried out this calculation for the ferromagnetic case using a direct expansion of the response function. Hohenberg *et al.*[12] have used the Martin *et al.*[29] technique for treating the planar ferromagnet for $T<T_c$. In both cases the calculations were extremely complicated and cumbersome. There are several reasons why these methods are inconvenient for calculating with $T<T_c$. The major complication is the proliferation of indices. For the antiferromagnet there are two independent correlation functions for $T \geq T_N$ while there are five nontrivial correlation functions for $T<T_N$ (see Sec. III). This comes about because the spontaneous staggered magnetization picks out a special direction. The difficulties arising from the large number of indices is particularly acute in the case of the Martin, Siggia, and Rose formalism where additional fields are introduced. We

attempted to carry out the Ma and Mazenko type calculation for the antiferromagnet and found that the complexity was increased by an order of magnitude over the ferromagnetic calculation which was itself extremely involved. We were able to isolate three main reasons for the technical difficulties below T_c.

(1) The simultaneous determinations of the dynamics and statics leads to unnecessary complication. The dynamic correlation functions depend, in a rough sense, on two main ingredients, the static susceptibility $\chi(q)$ and the generalized frequency and wave number dependent transport coefficients $\Gamma(q,\omega)$. The response-function calculation mixes these together. Since we know $\chi(q)$ from independent calculations it seems unnecessary to recalculate it from the equation of motion.

(2) The fluctuation-dissipation theorem takes on a more complicated form for $T<T_c$ than for $T\geqslant T_c$. While the result

$$\frac{\partial}{\partial t}C_{ij}(t)=-G_{ij}(t)+G_{ji}(-t) \tag{4.1}$$

[Eq. (2.45) in Ref. 10] relating the correlation and response function continues to be correct for $T<T_c$, the Fourier transform $G_{ij}(\omega)$ is not generally symmetric under interchange of i and j and $\omega\leftrightarrow-\omega$. In particular for the anitferromagnet the spontaneous staggered magnetization, via the mode coupling terms, breaks this symmetry which is valid for $T\geqslant T_N$. The fluctuation-dissipation theorem becomes in this more general case

$$C_{ij}(\omega)=\frac{2}{\omega}\,\mathrm{Im}G_{ij}^{(S)}(\omega)-\frac{2i}{\omega}\,\mathrm{Re}G_{ij}^{(A)}(\omega), \tag{4.2}$$

where $G^{(S)}$ and $G^{(A)}$ are the symmetric and antisymmetric parts of G under interchange of $i\leftrightarrow j$ and $\omega\leftrightarrow-\omega$. The above form for the fluctuation dissipation theorem can complicate matters considerably for $T<T_N$.

(3) We know that a convenient representation for the response function in the disordered phase (suppressing indices) is:

$$G^{-1}(\omega)=-i\omega/\Gamma(\omega)+\chi^{-1} \tag{4.3}$$

where $\Gamma(\omega)$ is the "physical-transport coefficient" or memory function and χ is the full static susceptibility. A direct diagrammatic expansion gives G^{-1} most conveniently in terms of the one-body irreducible self-energy $\Sigma(\omega)$:

$$G^{-1}(\omega)=-i\omega/\Gamma+(\chi^0)^{-1}-\Sigma(\omega). \tag{4.4}$$

Unlike $\Gamma(\omega)$, the self-energy $\Sigma(\omega)$ has no direct physical interpretation and several intermediate steps are required to relate $\Sigma(\omega)$ to $\Gamma(\omega)$.[30] For $T<T_c$, the determination of $\Gamma(\omega)$ from $\Sigma(\omega)$ becomes quite complicated.

We can circumvent these difficulties by working with that dynamical quantity for which $\Gamma(\omega)$ is essentially the self-energy. The appropriate quantity is the Laplace transform of the correlation function

$$\bar{C}(\omega)=-i\int_0^{+\infty}dt\,e^{+i\omega t}C(t), \tag{4.5}$$

where ω is assumed to have a small positive imaginary piece. It is easy to show using the fluctuation-dissipation theorem that in the disordered phase $\bar{C}(\omega)$ is related to the memory function $\Gamma(\omega)$, as defined by (4.3), by

$$\bar{C}(\omega)=\frac{\chi}{\omega+i\Gamma(\omega)\chi^{-1}}. \tag{4.6}$$

Note that the exact static susceptibility enters naturally. By working with $\bar{C}$ we can develop methods for calculating $\Gamma(\omega)$ directly. This will also allow us to circumvent the entire question of a fluctuation-dissipation theorem.

B. Memory-function formulation

We now want to outline a method for calculating $\Gamma(\omega)$ directly using ideas that have evolved in the treatment of the dynamics of fluids.[31] We intend to publish the details of the method elsewhere. Here, we will present a few ideas behind the method and the results relevant to the antiferromagnet.

Ma and Mazenko[10] have shown that the correlation function for a rather large class of dynamical models (including the isotropic antiferromagnet) can be written in the form[32]

$$C_{ij}(t)=\int\mathfrak{D}\,\varphi\;\delta\varphi_i\,e^{D_\varphi t}\delta\varphi_j W_\varphi \tag{4.7}$$

for $t\geqslant0$. In (4.7), φ_i represents the set of slowly varying fields: $\delta\varphi_i=\varphi_i-\langle\varphi_i\rangle$,

$$W_\varphi=e^{-F\varphi}\Big/\int\mathfrak{D}\,\bar{\varphi}\,e^{-F\bar{\varphi}} \tag{4.8}$$

is the static distribution function, and D_φ is the generalized Fokker-Planck operator

$$D_\varphi=-\sum_i\frac{\partial}{\partial\varphi_i}\left[V_i-L_i\left(\frac{\partial}{\partial\varphi_i}+\frac{\partial F}{\partial\varphi_i}\right)\right], \tag{4.9}$$

where V_i is the streaming term and L_i is a generalized transport coefficient. The above form for the correlation function is very similar to that for the usual correlation functions studied in the theory of classical fluids if we identify $-iD_\varphi$ with L the Liouville operator. The mapping is not complete because iD_φ is not Hermitian. Instead we have

$$\int\mathfrak{D}\,\varphi A(\varphi)D_\varphi B(\varphi)=\int\mathfrak{D}\,\varphi\,B(\varphi)\tilde{D}_\varphi A(\varphi), \tag{4.10}$$

where

$$\tilde{D}_\varphi = \sum_i \left[V_i - L_i\left(-\frac{\partial}{\partial\varphi_i} + \frac{\partial F}{\partial\varphi_i}\right)\right]\frac{\partial}{\partial\varphi_i} . \quad (4.11)$$

This lack of symmetry is not really a problem. We can develop the memory function methods for this case in almost complete analogy to the fluid case. We first introduce the Laplace transform

$$\bar{C}_{ij}(\omega) = -i\int_0^{+\infty} dt\, e^{+i\omega t} C_{ij}(t) ,$$

$$= -i\int_0^{+\infty} dt \int \mathfrak{D}\varphi\, \delta\varphi_i e^{i(\omega - iD_\varphi)t}\delta\varphi_j W_\varphi ,$$

$$= \int \mathfrak{D}\varphi\delta\varphi_i(\omega - iD_\varphi)^{-1}\delta\varphi_j W_\varphi \equiv \langle\varphi_i;\varphi_j\rangle_\omega . \quad (4.12)$$

The memory function $M(\omega)$ is defined in this general case by

$$\omega\bar{C}_{ij}(\omega) - \sum_k M_{ik}(\omega)\bar{C}_{kj}(\omega) = \chi_{ij} , \quad (4.13)$$

where $\chi_{ij} = C_{ij}(0)$ is the full static susceptibility. The essence of the method is to use operator identities like

$$(\omega - iD_\varphi)^{-1} = \omega^{-1} + \omega^{-1} iD_\varphi(\omega - iD_\varphi)^{-1} \quad (4.14)$$

to obtain an expression for $M(\omega)$ in terms of multifield correlation functions. In particular we find that M is a sum of two pieces:

$$M_{ij}(\omega) = M_{ij}^{(s)} + M_{ij}^{(c)}(\omega) , \quad (4.15)$$

where $M_{ij}^{(s)}$ is a static or ω-independent term given quite generally as

$$M_{ij}^{(s)} = \sum_k \langle(i\tilde{D}_\varphi\varphi_i)\delta\varphi_k\rangle\chi_{kj}^{-1} ,$$

$$= -i\sum_k \langle Q_{ik}[\varphi]\rangle\chi_{kj}^{-1} - iL_i\chi_{ij}^{-1} , \quad (4.16)$$

where the Q_{ik} are essentially the poisson brackets between the variables i and j and we have used Eq. (2.12) in Ma and Mazenko.[10] The second term in M^s, $-iL_i\chi_{ij}^{-1}$, is clearly just the usually zeroth-order damping term but with $(\chi_{ij}^0)^{-1}$ replaced by the full χ_{ij}^{-1}. For the antiferromagnet case where the index i has a spin label, an isospin label as well as a coordinate label, we arrive at the form

$$M_{ij,\alpha\beta}^{(s)}(q) = -i\lambda\epsilon_{ijz}\delta_{\alpha,-\beta}N\chi_j^{-1}(q) - iL^\alpha(q)\chi_j^{-1}(q)\delta_{ij}\delta_{\alpha,\beta} . \quad (4.17)$$

If we ignore $M^{(c)}$ then the equation for $\bar{C}_{ij,\alpha\beta}(q,\omega)$ is given by

$$[\omega + iL^\alpha(q)\chi_i^{-1}(q)]\bar{C}_{ij,\alpha\beta}(q,\omega) + i\lambda N\sum_l \epsilon_{ilz}\chi_l^{-1}(q)\bar{C}_{lj,-\alpha\beta}(q,\omega) = \chi_i(q)\delta_{ij}\delta_{\alpha\beta} . \quad (4.18)$$

In the disordered phase $N=0$ and

$$\bar{C}_{ij,\alpha\beta}(q,\omega) = \frac{\chi_i(q)\delta_{ij}\delta_{\alpha\beta}}{\omega + iL^\alpha(q)\chi_i^{-1}(q)} . \quad (4.19)$$

This is the zeroth-order result except for the χ_i's which are the exact static susceptibilities. In the ordered phase the term proportional to λ generates spin-wave modes with speed

$$C_s = \lim_{q\to 0}\frac{1}{q}\lambda N\{\chi_M^{-1}[\chi_N^T(q)]^{-1}\}^{1/2} . \quad (4.20)$$

This is the *exact* result one would construct using hydrodynamical arguments.[13] To the extent we replace χ_N^T and χ_N^L with their zeroth order in ϵ limits (4.19) reproduces the zeroth-order correlation functions discussed in Sec. III.

We turn now to the dynamical part of the memory function $M^{(c)}$. It can be shown that $M^{(c)}$ is given by

$$M_{ik}^{(c)}(\omega)\chi_{kj} = -\langle\delta I_i[\varphi];\delta I_j^\dagger[\varphi]\rangle_\omega + \langle I_i[\varphi];\delta\varphi_l\rangle_\omega \times \bar{C}_{lk}^{-1}(\omega)\langle\delta\varphi_k; I_j^\dagger[\varphi]\rangle_\omega , \quad (4.21)$$

where

$$I_i[\varphi] = iV_i - iL_i\partial F_I/\partial\varphi_i , \quad (4.22a)$$

$$I_i^\dagger[\varphi] = iV_i + iL_i\partial F_I/\partial\varphi_i . \quad (4.22b)$$

Here, and in the rest of this section, we use the convention that one sums over repeated indices. We have also introduced the notation

$$\langle A;B\rangle_\omega = \langle\delta A(\omega - iD_\varphi)^{-1}\delta B\rangle \quad (4.23)$$

with A and B arbitrary functions of the field variables. F_I is the "interaction" part of a Landau-Ginzburg free energy. Note that $M^{(c)}$ depends only on the nonlinear terms in the equation of motion.

If we now assume canonical forms for mode-coupling and free-energy interactions, Eq. (4.21) takes the form:

$$M_{ik}^{(c)}(\omega)\chi_{kj} = \tilde{V}_{ils}\tilde{V}_{juv}G_{ls,uv}(\omega) + \tilde{V}_{ils}L_j u_{juvw}G_{ls,uvw}(\omega) - L_i u_{ilst}\tilde{V}_{juv}G_{lst,uv}(\omega) - L_iL_j u_{ilst}u_{juvw}G_{lst,uvw}(\omega) \quad (4.24)$$

where $\tilde{V}_{ils}$ which represents the mode-coupling terms is symmetric in l and s, and u_{ijkl} is completely symmetric. The G's in this equation are the higher-order correlation functions defined by:

$$G_{ls\ldots,uv\ldots}(\omega) = \langle\varphi_l\varphi_s\ldots;\varphi_u\varphi_v\ldots\rangle_\omega - \langle\varphi_l\varphi_s\ldots;\varphi_r\rangle_\omega\bar{C}_{rp}^{-1}(\omega)\langle\varphi_p;\varphi_u\varphi_v\ldots\rangle_\omega . \quad (4.25)$$

If we are working in the disordered phase then the first term in Eq. (4.24) is of $O(\epsilon)$ and gives the usual mode-coupling contributions, the terms proportional to $\tilde{V}$ and u are of $O(\epsilon^{3/2})$, and the term proportional to u^2 give $O(\epsilon^2)$ corrections. The u^2 term give the leading dynamical contributions in the time-dependent Ginzburg-Landau models (TDGL).[33] This equation for $M^{(c)}$ can be used to develop a perturbation theory expansion for M in powers of λ and u.[34] The details will be presented elsewhere.

C. Calculation of $M_{ij}^{(c)}$ to $O(\epsilon)$ for $T<T_N$

Our main concern in this paper is the analysis of $M_{ij}^{(c)}$ for $T<T_N$. FM, who worked above and at T_N, needed to keep only the $\tilde{V}^2$ term in the expression for $M^{(c)}$. For $T<T_N$ we must be more careful. The reason is that the 5 and 6 point functions $G_{lst,uv} G_{lst,uvw}$ possess disconnected pieces corresponding to $\varphi=N$ or what we will call, in analogy to the case of helium, condensate insertions. Since each condensate insertion is of $O(\epsilon^{-1/2})$ we can obtain overall contributions of $O(\epsilon)$.

After taking into account these condensate insertions and the symmetries of the various correlation functions we can write correct to $O(\epsilon)$:

$$M_{ik}^{(c)}(\omega)\chi_{kj}=[\tilde{V}_{ils}-3L_i N_t u_{ilst}]G_{ls,uv}(\omega)\times[\tilde{V}_{juv}+3L_j N_w u_{juvw}] . \tag{4.26}$$

To complete our analysis we must evaluate $G_{ls,uv}(\omega)$ to zeroth order in ϵ. We outline here two methods for carrying out this analysis. The first approach is to consider the correlation functions $G_{ls,uv}(t)$ in real time. Since we need only consider results of zero order in ϵ, the average is Gaussian, and the four-point correlations may be evaluated pairwise. The result for the memory function is:

$$M_{ik}^{(c)}(\omega)\chi_{kj}=2(\tilde{V}_{ils}-3L_i N_t u_{ilst})\times\int\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\frac{C_{lu}(\omega_1)C_{sv}(\omega_2)}{\omega-\omega_1-\omega_2}\times(\tilde{V}_{juv}+3L_j N_w u_{juvw}) \tag{4.27}$$

where $C_{lu}(\omega)$ is the Fourier transform of the zeroth order correlation functions. This expression is a standard looking mode-coupling result. We introduce it since we will need it in Sec. VI.

The second method is more general and also more convenient from a calculational point of view. This approach is a simple extension of the analysis of C_{ij} itself. Using the operator identity (4.14) $G_{ls,uv}(\omega)$ may be expressed in terms of a static correlation function plus higher-order five- and six-point correlation functions. After taking into account the condensate insertions and neglecting those terms which are of order $\epsilon^{1/2}$ or higher, we arrive at the closed matrix equation for $G_{ls,uv}(\omega)$:

$$[\omega+i(\tilde{L}_i+\tilde{L}_j)]G_{ij,kl}(\omega)-i(\delta_{ip}\delta_{jq}+\delta_{iq}\delta_{jp})2\tilde{V}_{pmn}N_m G_{nq,kl}(\omega)+(\delta_{ip}\delta_{jq}+\delta_{iq}\delta_{jp})iL_p u_{pmns}3N_m N_n G_{sq,kl}(\omega)=\tilde{G}_{ij,kl} , \tag{4.28}$$

where

$$\tilde{L}_i=L_i\chi_i^{-1} \tag{4.29}$$

and $\tilde{G}_{ij,kl}$ is the static counterpart of $G_{ij,kl}(\omega)$. For simplicity we work in a basis where χ is diagonal.

It is now a straightforward although tedious matter to invert the matrix equations and then utilize Eq. (4.26) to obtain an explicit representation for the memory functions. We summarize the results. The labels, σ, σ', s, s' denote spin components in the helicity basis while all other Greek letters represent isospin indices. Then to $O(\epsilon)$, $M^{(c)}$ is given by[35]

$$\Gamma_{\delta\delta'}^{\sigma\sigma'}(k_1k_2,\omega)=\sum_{\mu,s}M_{\delta\mu,\sigma s}^{(c)}(k_1k_2,\omega)\chi_{\mu\delta'}^{s\sigma'}(k_2) , \tag{4.30}$$

$$\Gamma_{\delta\delta'}^{\sigma\sigma'}(k_1k_2,\omega)=2I_{\delta\alpha\beta}^{\sigma ss'}(k_1,q_1q_2)D_{s^2s'^2}^{-1}(q_1q_2)\langle\alpha\beta|N_{ss'}(q_1q_2)|\alpha'\beta'\rangle\tilde{C}_\mu^s(q_1)\tilde{C}_\nu^{s'}(q_2)I_{\delta'\mu\nu}^{\dagger,\sigma,-s,-s'}(k_2,-q_1,-q_2) , \tag{4.31}$$

where

$$I_{\delta\alpha\beta}^{\sigma ss'}(k_1,q_1q_2)=[E_{\sigma ss'}\tfrac{1}{2}W_{\delta\alpha\beta}(k_1q_1)+3L_\delta(k_1)u\delta_{\delta,N}\delta_{\alpha,N}\delta_{\beta,N}b_{\sigma ss'0}]\,\delta(k_1-q_1-q_2) , \tag{4.32}$$

$$I^\dagger=-I^* , \tag{4.33}$$

$$E_{\sigma ss'}=\delta_{\sigma,0}is\delta_{s,-s'}+i\sigma(\delta_{s,0}\delta_{\sigma,s'}-\delta_{s',0}\delta_{\sigma,s}) , \tag{4.34}$$

is ϵ_{ijk} transformed into the helicity basis,

$$b_{\sigma ss'0}=\tfrac{1}{3}(\delta_{\sigma,s}\delta_{s',0}+\delta_{\sigma,s'}\delta_{s,0}+\delta_{\sigma,0}s'^2\delta_{s,-s'}+\delta_{\sigma,0}\delta_{s',0}\delta_{s,s'}) , \tag{4.35}$$

$$W_{\delta\alpha\beta} = \sigma_{\alpha\beta\gamma}\lambda|\delta_{\alpha,-\gamma}[\chi_\gamma^0(q-q_1)]^{-1} - \delta_{\alpha,-\beta}[\chi_\beta^0(q_1)]^{-1}, \tag{4.36}$$

$$\sigma_{\alpha\beta\gamma} = \delta_{\alpha,N}\delta_{\beta,-\gamma} + \delta_{\alpha,M}\delta_{\beta,\gamma}, \tag{4.37}$$

$$\begin{aligned} D_{s^2s'^2}(q_1q_2) = {} & A_{\alpha\beta}^{ss'}A_{-\alpha\beta}^{ss'}A_{\alpha,-\beta}^{ss'}A_{-\alpha,-\beta}^{ss'} - s^2B(q_1)(A_{\alpha,\beta}^{ss'}A_{-\alpha,\beta}^{ss'} + A_{\alpha,-\beta}^{ss'}A_{-\alpha,-\beta}^{ss'}) - s'^2B(q_2)(A_{\alpha\beta}^{ss'}A_{\alpha,-\beta}^{ss'} + A_{-\alpha,\beta}^{ss'}A_{-\alpha,-\beta}^{ss'}) \\ & + [s^2B(q_1) - s'^2B(q_2)]^2, \end{aligned} \tag{4.38}$$

$$\begin{aligned} \langle \alpha\beta | N_{ss'}(q_1q_2) | \alpha'\beta' \rangle = {} & \delta_{\alpha\alpha'}\delta_{\beta\beta'}[A_{-\alpha,\beta}^{ss'}A_{\alpha,-\beta}^{ss'}A_{-\alpha,-\beta}^{ss'} - s^2B(q_1)A_{-\alpha,\beta}^{ss'} \\ & - s'^2B(q_2)A_{\alpha,-\beta}^{ss'}] + b_\alpha\delta_{-\alpha,\alpha'}\delta_{\beta\beta'}[-A_{\alpha,-\beta}^{ss'}A_{-\alpha,-\beta}^{ss'} + s^2B(q_1) - s'^2B(q_2)] \\ & + C_\beta\delta_{\alpha\alpha'}\delta_{-\beta,\beta'}[-A_{-\alpha,\beta}^{ss'}A_{-\alpha,-\beta}^{ss'} + s'^2B(q_2) - s^2B(q_1)] + \delta_{-\alpha,\alpha'}\delta_{-\beta,\beta'}b_\alpha C_\beta(A_{\alpha,-\beta}^{ss'} + A_{-\alpha,\beta}^{ss'}), \end{aligned} \tag{4.39}$$

$$A_{\alpha\beta}^{ss'} = [\omega + i\tilde{L}_\alpha^s(q_1) + i\tilde{L}_\beta^{s'}(q_2)], \tag{4.40}$$

$$b_\alpha = -s\lambda r_{-\alpha}(q_1)N, \tag{4.41}$$

$$C_\alpha = -s'\lambda r_{-\alpha}(q_2)N, \tag{4.42}$$

and

$$s^2B(q_1) = b_\alpha b_{-\alpha} = s^2\lambda^2 r_\alpha(q_1)r_{-\alpha}(q_1)N^2, \tag{4.43}$$

$$\tilde{L}_\alpha^s(q) = \Gamma_\alpha(q)\{[\chi_\alpha^0(q)]^{-1} + \delta_{\alpha,N}uN^2(1+2\delta_{s,0})\}. \tag{4.44}$$

Note that D is independent of the isospin indices. We can then easily find any component of M^c by doing the internal sums over α, β, s, and s' in Eq. (4.31).

The rather complex nature of these results emphasizes the difficulties one might encounter if a more conventional approach were used.

V. TRANSVERSE CORRELATION FUNCTIONS

The general structure of the transverse part of the dynamic memory function is given by Eq. (4.31). The transverse memory functions are related to the transverse correlation functions, using Eq. (4.13), by the matrix equation:

$$\sum_\beta \{\omega\delta_{\alpha\beta} - \lambda N[\chi_\beta^T(q)]^{-1}\delta_{\alpha,-\beta} + i\tilde{\Gamma}_{\alpha\beta}^T(q,\omega) \times [\chi_\beta^T(q)]^{-1}\}\overline{C}_{\beta\mu}^T(q,\omega) = \chi_\alpha^T(q)\delta_{\alpha\mu}, \tag{5.1}$$

where Eq. (4.17) has been utilized for the static or ω-independent part of the memory function. We furthermore have

$$\tilde{\Gamma}_{\alpha\beta}^T(q,\omega) = \Gamma_\alpha(q)\delta_{\alpha\beta} + i\Gamma_{\alpha\beta}^T(q,\omega), \tag{5.2}$$

where Γ_α indicates the bare transport coefficient and $\Gamma_{\alpha\beta}^T(q,\omega)$ is the ω-dependent part of the memory function given by Eq. (4.31).

We have evaluated the function $\Gamma_{\alpha\beta}^T(q,\omega)$ in the hydrodynamic limit of small q, and $\omega=0$. After performing certain elementary exponentiations, we have in this limit:

$$\tilde{\Gamma}_{MM}^T(q,\omega) = \Gamma_M q^2(\Lambda\xi)^{\epsilon/2}(1-\epsilon a_{MM}), \quad a_{MM} = 0.4200 \tag{5.2a}$$

$$\tilde{\Gamma}_{MN}^T(q,\omega) = -(\Gamma_M\Gamma_N)^{1/2}qx\epsilon a_{MN}, \quad a_{MN} = 0.0622 \tag{5.2b}$$

$$\tilde{\Gamma}_{NN}^T(q,\omega) = \Gamma_N(\Lambda\xi)^{\epsilon/2}(1-\epsilon a_{NN}), \quad a_{NN} = 0.3863 \tag{5.2c}$$

ξ is the correlation length, Λ a wave-number cutoff, while $x = q\xi$.

Utilizing these expressions for the memory function, it is a simple matter to invert the matrix equation (5.1) and calculate the transverse correlation function in the hydrodynamic limit:

$$\overline{C}_{NN}^T(q,\omega) = \frac{(\omega + i\chi_M^{-1}\tilde{\Gamma}_{MM}^T)\chi_N^T}{D(q,\omega)}, \tag{5.3a}$$

$$\overline{C}_{NM}^T(q,\omega) = \overline{C}_{MN}^T(q,\omega) = \frac{\lambda N + \tilde{\Gamma}_{NM}^T}{D(q,\omega)}, \tag{5.3b}$$

$$\overline{C}_{MM}^T(q,\omega) = \frac{[\omega + i(\chi_N^T)^{-1}\tilde{\Gamma}_{NN}^T]\chi_M}{D(q,\omega)}, \tag{5.3c}$$

$$D(q,\omega) = [\omega + i(\chi_N^T)^{-1}\tilde{\Gamma}_{NN}^T][\omega + i\chi_M^{-1}\tilde{\Gamma}_{MM}^T] - \chi_M^{-1}(\chi_N^T)^{-1}(\lambda N + \tilde{\Gamma}_{NM}^T)^2. \tag{5.3d}$$

From the roots of $D(q,\omega)$ we extract the spin-wave velocity and spin-wave damping coefficient valid to order ϵ:

$$D(q,\omega) = (\omega - \omega_+)(\omega - \omega_-), \tag{5.4a}$$

$$\omega_\pm = \pm c_s q - \tfrac{1}{2}(iq^2D_s), \tag{5.4b}$$

$$c_s^2 = R\Gamma_N^2\xi^{-2}(\Lambda\xi)^\epsilon\left[1 + \tfrac{1}{11}\epsilon(1+\tfrac{1}{3}x^2) - 2\left(\frac{u}{f}\right)^{1/2}x^2\epsilon a_{MN}\right], \tag{5.4c}$$

$$D_s = \Gamma_N(\Lambda\xi)^{\epsilon/2}(1 + a + \tfrac{1}{11}\epsilon - \epsilon a_{NN} - \epsilon a_{MM}). \tag{5.4d}$$

Note that c_s^2 is valid to order x^2, while D_s is valid to zero order in x^2.

It will be useful to characterize the spin-wave velocity and spin-wave attenuation coefficient by the dimensionless parameters b_s and d_s defined as:

$$\lim_{x\to 0} c_s = \Gamma_N \xi^{-1}(\Lambda\xi)^{\epsilon/2} b_s \tag{5.5a}$$

$$\lim_{x\to 0} D_s = \Gamma_N (\Lambda\xi)^{\epsilon/2} d_s . \tag{5.5b}$$

From Eqs. (5.4), we have for b_s and d_s,

$$b_s = \sqrt{R}(1 + \tfrac{1}{22}\epsilon) \tag{5.6}$$

$$d_s = 1 + a + \tfrac{1}{11}\epsilon - \epsilon a_{NN} - \epsilon a_{MM} . \tag{5.7}$$

Using our results given in Sec. II for a, f, and u to the necessary order in ϵ, we have for b_s, and d_s, valid to $O(\epsilon)$

$$b_s = \sqrt{\tfrac{11}{3}}\left(1 + \frac{\epsilon}{2}(f_2 + a_1 - u_2 + \tfrac{1}{11})\right) = 1.981 \quad (d=3) \tag{5.8a}$$

$$d_s = \tfrac{4}{3}[1 + \tfrac{3}{4}\epsilon(a_1 + \tfrac{1}{11} - a_{NN} - a_{MM})] = 1.424 \quad (d=3). \tag{5.8b}$$

The spin-wave frequency and spin-wave attenuation both have the dimensions of a frequency. Consequently, we may form the universal dimensionless ratio:

$$\lim_{x\to 0} \frac{D_s q^2}{c_s q x} = \overline{D} \tag{5.9}$$

$$\overline{D} = \frac{4}{\sqrt{33}}[1 - \epsilon(\tfrac{1}{2}f_2 - \tfrac{1}{4}a_1 - \tfrac{1}{2}u_2 - \tfrac{1}{44} + \tfrac{3}{4}a_{NN} + \tfrac{3}{4}a_{MM})]$$
$$= 0.720 \text{ (for } d=3). \tag{5.10}$$

While this ratio is experimentally determinable we have found no such measurements reported in the literature.

We now discuss the order ϵ results for the shape functions. In order to extract the shape functions, we define the characteristic frequency $\omega_c = qc_s$. Furthermore the Fourier transform of the correlation function $C(q,\omega)$ is related to the Laplace transform $\overline{C}(q,\omega)$ by the simple relation

$$C(q,\omega) = -2\,\mathrm{Im}\overline{C}(q,\omega). \tag{5.11}$$

Using the dynamical scaling form of Eqs. (3.20)–(3.22) we have:

$$f_{NN}^T(x,\nu) = 2xa\sqrt{R^{-1}}\left[\frac{\nu^2}{a}(1 + \tfrac{1}{22}\epsilon - \epsilon a_{NN}) + 1 - \epsilon(\tfrac{1}{22} + a_{MM})\right][D(x,\nu)]^{-1} \tag{5.12a}$$

$$f_{NM}^T(x,\nu) = 2x\nu(1+a)\sqrt{R^{-1}}\left[1 + \frac{\epsilon}{22}\left(\frac{1-a}{1+a}\right) - \frac{\epsilon a_{NN}}{1+a} - \frac{\epsilon a a_{MM}}{1+a}\right][D(x,\nu)]^{-1} \tag{5.12b}$$

$$f_{MM}^T(x,\nu) = 2xa\sqrt{R^{-1}}\left[\nu^2\left(1 - \frac{\epsilon}{22} - \epsilon a_{MM}\right) + \frac{1}{a} - \frac{\epsilon}{a}\left(a_{NN} - \frac{1}{22}\right)\right][D(x,\nu)]^{-1} \tag{5.12c}$$

$$D(x,\nu) = \nu^4 - 2\nu^2\left[1 + \frac{ux^2}{f}(1 - \epsilon a_{NN} - \epsilon a_{MM}) - \frac{x^2 u}{2fa}(1+a)^2\left(1 + \frac{\epsilon}{11}\left(\frac{1-a}{1+a}\right) - \frac{2\epsilon a_{NN}}{1+a} - \frac{2\epsilon a_{MM}}{1+a}\right)\right]$$
$$+ \frac{2ux^2}{f}(1 - \epsilon a_{NN} - \epsilon a_{MM}) + 1. \tag{5.12d}$$

Again, in order to explicitly evaluate these functions, the results of Eqs. (2.7) must be utilized. Figures 6–8 summarize the order ϵ results for the transverse correlation functions in the range $x \lesssim 1$ where our calculations are valid. Qualitatively, the order ϵ results have the same feature as the zero order results. There is a sharply peaked spin-wave excitation, centered roughly about $\nu = 1$. However, a more detailed comparison does reveal some differences. In Figs. 9 and 10, we compare the peak positions for Eqs. (5.12) with the zero-order correlation functions. In the case of f_{MM}^T, qualitative differences appear. For $f_{MM}^{T,0}$ the peak position decreases monatomically with increasing x and eventually disappears. However, for Eq. (5.12c), the peak position increases with increasing x. Of course, our results are no longer valid for $x > 1$, but we can interpolate between our hydrodynamic equation and the known properties of f_{MM}, as calculated by FM at T_N, who found no evidence of any spin-wave remnants. Consequently, we expect the spin-wave position for f_{MM} to increase with increasing x, reach some maximum position and then decrease with increasing x, eventually disappearing altogether. The disappearance of the spin-wave peak in f_{MM}^T for finite x is also predicted by the zero-order results. This suggests that order ϵ corrections are qualitatively important in determining the location of the spin-wave peak in f_{MM}^T for small x, but do not qualitatively affect the persistence of the peaks.

A comparison of peak height between zero-order and order-ϵ results for f_{MM}^T in Fig. 11 shows a substantial increase in peak height when ϵ cor-

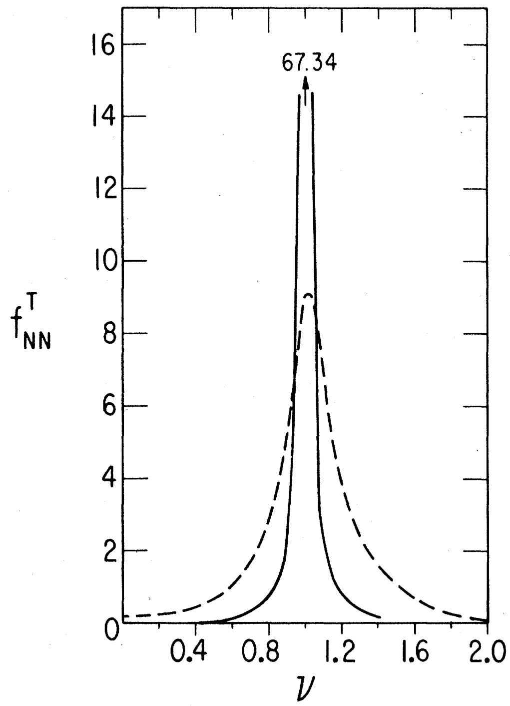

FIG. 6. Order ϵ, staggered-magnetization transverse shape function $f^T_{NN}(x,\nu)$, as a function of ν for $x=0.1$ (solid line) and $x=0.75$ (dashed line).

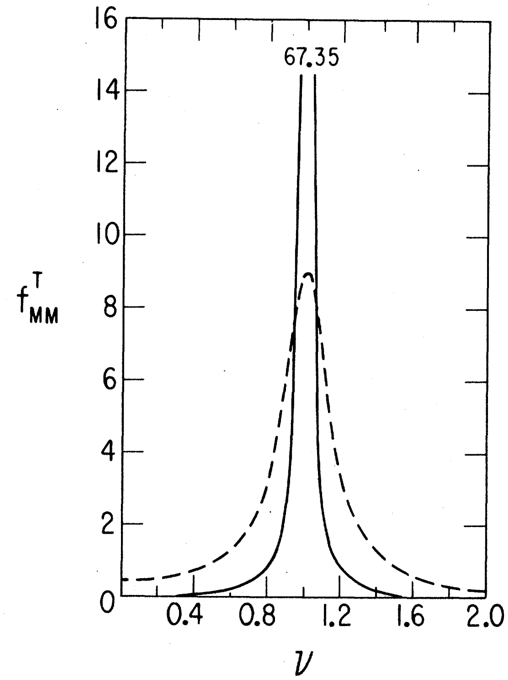

FIG. 8. Order ϵ, magnetization transverse shape function $f^T_{MM}(x,\nu)$ as a function of ν for $x=0.1$ (solid line) and $x=0.75$ (dashed line).

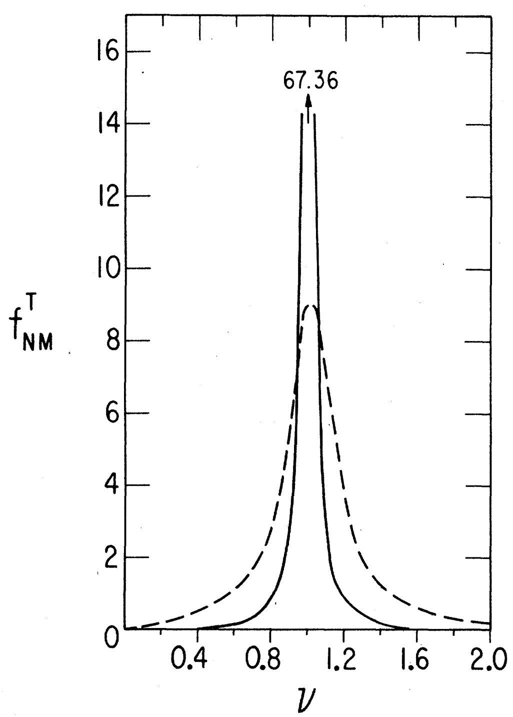

FIG. 7. Order ϵ, off-diagonal transverse shape function, $f^T_{MN}(x,\nu)$, as a function of ν for $x=0.1$ (solid line) and $x=0.75$ (dashed line).

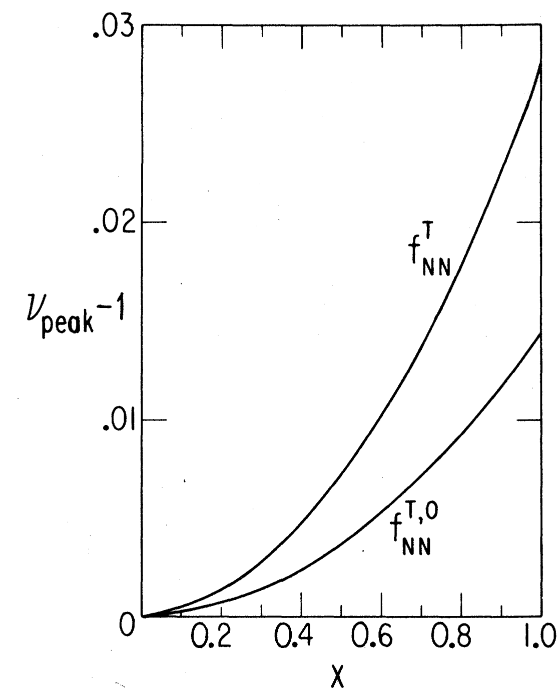

FIG. 9. Spin-wave location ν_{peak} of the order-parameter transverse shape function is compared for the zero-order $f^{T,0}_{NN}$, and order ϵ, f^T_{NN}, results in the hydrodynamic region $x<1$.

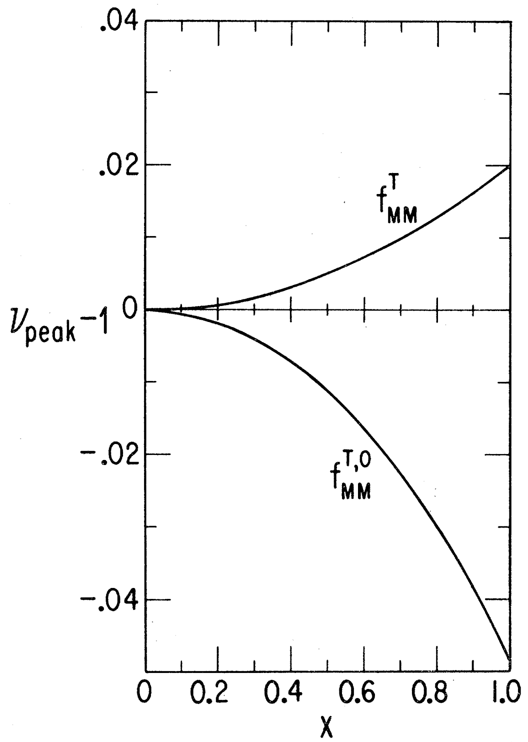

FIG. 10. Spin-wave location ν_{peak} of the magnetization transverse shape functions is compared for the zero order, $f^{T,0}_{MM}$ and order ϵ, f^T_{MM}, results in the hydrodynamic region $x<1$.

rections are considered. While quantitative predictions are not possible, this enhancement suggests that the spin-wave peaks in f^T_{MM} will persist for higher x values than is predicted by the zero-order results.

For f^T_{NN} both the order ϵ and zero-order results demonstrate a positive shift in peak position for increasing x (see Fig. 9). However, in the case of the order-ϵ correlation function, this shift is enhanced. If we again interpolate between order-ϵ hydrodynamic results and the FM results at T_N, we would expect the peak position to eventually decrease with increasing x but never disappear entirely as long as $T \lesssim T_N$. This should be compared to the zero order in ϵ result $f^{T,0}_{NN}$ where the peaks shift positively for $x \lesssim 5.5$. For x greater than this, the shift is negative and for $x \approx 12.8$, the peaks disappear entirely. This suggests that for f^T_{NN} the $O(\epsilon)$ results are relatively unimportant for determining peak location but are very important in determining the persistence of the peaks as a function of x.

A comparison of peak height between zero order and order ϵ results for f^T_{NN} in Fig. 12 shows a substantial increase in peak height when ϵ corrections are considered for small x. This enhancement is, of course, not unexpected and is consistent with the persistence of the peaks at T_N as calculated by FM.

We have not included any quantitative comparison between first- and zero-order ϵ results for the off-diagonal correlation function f^T_{MN} since this does not seem to be an experimentally accessible quantity. However, we do expect the peaks to exist for all x as long as $T<T_c$. As x becomes larger the spectrum will become flatter, eventually going to zero everywhere as $T \to T_N$. For $T>T_N$, this correlation function will then remain zero.

In conclusion then, the order-ϵ transverse correlation functions reproduce for the most part the same qualitative features of the zero-order correlation function for $x \lesssim 1$. We expect, however, order-ϵ corrections to play an important role in two ways: They determine the spin-wave peak location in f^T_{MM}, and control the persistence of the spin wave peaks in f^T_{NN} for large x. In particular, the determination of f^T_{NN} to order ϵ for all x would be a very interesting calculation. This function would presumably demonstrate explicitly the cross-

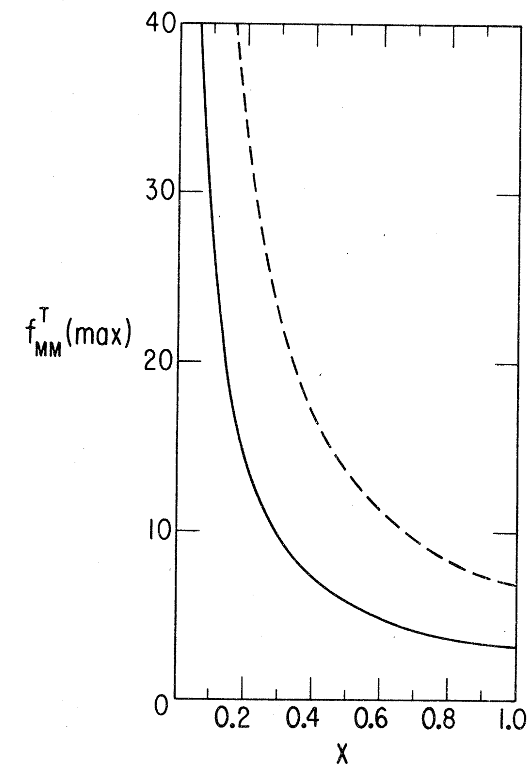

FIG. 11. Maximum value f^T_{MM} (max) of the magnetization transverse shape function is compared for the zero-order, $f^{T,0}_{MM}$ (solid line) and order-ϵ, f^T_{MM} (dashed line) results in the hydrodynamic region $x<1$.

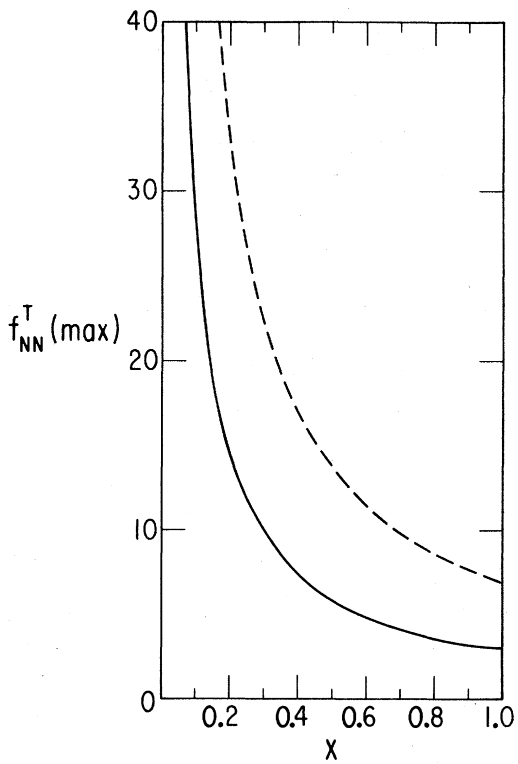

FIG. 12. Maximum value f^T_{MM} (max) of the staggered-magnetization transverse shape function is compared for the zero-order, $f^{T,0}_{NN}$ (solid line) and order-ϵ, f^T_{NN} (dashed line) results in the hydrodynamic region $x<1$.

over behavior from a pure spin-wave spectrum to the "sloppy" spin-wave spectrum calculated by FM at T_N.

VI. LONGITUDINAL MAGNETIZATION CORRELATION FUNCTION

We obtain the memory function associated with the longitudinal magnetization correlation function correct to $O(\epsilon)$ by taking the $\sigma=\sigma'=0$, $\delta=\delta'=M$ component of $\Gamma^{\sigma\sigma'}_{\delta\delta'}$ given by Eq. (4.32) and doing the internal sums. We obtain the relatively simple result ($\Gamma^L_M=\Gamma^{00}_{MM}$),

$$\Gamma^L_M(q,\omega) = \frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\{[\chi^0_N(k-q)]^{-1}-[\chi^0_N(k)]^{-1}\}^2 2\chi^+_N(k)\chi^+_N(k-q)}{D_{++}(k,k-q)} \times [A^{++}_{MN}A^{++}_{NM}A^{++}_{MM}-B(k)A^{++}_{MN}-B(k-q)A^{++}_{NM}], \tag{6.1}$$

where the A's, B, and D_{++} are defined by Eqs. (4.40), (4.43), and (4.38). We have, thus far, evaluated $\Gamma^L_M(q,\omega)$ explicitly only in certain limiting, but interesting cases. The first case we study is that of fixed frequency and $q\to 0$:

$$\overline{\Gamma}^L_M(0,\omega)=\lim_{q\to 0}\overline{\Gamma}^L_M(q,\omega), \tag{6.2a}$$

$$\overline{\Gamma}^L_M(q,\omega)=\frac{1}{q^2}\tilde{\Gamma}^L_M(q,\omega), \tag{6.2b}$$

where $\tilde{\Gamma}^L_M(q,\omega)$ is the full generalized transport coefficient

$$\tilde{\Gamma}^L_M(q,\omega)=\Gamma_M q^2+i\Gamma^L_M(q,\omega). \tag{6.2c}$$

The real part of $\overline{\Gamma}^L_M(0,\omega)$ has a direct physical interpretation. We can see this by noting the identity

$$\mathrm{Re}\overline{\Gamma}^L_M(0,\omega)=\lim_{q\to 0}\frac{\omega^2}{q^2}C^L_M(q,\omega), \tag{6.3}$$

where $C^L_M(q,\omega)$ is the Fourier transform of $C^L_M(q,t)$ and is related to the Laplace transform by

$$C^L_M(q,\omega)=-2\,\mathrm{Im}\overline{C}^L_M(q,\omega). \tag{6.4}$$

Using the continuity equation

$$\frac{\partial M_i}{\partial t}+\nabla\cdot\vec{J}_i=0, \tag{6.5}$$

where $\vec{J}$ is the spin-current density, we see using Eq. (6.3) that $\mathrm{Re}\overline{\Gamma}^L_M(0,\omega)$ is just the Fourier transform of the spin-current autocorrelation function. We can calculate, correct to first order in ϵ, in the limit of small ω

$$\overline{\Gamma}^L_M(0,\omega)=\Gamma_M+\frac{\Gamma_M f K_4}{2}\left[\tfrac{1}{2}\ln(\Lambda\xi)^2-\frac{1}{2(1+a)}\ln(-i\nu/2)+\frac{a}{4(1+a)}\ln(a/R^2)+\tfrac{1}{4}\ln a+\frac{1}{4(1+a)}\ln\left(\frac{(1+a)^2}{4a}\right)\right]. \tag{6.6}$$

The second case of interest is fixed q and $\omega\to 0$. Then, for small q,

$$\overline{\Gamma}^L_M(q,0)=\Gamma_M(\Lambda\xi)^{\epsilon/2}(1-\tfrac{3}{32}\epsilon\ln x^2+\epsilon K), \tag{6.7}$$

where

$$K=\tfrac{3}{16}(-\tfrac{5}{6}\ln 11-\tfrac{1}{2}\ln 3-\tfrac{1}{4}+3\ln 2) =-0.135 \tag{6.8}$$

and we have exponentiated a $\ln(\Lambda\xi)$ term in a manner compatible with scaling. The key point here is that for all temperatures below T_N there are logarithms that blow up as q and ω both go to zero. These divergent processes are representative of a whole class of processes that occur below T_N and for which the ϵ expansion and renormalization group *cannot* give you the physics without supplemental information. The difficulty can be summarized as follows. The renormalization group can tell us that $\overline{\Gamma}_M(q,0)\chi^{-1}_M$ is of the form

$$\overline{\Gamma}_M(q,0)\chi_M^{-1} = q^{z-2}\Omega_M^L(x) \quad (6.9)$$

and in this case $z=\frac{1}{2}d$. What the renormalization group cannot tell us is the functional dependence of Ω_M^L on x. In our present case we can rewrite our result for $\overline{\Gamma}_M^L(q,0)$ as

$$\overline{\Gamma}_M(q,0)\chi_M^{-1} = (\Lambda\xi)^{\epsilon/2}\overline{\Omega}_M^L(x) \ , \quad (6.10)$$

where

$$\overline{\Omega}_M^L(x) = \left(1+\epsilon K - \frac{3\epsilon}{32}\ln x^2\right) . \quad (6.11)$$

We do not know how to treat the $\ln x$ term—should it be exponentiated and if so how? There are several possible ways of proceeding at this point. One can follow the Brézin *et al.*[25] treatment of the longitudinal order parameter correlation function and go to $O(\epsilon^2)$ and assume there is a single power-law divergence. One can then unambiguously exponentiate the logarithm terms. One can also carry out a $1/n$ expansion calculation where one does not expand in ϵ. This gives one the correct qualitative behavior in the case of χ_N^L and we will discuss this approach in the next section when we treat the longitudinal fluctuations of the order-parameter. In our situation here it is relatively clear that a second order in ϵ calculation would be extremely difficult and we do not want to tackle the problem of solving the infinite component generalization of our model.[36] We do have one technique available for treating these new singularities. This technique is a sort of generalized mode-coupling approach. This method involves three basic assumptions: (i) We evaluate the interals in the memory function expression Eq. (6,1) in d dimensions (as opposed to evaluating integrals in four dimensions). (ii) We assume that we can replace the correlation functions in the integrand with their hydrodynamical forms. Since we are focussing on divergences appearing for small q and ω this assumption should give the correct small q and ω dependence. (iii) We use ϵ-expansion techniques to evaluate the various ratios appearing in the calculation. A similar approach has been taken by Hohenberg, *et al.*[12] in treating the λ transition in helium. We will discuss the reliability of these approximations as we proceed. These assumptions, together with Eq. (4.27), allow us to write:

$$\Gamma_{M,H}^L(q,\omega) = \frac{\lambda^2}{2}\int\frac{d^dk}{(2\pi)^d}\int_0^{+\infty}dt\, e^{+i\omega t}\{[\chi_N^T(k-q)]^{-1}-[\chi_N^T(k)]^{-1}\}^2 \times C_{N,H}^T(k,t)C_{N,H}^T(q-k,t) \ , \quad (6.12)$$

where we assume the transverse order-parameter correlation functions have the hydrodynamical form

$$C_{N,H}^T(q,t) = \chi_N^T(q)\cos(c_s qt)e^{-D_s q^2 t/2} \quad (6.13)$$

and c_s and D_s are the spin-wave velocity and damping. We can use the expressions for c_s and D_s determined in the last section to $O(\epsilon)$. The hydrodynamical expression for C_N^T gives the frequency spectrum correctly for small q and $\omega\approx c_s q$. For finite q and for frequencies away from $\omega=c_s q$ there will be errors. We find, for example, if we replace c_s and D_s with their zeroth order in ϵ expressions and Fourier transform over time we obtain

$$C_{N,H}^T(q,\omega) = C_{NN}^{T,0}(q,\omega) + \frac{\chi_N^T(q)}{\omega_c(q)}\frac{x\overline{D}[\nu^2-1+\frac{1}{4}(\overline{D}x)^2]}{D_0(x,\nu)} \ , \quad (6.14)$$

where $\overline{D}=d_s/b_s$, $D_0(x,\nu)$ is given by Eq. (3.26) and $C_{NN}^{T,0}(q,\omega)$ is the $O(1)$ expression given by Eq. (3.20). We see that for small x and $\nu\to 1$ the difference between the two vanishes. In general

$$\frac{C_{N,H}^T(q,\omega)}{C_{NN}^{T,0}(q,\omega)} = \frac{1+\nu^2+(\overline{D}x)^2}{\frac{1}{2}+\frac{3}{2}\nu^2+\frac{3}{22}x^2} \quad (6.15)$$

and

$$\frac{C_{N,H}^T(0,0)}{C_{NN}^{T,0}(0,0)} = 2 \ . \quad (6.16)$$

Thus we see that there will be numerical differences between the use of $C_{N,N}^{T,0}$ and $C_{N,H}^T$ when we do integrals over q and ω. For now let us concentrate on $\tilde{\Gamma}_{M,H}^L$. After inserting $C_{N,H}^T$, doing the time integral and making some rearrangements (including letting $\vec{k}\to\vec{k}+\frac{1}{2}\vec{q}$) we obtain

$$\Gamma_{M,H}^L(q,\omega) = 2\lambda^2\rho_s^2\int\frac{d^dk}{(2\pi)^d}\frac{(\vec{k}\cdot\vec{q})^2}{Q^4-(\vec{k}\cdot\vec{q})^2}(D_sQ^2-i\omega) \times\frac{(D_sQ^2-i\omega)^2+2c_s^2Q^2}{[(D_sQ^2-i\omega)^2+2c_s^2Q^2]^2-4c_s^4[Q^4-(\vec{k}\cdot\vec{q})^2]} , \quad (6.17)$$

where we have used $\chi_N^T(q)=\rho_s/q^2$ and $Q^2=k^2+\frac{1}{4}q^2$. Consider first the limit

$$\overline{\Gamma}_{M,H}^L(0,\omega) = \lim_{q\to 0}\frac{1}{q^2}\tilde{\Gamma}_{M,H}^L(q,\omega) \ .$$

It is a straightforward calculation to show for small ω and $d=4$ that

$$\overline{\Gamma}_{M,H}^L(0,\omega) = \Gamma_M\left[1+\frac{fK_4}{8d_s}\ln\left(\frac{d_s^3(\Lambda\xi)^4}{4b_s^2(-i\nu)}\right)\right] \ . \quad (6.18)$$

This result serves as a check on the accuracy of the use of the hydrodynamical forms for the correlation function C_N^T in $\Gamma_{M,H}^L$. We can compare the above result with the corresponding ϵ-expansion result where the correct, to $O(\epsilon)$, C_N^T were used. From Eq. (6.6) we easily find, keeping only logarithmic terms, that

$$\bar{\Gamma}_M^L(0,\omega)=\Gamma_M\left[1+fK_4\left(\frac{1}{4}\ln(\Lambda\xi)^2-\frac{1}{4(1+a)}\ln(-i\nu/2)\right)\right] \tag{6.19}$$

which we are to compare with

$$\bar{\Gamma}_{M,H}^L(0,\omega)=\Gamma_M\left\{1+fK_4\left[\frac{1}{4d_s}\ln(\Lambda\xi)^2-\frac{1}{8d_s}\ln\left(-\frac{i\nu}{2}\right)+O(1)\right]\right\}. \tag{6.20}$$

Inserting $d_s=\frac{4}{3}+O(\epsilon)$, $a=\frac{1}{3}+O(\epsilon)$, $\rho_s=1+O(\epsilon)$, we find that the coefficient of $\ln(\Lambda\xi)^2$ in $\bar{\Gamma}_{M,H}^L$ is $\frac{3}{16}$ compared to $\frac{1}{4}$ in the ϵ expansion and the coefficient of $\ln(-i\nu/2)$ for $\bar{\Gamma}_{M,H}^L$ is $\frac{3}{32}$ compared to $\frac{3}{16}$ in the ϵ-expansion. Note that differences of 50% are occurring.

Our main reason for looking at $\Gamma_{M,H}^L$ is that we can work in dimensions less than four. We obtain in that case,

$$\bar{\Gamma}_{M,H}^L(0,\omega)=\Gamma_M\rho_s^2\frac{fK_d\pi}{dd_s}(\Lambda\xi)^{\epsilon/2}\times\left[\delta_1\left(\frac{|\nu|}{d_s}\right)^{-\epsilon/2}+\delta_2\left(\frac{2b_s}{d_s}\right)^{-\epsilon}\right], \tag{6.21a}$$

where

$$\delta_1=[\sin\tfrac{1}{4}(\pi d)]^{-1}+i[\sin\pi\tfrac{1}{4}(d-2)]^{-1} \tag{6.21b}$$

$$\delta_2=[\sin(d-2)\tfrac{1}{4}\pi]^{-1}$$

$$\nu=\omega/\omega_c \tag{6.21c}$$

and

$$\omega_c=\Gamma_N\xi^{-2}(\Lambda\xi)^{\epsilon/2}.$$

The most interesting feature of this result is that $\bar{\Gamma}_{M,H}^L(\omega)$ diverges for small ω as $\omega^{-\epsilon/2}$. Remembering that the real part of $\bar{\Gamma}_M^L(\omega)$ is related to the spin-current autocorrelation function $J(t)$ via Eq. (6.3) we obtain

$$J(t)=\mathrm{Re}\chi_M^{-1}\int\frac{d\omega}{2\pi}e^{-i\omega t}\bar{\Gamma}_{M,H}^L(\omega)=\chi_M^{-1}\Gamma_M\frac{B_0}{\tau_0}(\Lambda\xi)^{1-3\epsilon/2+\epsilon^2/4}\left(\frac{\tau_0}{t}\right)^{1-\epsilon/2} \tag{6.22a}$$

where

$$\tau_0=(\Gamma_N\Lambda^2)^{-1} \tag{6.22b}$$

and

$$B_0=\rho_s^2\frac{fK_d}{dd_s}(d_s)^{\epsilon/2}\Gamma(1-\epsilon/2)\times\left(\frac{\cos\epsilon\pi/4}{\sin[(d-2)\pi/4]}+\frac{\sin\epsilon\pi/4}{\sin\pi d/4}\right). \tag{6.22c}$$

In three dimensions we see that $J(t)\sim t^{-1/2}$ and, using lowest order in ϵ expressions for fK_d, d_s, and ρ_s, we find that

$$B_0=\left(\frac{\pi}{3}\right)^{1/2}. \tag{6.23}$$

We see that the coupling of the longitudinal modes into the long-lived transverse modes leads to a long time tail. If hydrodynamics were valid we could use the Green-Kubo relation[14] between the time integral of the current correlation function and the spin diffusion coefficient. The breakdown of hydrodynamics is indicated by the nonintegrability of $J(t)$.

We next turn to the case of q fixed and $\omega\to 0$. The integrals in this case are more difficult so we limit the analysis to $d=3$. We obtain in this case, for small q, the result

$$\Gamma_{M,H}^L(q,0)=\Gamma_M(\Lambda\xi)^{1/2}g_0(x^{-1/2}+g_1), \tag{6.24a}$$

where

$$g_0=\frac{f}{20\pi(b_sd_s/2)^{1/2}}, \tag{6.24b}$$

$$g_1=\frac{5}{6}\left(\frac{d_s}{2b_s}\right)^{1/2}. \tag{6.24c}$$

We can evaluate g_0 to lowest order in ϵ

$$g_0=\tfrac{2}{5}\pi(\tfrac{3}{2})^{1/2}(\tfrac{3}{11})^{1/4}+O(\epsilon)=1.112+O(\epsilon), \tag{6.24d}$$

while

$$g_1=\tfrac{5}{6}(\tfrac{2}{3})^{1/2}(\tfrac{3}{11})^{1/4}+O(\epsilon)=0.492+O(\epsilon). \tag{6.24e}$$

The first order in ϵ corrections for g_0 and g_1 are $g_0=0.2638+O(\epsilon^2)$ and $g_1=0.500+O(\epsilon^2)$. The correction for g_0 is substantial. We see, as expected, that the physical transport coefficient diverges as $q\to 0$ and with the power law $q^{-1/2}$ in 3 dimensions. We can use this result to interpret our previous ϵ-expansion result for $\bar{\Gamma}_M^L(q,0)$ given by Eq. (6.7). If we assume that $\bar{\Gamma}_M^L(q,0)$ is of the form

$$\bar{\Gamma}_M^L(q,0)=\Gamma_M(\Lambda\xi)^{\epsilon/2}g_0(x^{-\epsilon/2}+g_1), \tag{6.25}$$

then in an ϵ expansion we have

$$\bar{\Gamma}_M^L(q,0)=\Gamma_M(\Lambda\xi)^{\epsilon/2}g_0\left[1+g_1-\frac{\epsilon}{2}\ln x+O(\epsilon^2)\right]. \tag{6.26}$$

We see from Eq. (6.7) that to $O(\epsilon)$

$$g_0(1+g_1)=1+\epsilon K, \tag{6.27a}$$

$$g_0=\tfrac{3}{8}+O(\epsilon). \tag{6.27b}$$

Since we only know g_0 to zero order in ϵ we also have

$$g_1=\tfrac{5}{3}+O(\epsilon). \tag{6.28}$$

We know $g_0(1+g_1)$ to $O(\epsilon)$. In Table II we compare the mode-coupling expressions for g_0, g_1 and g_0(1

TABLE II. Comparison of parameters calculated in the ϵ expansion and by mode coupling.

	Direct ϵ expansion	Mode coupling	
g_0	$0.375+O(\epsilon)$	$1.112+O(\epsilon)$	$0.2638+O(\epsilon^2)$
g_1	$1.667+O(\epsilon)$	$0.492+O(\epsilon)$	$0.500\ +O(\epsilon^2)$
$g_0(1+g_1)$	$0.865+O(\epsilon)$	$1.659+O(\epsilon)$	$0.396\ +O(\epsilon^2)$
x_c	$0.36\ +O(\epsilon)$	$4.131+O(\epsilon)$	$4.00\ \ +O(\epsilon^2)$

$+g_1$) with the direct ϵ-expansion results. The value of g_1 is particularly important since it determines the regime in x where the $x^{-1/2}$ term predominates. We define a cross-over value of x

$$x_c=g_1^{-2}. \tag{6.29}$$

We see from the table that there is a huge difference between the x_c from the ϵ-expansion and from mode coupling, and that in general the agreement between the parameters calculated is poor.

There is another limit in which we can evaluate $\overline{\Gamma}^L_{M,H}(q,\omega)$. This is in the limit $\omega=c_s q\nu$ where $q\to 0$ and ν is fixed. We find in this case that

$$\Gamma^L_{M,H}(q,\omega)=\Gamma_M(\Lambda\xi)^{\epsilon/2}\frac{g_0}{\sqrt{x}}I(\nu), \tag{6.30a}$$

where

$$I(\nu)=[(8\nu^2+3)(\sigma-\sigma')+4\nu(\sigma+\sigma')]\left[-\frac{(1+i)}{3\sqrt{2}}\right], \tag{6.30b}$$

$$\sigma=[\tfrac{1}{2}(\nu-1)]^{1/2},\quad \sigma'=[\tfrac{1}{2}(\nu+1)]^{1/2}. \tag{6.30c}$$

The quantity that enters conveniently into the correlation function is

$$\frac{q^2\overline{\Gamma}^L_M(q,\omega)\chi_M^{-1}}{\omega_c}=\sqrt{x}\ A_0 I(\nu), \tag{6.31}$$

where

$$A_0=\frac{a g_0}{b_s}=\frac{af}{20\pi b_s(b_s d_s/2)^{1/2}}. \tag{6.32}$$

To lowest order in ϵ

$$A_0=\frac{\pi}{5}\left(\frac{6}{11}\right)^{1/2}\frac{1}{(33)^{1/4}}=0.194+O(\epsilon). \tag{6.33}$$

Again, due to the large shift in f, $O(\epsilon)$ corrections lead to a substantial change in A_0 in three dimensions,

$$A_0=0.0833+O(\epsilon^2).$$

The correlation function is then given by

$$C^L_M(q,\omega)=\frac{\chi_M}{\omega_c}f^L_M(x,\nu), \tag{6.34}$$

where $\omega_c=c_s q$ and the shape function is given by

$$f^L_M(x,\nu)=\frac{2A_0\sqrt{x}\,I'}{(\nu-A_0\sqrt{x}I'')^2+(A_0\sqrt{x}I')^2} \tag{6.35a}$$

where

$$I'(\nu)=\begin{cases}\dfrac{(8\nu^2+3)(\sigma'-\bar\sigma)-4\nu(\sigma+\sigma')}{3\sqrt{2}}, & \nu>1 \quad (6.35b)\\[2ex] \dfrac{(\sigma'+\sigma)(8\nu^2+3-4\nu)}{3\sqrt{2}}, & \nu<1 \quad (6.35c)\end{cases}$$

$$I''(\nu)=\begin{cases}\dfrac{(8\nu^2+3)(\sigma'-\sigma)-4\nu(\sigma+\sigma')}{3\sqrt{2}}, & \nu>1 \quad (6.35d)\\[2ex] \dfrac{(\sigma'-\bar\sigma)(8\nu^2+3-4\nu)}{3\sqrt{2}}, & \nu<1 \quad (6.35e)\end{cases}$$

and

$$\bar\sigma=\left(\frac{1-\nu}{2}\right)^{1/2}. \tag{6.35f}$$

We can then work out the limiting case

$$f^L_M(x,0)=\frac{2}{A_0\sqrt{x}}. \tag{6.36}$$

In Fig. 13, we show numerical results for $f^L_M(x,\nu)$ as a function of x. An expanded scale of this graph would show there is a definite cusp at $\nu=1$ for $x\sim 1$ reflecting the beating of two spin-wave modes. This cusp may be an artifact of our approximation and, in any event, appears unobservable. As we go to small values of x the cusp is suppressed and a central Lorentzian peak sharpens up. This cross-over occurs for $x<x_c$ (remembering how-

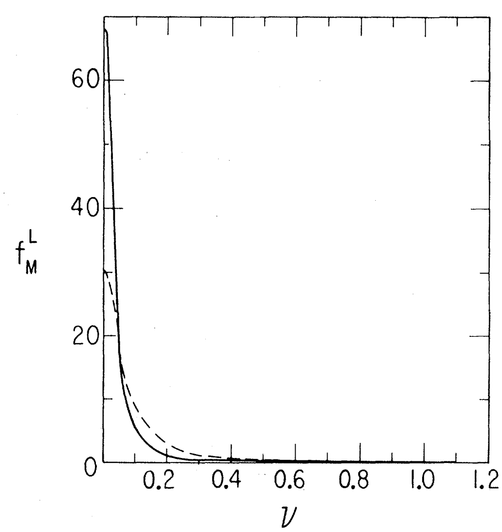

FIG. 13. Longitudinal magnetization correlation function $f^L_M(x,\nu)$ is graphed as a function of ν for $x=0.1$ (solid line) and $x=0.5$ (dashed line).

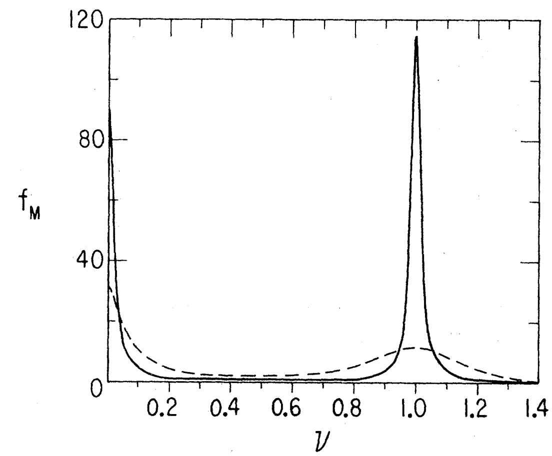

FIG. 14. Total magnetization correlation function $f_M(x,\nu)$ is graphed as a function of ν for $x=0.05$ (solid line) and $x=0.5$ (dashed line).

ever that our equation is valid only for small x). For very small x all of the weight in $f_M^L(x,\nu)$ will be near $\nu=0$ so we can replace I' and I'' with their $\nu=0$ values and

$$f_M^L(x,\nu)=\frac{2A_0\sqrt{x}}{\nu^2+(A_0\sqrt{x})^2}, \tag{6.37}$$

for $x\ll 1$. In the final asymptotic hydrodynamic region ($x\ll 1$) the total magnetization function is of the form

$$C_M(q,\omega)=\frac{\chi_M}{\omega_c}f_M(x,\nu), \tag{6.38}$$

where the shape function is

$$f_M(x,\nu)=2f_M^T(x,\nu)+f_M^L(x,\nu). \tag{6.39}$$

We have plotted $f_M(x,\nu)$ in Fig. 14 for various values of x. We have used Eq. (6.37) for f_M^L and the zeroth in ϵ result for f_M^T. The main point is that for small x the central and spin-wave peaks are sharp and well separated.

VII. LONGITUDINAL-ORDER-PARAMETER CORRELATION FUNCTION

Study of the longitudinal-order-parameter correlation function leads to a number of difficulties in the hydrodynamical regime. This is because of the appearance of a new class of diagrams in our analysis. On the one hand we could ignore these difficulties since the longitudinal correlation function is probably not observable in the hydrodynamical region due to strong transverse fluctuations which will mask the longitudinal mode.[10] On the other hand these new graphs indicate (i) new difficulties in interpreting the ϵ expansion, and (ii) a complete breakdown in the mode-coupling approach. Since our approach has been based on the ϵ-expansion and we used mode-coupling ideas in the last section we will investigate some of the reasons for (i) and (ii) above. The difficulties are associated with the appearance of strong q and ω divergences in $\overline{\Gamma}_N^L$ for small q and ω to $O(\epsilon)$. We can see the difficulty by looking back to our general expression for Γ given by Eq. (4.31). We have, after doing various spin sums,

$$\Gamma_{NN}^L(q,\omega)=\sum_{i=1}^{4}\Gamma_{N,i}^L(q,\omega), \tag{7.1}$$

where

$$\Gamma_{N,1}^L(q,\omega)=2\lambda^2r^2\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{C}_N^+(k)\tilde{C}_M^+(q-k)A_{NN}^{++}}{D_T(k,q-k)}\{A_{MM}^{++}A_{MN}^{++}-\lambda^2N^2r[\tilde{\chi}_N^{-1}(k)-\tilde{\chi}_N^{-1}(q-k)]\}, \tag{7.2}$$

$$\Gamma_{N,2}^L(q,\omega)=8\lambda^2riuN^2\Gamma_N\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{C}_N^+(k)}{D_T(k,q-k)}[-A_{MM}^{++}A_{MN}^{++}+B(q-k)-B(k)] \tag{7.3}$$

$$\Gamma_{N,3}^L(q,\omega)=-4(uN)^2\Gamma_N^2\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{C}_N^+(k)\tilde{C}_N^+(q-k)}{D_T(k,q-k)}A_{MN}^{++}[A_{NM}^{++}A_{MM}^{++}-2B(k)] \tag{7.4}$$

$$\Gamma_{N,4}^L(q,\omega)=-18(uN)^2\Gamma_N^2\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{C}_N^0(k)\tilde{C}_N^0(q-k)}{A_{NN}^{00}} \tag{7.5}$$

where

$$D_T(k,q-k)=A_{MN}^{++}A_{NN}^{++}A_{MM}^{++}A_{NM}^{++}-B(k)(A_{MN}^{++}A_{NN}^{++}+A_{MM}^{++}A_{NM}^{++})-B(q-k)(A_{MN}^{++}A_{MM}^{++}+A_{NN}^{++}A_{NM}^{++})+[B(k)-B(q-k)]^2. \tag{7.6}$$

Things are particularly simple in the $q=0$ limit where, introducing the dimensionless frequency $\nu=\omega\xi^2/\Gamma_N(uN^2=\xi^{-2})$, we obtain

$$\Gamma_{N,1}^L(0,\omega)=faK_4\Gamma_N\int_0^{(\Lambda\xi)^2}\frac{dy(\nu+2iay)(\nu+2iy)}{[\nu+i(1+a)y][(\nu+2iay)(\nu+2iy)-4Ry]}, \tag{7.7}$$

$$\Gamma^L_{N,2}(0,\omega)=-4ifa\Gamma_N K_4\int_0^{(\Lambda\xi)^2}\frac{dy\,(\nu+2iay)}{[\nu+i(a+1)y][(\nu+2iay)(\nu+2iy)-4Ry]}, \quad (7.8)$$

$$\Gamma^L_{N,3}(0,\omega)=-2u\Gamma_N K_4\int_0^{(\Lambda\xi)^2}\frac{dy[(\nu+i(1+a)y)(\nu+2iay)-2Ry]}{y\,[\nu+i(1+a)y]\{[\nu+i(1+a)y](\nu+2iay)-4Ry\}}, \quad (7.9)$$

$$\Gamma^L_{N,4}(0,\omega)=-9\Gamma_N uK_4\int_0^{(\Lambda\xi)^2}\frac{dy\,y}{(y+2)^2[\nu+2i(y+2)]}. \quad (7.10)$$

Then for small ν, we have

$$\Gamma^L_{N,1}(0,0)=-i\frac{faK_4\Gamma_N}{1+a}\ln[(\Lambda\xi)^2u/f] \quad (7.11)$$

$$\Gamma^L_{N,4}(0,0)=i\tfrac{9}{8}\Gamma_N uK_4 \quad (7.12)$$

and these terms give us no problem. We note however that for small ν

$$\Gamma^L_{N,2}(0,\omega)=-4ifa\Gamma_N K_4\left(-\frac{(1+3a)}{4(1+a)}R\ln\nu+\text{const}\right). \quad (7.13)$$

Therefore, $\Gamma^L_{N,2}(0,\omega)$ blows up logarithmically and appears similar in structure to $\Gamma^L_M(0,\omega)$. $\Gamma^L_{N,3}(0,\omega)$ is more complicated. Note that the $\Gamma^L_{N,3}$ term, which comes from a self-energy-like diagram with two four-point vertices and two "condensate" insertions is infinite due to the bare $1/y$ term in the integrand in Eq. (7.9). We cannot set $q=0$ in $\Gamma^L_{N,3}$. If we set $\omega=0$ and evaluate $\Gamma^L_{N,3}(q,0)$ for small $x=q\xi$ we obtain

$$\Gamma^L_{N,3}(q,0)=-\frac{2i\Gamma_N uK_4}{\sqrt{R}x}(\ln x+\text{const}). \quad (7.14)$$

Thus, this term diverges as $(\ln x)/x$ for small x and its interpretation within an ϵ expansion is ambiguous without further information about the physics in this situation. It is clear from the above analysis that the trouble comes from TDGL-type contributions rather than from mode coupling terms so, in order to see what is going on, we will, for the moment, restrict our analysis to a TDGL model with a nonconserved order parameter. This corresponds to simply setting $\lambda=0$ in our model. This has the effect of eliminating the transverse spin-wave modes. In this case, the contribution to $\Gamma^L_N(q,\omega)$ corresponding to $\Gamma^L_{N,3}$ above is just

$$\Gamma^L_{N,T}(q,\omega)=-4(uN)^2\Gamma_N^2\int\frac{d^dk}{(2\pi)^d}\tilde{C}^+_N(k)\tilde{C}^+_N(q-k)(-i)\int_0^{+\infty}dt\,e^{+i\omega t}e^{-\Gamma_N t[k^2+(k-q)^2]}, \quad (7.15a)$$

$$=-4(uN\Gamma_N)^2\int\frac{d^dk}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k-q)^2}\frac{1}{\omega+i\Gamma_N[k^2+(k-q)^2]}, \quad (7.15b)$$

where we have written the integral in d dimensions. If we restrict ourselves to $\omega=0$ we find in four dimensions,

$$\Gamma^L_{N,T}(q,0)=i\Gamma_N(4\ln2)uK_4/x^2, \quad (7.16)$$

while in three dimensions,

$$\Gamma^L_{N,T}(q,0)=i\Gamma_N\frac{(uN)^2}{q^3}(1-4/\pi^2). \quad (7.17)$$

Note that the divervence for $d=4$ for the TDGL model is stronger (x^{-2} compared to $x^{-1}\ln x$) than for the model with spin waves. If one stops here one would conclude that for TDGL models the physical kinetic coefficient for the longitudinal-order-parameter correlation function diverges as $q^{-(6-d)}$ for small q. For reasons, which we now discuss, we believe this conclusion to be false and indicative of the problems of using the ϵ expansion in treating $\tilde{\Gamma}^L_N$. We can see this by generalizing our TDGL model to n-components[27] and calculating $\tilde{\Gamma}^L_N$ to lowest order in a $1/n$ expansion. This calculation and the general results for the correlation function are discussed in the Appendix. The result for $\tilde{\Gamma}^N_L$ is

$$\tilde{\Gamma}^L_{N,n}(q,\omega)=\Gamma_N\Big(1-i\frac{(n-1)}{2}\frac{\tilde{\Gamma}^L_{N,T}(q,\omega)}{\Gamma_N}\times[1+u(n-1)\Pi(q,0)]^{-1}\times[1+u(n-1)\Pi(q,\omega)]^{-1}\Big)^{-1} \quad (7.18)$$

where $\Pi(q,\omega)$ is defined by Eq. (A2) in the Appendix. $\tilde{\Gamma}^L_{N,T}(q,\omega)$ is the same as in Eq. (7.15a) above. We see that for $n=3$ and to lowest order in $u\sim\epsilon$ we regain the $O(\epsilon)$ results discussed above,

$$\tilde{\Gamma}^L_{N,T}(q,\omega)=\Gamma_N+i\Gamma^L_{N,T}(q,\omega)+O(\epsilon^2). \tag{7.19}$$

In the $1/n$ expansion one sums up a string of bubbles like that contributing to $\Gamma^L_{N,3}$. The important point here is that for $un/q \gg 1$ and $\omega=0$ we obtain for $d=3$ the result

$$\tilde{\Gamma}^L_{N,T}(q,0)=\frac{\Gamma_N(\pi x)}{(2+\pi x)}, \tag{7.20}$$

where $x=nq/16N^2\equiv q\xi$. We see that as $x\to 0$ $\tilde{\Gamma}^L_{N,T}(q,0)$ not only does not diverge, but goes to zero. We conclude from this that it is essential to sum up the string of bubbles with transverse propagators if one is to obtain sensible results. This is, of course, equally true for the longitudinal static susceptibility where one must sum the static bubbles in order to obtain the $q^{-\epsilon}$ result for small q. We have not yet attempted to carry out a calculation for λ finite where we sum the bubbles. This would be a very tedious calculation and we expect that the result will be somewhat similar to that for the TDGL model except that $\tilde{\Gamma}^L_N$ will be proportional to $(\Lambda\xi)^{\epsilon/2}$ rather than $(\Lambda\xi)^{-\eta}$ as in the TDGL model. The analysis of the TDGL model in the large n limit shows that the shape function deviates from a Lorentzian in the hydrodynamic limit, but does not show any pronounced structure.

VIII. DISCUSSION

We have found that the critical dynamics of antiferromagnets in the ordered phase are rich in features and effects not found above T_N. The transverse correlation functions are dominated by spin waves and the associated pole structure is in agreement with the predictions of hydrodynamics in the appropriate limit. As $x=q\xi$ increases and one leaves the hydrodynamical region we find that the positions of the spin-wave peaks for the transverse magnetization and staggered-magnetization correlation functions differ significantly. The main feature is that the spin-wave peaks in the staggered magnetization are persisting to much larger values of x than for the magnetization. This is in agreement with our previous calculation as $T=T_N$ where we found highly damped spin wave modes in the staggered magnetization correlation function. The most interesting result from our treatment of the longitudinal magnetization correlation function is that hydrodynamics breaks down in the hydrodynamical region. This is manifested in the wavenumber dependent spin-diffusion coefficient going as $q^{-\epsilon/2}$ for small q. The longitudinal-order-parameter correlation function is very difficult to treat within an ϵ expansion. Our best "guess" is that if we performed a calculation resumming certain bubble diagrams, as discussed in the last section, we would find that the mode is relaxational in nature.

The most striking qualitative feature below T_N, besides the spin waves, is the coupling of the transverse modes into the longitudinal modes. The first theoretical manifestation of these strong couplings was the prediction that the static longitudinal-order-parameter correlation function will diverge as $q^{-\epsilon}$ for small q. Unfortunately, this divergence has not yet been directly observed.[37] It is difficult to measure $\chi^L_N(q)$ directly because one normally measures the total correlation function $=2\chi^T_N(q)+\chi^L_N(q)$ and since $\chi^T_N(q)$ diverges more strongly than $\chi^L_N(q)$ for small q it is very difficult to extract $\chi^L_N(q)$ for small q. It is for essentially the same reason that one can probably not measure the dynamical longitudinal-order-parameter correlation function and we therefore chose not to carry out a more elaborate calculation of C^L_N. From an experimental point of view the most promising way of seeing the coupling of the Nambu-Goldstone modes into the longitudinal modes is via an inelastic neutron-scattering measurement of the magnetization correlation functions. In this case, the longitudinal and transverse correlation functions will come in with equal weighting ($\chi^M_T=\chi^M_L=$ constant as $q\to 0$) and one can hope to resolve the longitudinal correlation function. Thus one could hope to see the predicted strong coupling of transverse and longitudinal modes. Indeed after our work on the longitudinal magnetization was completed on inelastic neutron-scattering experiment was undertaken at Brookhaven[38] to measure C^L_M. Preliminary results are consistent with our predictions that hydrodynamics does not hold and the width goes as $q^{3/2}$.

Another possible "experiment" is a computer molecular dynamics experiment in the ordered phase and a direct calculation of the spin-current autocorrelation function. This should show the long time $t^{-1+\epsilon/2}$ behavior directly.

While we have been able to draw a number of conclusions from our calculations which seem believable it should also be clear that we have been neither exhaustive nor, in some cases, are our calculations completely reliable. There are several further calculations one could carry out within the $O(\epsilon)$ approximations discussed here. In particular we have not investigated the shape functions for intermediate values of $x=q\xi$ except to zeroth order in ϵ. Thus we have not addressed the interesting question of the transition from spin-waves for $x\ll 1$ to the fluctuations induced peaks at $T=T_N$. A more pressing question however surrounds the ϵ expansion. The expansion converges very slowly with respect to the matching values of a and f. An even more difficult problem arises due to the $q^{-\epsilon}$ type singularities. The interpretation of the ϵ expansion in these cases requires considerable care. It would be most useful to develop some method complementary to the ϵ expansion for calculating the critical dynamics of isotropic antiferromagnets. A real-space renormalization-group approach for example would be most welcome.

ACKNOWLEDGMENT

We would like to thank Professor Paul Horn for many useful and illuminating discussions.

APPENDIX: CALCULATION OF $C_L(q,\omega)$ IN THE $1/n$ EXPANSION

Starting with the TDGL n-vector model, the $O(1)$ contribution to the longitudinal propagator is given by[27]

$$G_L^{-1}(q,\omega) = -\frac{i\omega}{\Gamma_N} + q^2 + \frac{2uN^2}{1+u(n-1)\Pi(q,\omega)}, \quad (A1)$$

and

$$\Pi(q,\omega) = i\int \frac{d^dk}{(2\pi)^d}\,\frac{\chi_N^T(k-q)\chi_N^T(k)[L_N(k)+L_N(k-q)]}{\omega + i[L_N(k)+L_N(k-q)]} \quad (A2)$$

with

$$L_N(q) = \Gamma_N q^2.$$

We want to use G_L^{-1} to calculate the memory function $\tilde{\Gamma}_N^L(q,\omega)$ which is defined by

$$G_L^{-1}(q,\omega) = \frac{-i\omega}{\tilde{\Gamma}_N^L(q,\omega)} + \chi_L^{-1}(q) \quad (A3)$$

where the full longitudinal static susceptibility is given by

$$\chi_L^{-1}(q) = q^2 + 2uN^2[1+u(n-1)\Pi(q,0)]^{-1}, \quad (A4)$$

and where

$$\Pi(q,0) = \int \frac{d^dk}{(2\pi)^d}\,\chi_N^T(k-q)\chi_N^T(k). \quad (A5)$$

We find immediately that $\tilde{\Gamma}_N^L$ is given to $O(1/n)$ by

$$\tilde{\Gamma}_N^L(q,\omega) = \Gamma_N\left[1 + \frac{2\Gamma uN^2}{-i\omega}\left(\frac{1}{1+u(n-1)\Pi(q,\omega)} - \frac{1}{1+u(n-1)\Pi(q,0)}\right)\right]^{-1} \quad (A6)$$

which leads directly to Eq. (7.18) above. If we restrict the analysis to three dimensions we find

$$\Pi(q,\omega) = \frac{i}{4\pi q}\ln\left(\frac{1+i(1-2i\nu)^{1/2}}{-1+i(1-2i\nu)^{1/2}}\right) \equiv \frac{i}{4\pi q}h(\nu) \quad (A7a)$$

$$\nu = \omega/q^2\Gamma_N \quad (A7b)$$

$$\Pi(q,0) = \frac{1}{8q}. \quad (A7c)$$

We then obtain for $u(n-1)/q \gg 1$

$$\tilde{\Gamma}_N^L(q,\omega) = \Gamma_N\left[1 + (-i\nu x)^{-1}\left(\frac{\pi}{2ih(\nu)} - 1\right)\right]^{-1} \quad (A8)$$

$$[\chi_N^L(q)]^{-1} = q^2(1+1/x), \quad (A9)$$

where $x = nq/16N^2$. Since $h(\nu) = -i\pi/2 + \nu + O(\nu^2)$ we easily obtain in the small-ν limit

$$\tilde{\Gamma}_N^L(q,0) = \frac{\Gamma_N \pi x}{2+\pi x} \quad (A10)$$

while in the small-x limit

$$\tilde{\Gamma}_N^L(q,\omega) = \Gamma_N(-ix\nu)\left(\frac{\pi}{2ih(\nu)} - 1\right)^{-1}. \quad (A11)$$

The correlation function can then be written in the scaling form

$$C_N^L(q,\omega) = \frac{\chi_N^L(q)}{\omega_c(q)} f_N^L(x,\nu), \quad (A12)$$

where $\omega_c(q) = \Gamma_N q^2$ and we have the two limits of interest

$$f_N^L(\infty,\nu) = 2(1+\nu^2)^{-1} \quad (A13)$$

and

$$f_N^L(0,\nu) = \frac{4}{\pi\nu}\mathrm{Re}\,h(\nu). \quad (A14)$$

A plot of $f_N^L(0,\nu)$ looks quite Lorentzian in shape.

---

*This work was supported in part by the National Science Foundation (Grant No. NSF DMR 76-21298), The Materials Research Research Laboratory of the NSF, and the Louis Block Fund of the University of Chicago.
†Supported by an IBM predoctoral fellowship.
‡Present address: Shell Oil Company, P. O. Box 481, Houston, Texas 77001.

[1]R. Freedman and G. F. Mazenko, Phys. Rev. Lett. 34, 1575 (1975).
[2]R. Freedman and G. F. Mazenko, Phys. Rev. B 13, 4967 (1976).
[3]P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. (to be published). This is a comprehensive and up-to-date review of the theory of dynamical critical phenomena.
[4]F. Wegner, Z. Phys. 218, 265 (1968); and 216, 433 (1968).
[5]A. Tucciarone, H. Y. Lau, L. M. Corliss, A. Delapalme, and J. W. Hastings, Phys. Rev. B 4, 3206 (1971).

[6]H.-K. Janssen (unpublished) has studied the n-component model of L. Sasvári and P. Szépfalusy, IUPAP International Conference on Statistical Physics, Budapest, 1975 (unpublished).

[7]L. Sasvári and P. Szépfalusy, see Ref. 6. The n-component model introduced by these authors reduces to our model for $n=3$.

[8]The exponentiation procedure discussed in Ref. 2 is a good example. If the shape function is assumed to be Lorentzian then the interpretation of logarithms is simple. For more complicated line shapes one must come up with more complicated interpolation schemes.

[9]S. Ma and G. F. Mazenko, Phys. Rev. Lett. 33, 1384 (1974).

[10]S. Ma and G. F. Mazenko, Phys. Rev. B 11, 4077 (1975).

[11]E. D. Siggia, Phys. Rev. B 13, 3218 (1976).

[12]P. C. Hohenberg, B. I. Halperin, and E. D. Siggia, Phys. Rev. B 14, 2865 (1976).

[13]B. I. Halperin and P. C. Hohenberg, Phys. Rev. Lett. 19, 700 (1967); B. I. Halperin and P. C. Hohenberg, Phys. Rev. 177, 952 (1969); B. I. Halperin and P. C. Hohenberg, Phys. Rev. 188, 898 (1969).

[14]D. Forster, *Hydrodynamics Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, New York, 1975).

[15]K. Kawasaki and H. Mori, Prog. Theor. Phys. (Kyoto) 25, 1043 (1961), and 38, 1052 (1967); J. L. Lebowitz and O. Penrose, Phys. Rev. Lett. 35, 549 (1975); V. G. Vaks, A. I. Larkin, and S. A. Pikin, Zh. Eksp. Teor. Fiz. 53, 1089 (1967) [Sov. Phys.-JETP 26, 647 (1968)]; M. E. Fisher, M. N. Barber, and D. Jasnow, Phys. Rev. A 8, 1111 (1973); and Refs. 25–27 below.

[16]J. Villain, Solid State Commun. 8, 31 (1970). See also K. H. Michel and F. Schwabl, Z. Physik 240, 354 (1970).

[17]B. I. Halperin, P. C. Hohenberg, and E. Siggia, Phys. Rev. B 13, 1299 (1976).

[18]S. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976).

[19]K. G. Wilson, Phys. Rev. Lett. 28, 548 (1972).

[20]C. Joukoff-Piette and P. Résibois, Phys. Lett. A 42, 531 (1973).

[21]D. L. Huber and D. A. Krueger, Phys. Rev. Lett. 24, 111 (1970).

[22]R. Nathans, F. Menzinger, and S. Pickart, J. Appl. Phys. 39, 1237 (1968).

[23]H. Lau, L. Corliss, A. Delapalme, J. Hastings, R. Nathans, and A. Tucciarone, Phys. Rev. Lett. 23, 1225 (1969).

[24]A. Tucciarone, J. M. Hastings, and L. M. Corliss, Phys. Rev. B 8, 1103 (1973).

[25]E. Brézin, D. J. Wallace, and K. G. Wilson, Phys. Rev. B 7, 232 (1973).

[26]D. R. Nelson, Phys. Rev. B 13, 2222 (1976).

[27]G. F. Mazenko, Phys. Rev. B 14, 3933 (1976).

[28]Our analysis here follows that of G. Grinstein, S. Ma, and G. F. Mazenko, Phys. Rev. B 15, 258 (1977).

[29]P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973).

[30]In the Ma and Mazenko (Ref. 10) approach one calculates $\Sigma(\omega)$ and from this one can extract $\Gamma(\omega)$ [see, for example Eqs. (4.9), (4.14) and (4.16) in Freedman and Mazenko (Ref. 2)]. The problems involved in calculating $\Gamma(\omega)$ directly using the Martin, Siggia, and Rose (Ref. 29) formalism are more severe. This is because the propagator in that formalism does not have a direct physical interpretation, does not satisfy a fluctuation dissipation theorem, and one is forced to introduce vertex corrections in order to relate $\Gamma(\omega)$ to the self-energy in the Martin, Siggia, and Rose formalism [see Halperin, Hohenberg and Siggia (Ref. 7)].

[31]G. F. Mazenko, Phys. Rev. A 9, 360 (1974). See also G. F. Mazenko and S. Yip, in *Modern Theoretical Chemistry*, edited by B. J. Berne (Plenum, New York) (to be published).

[32]See Eq. (2.43) in Ref. 10.

[33]B. I. Halperin, P. C. Hohenberg, and S. Ma, Phys. Rev. Lett. 29, 1548 (1972), Phys. Rev. B 10, 139 (1974) and Phys. Rev. B 13, 4119 (1976).

[34]In particular we have used these methods to carry out the matching calculation mentioned above.

[35]It is a simple matter to obtain M^c from Γ by multiplying by χ^{-1}. We find it more convenient to present the results in terms of Γ.

[36]Presumably the appropriate generalization is that of Sasvári and Szépfalusy (Ref. 7). It would be most interesting if a $1/n$ type calculation could be carried out for their model for $T < T_N$.

[37]M. Barmatz, P. C. Hohenberg, and A. Kornblit, Phys. Rev. B 12, 1947 (1975) and Ref. 27.

[38]P. Horn, L. Corliss, and J. Hastings (private communication).