Some symmetry properties of renormalization-group transformations

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For a Landau-Ginzburg-Wilson Hamiltonian of any given symmetry we show how one can find a group G_T of orthogonal transformations in parameter space, which commute with renormalization-group transformations. Then a renormalization-group transformation may be expanded into covariants of G_T. We also present a systematic procedure for finding fixed points; they are most likely to decouple the Hamiltonian or to increase its symmetry. The merit of the conclusions obtained is illustrated using an example of a system with C_4 symmetry. Agreement with the results of ϵ-expansion calculations has been found.

I. INTRODUCTION

Since its appearance, the renormalization group (RG) has proved to be a very useful tool in the modern theory of critical phenomena as well as in other areas of theoretical physics.[1-3] Since Kadanoff's[4] and Wilson's[5] pioneer work, our understanding of RG theory has broadened in a qualitative as well as formal way. There is a large amount of literature on the RG theory.[6] Particularly, there have already been several papers concerned with the study of classes of equivalent Hamiltonians (with respect to universality). Also several papers concerning classes of equivalent RG transformations have appeared. Jona-Lasinio[7] has been interested in studying such RG transformations, especially those related by diffeomorphisms, for which he has established some theorems. Wegener[8] and Green[9] have established several similar theorems (called invariance theorems) for RG transformations generated by a flow vector.

It is interesting to study the connection between classes of equivalent Hamiltonians and classes of RG transformations. Korzhenevskiĭ[10] has studied the example of a two-component field of cubic symmetry. He studied the connection between the properties of the RG transformation and symmetry of the Hamiltonian.

We have been motivated by the many results of RG analysis for multicomponent order parameters[11-14] to investigate apparent symmetry in RG equations, its origin and its consequences. We have found that there is a group of symmetry operations in parameter space under which RG transformations remain invariant. We determine the relationship of this group of symmetry transformations to the symmetry group of the Hamiltonian. We also show the relation of this group to the structure of RG equations. We formulate a systematic procedure for obtaining fixed points of RG transformations, based on the symmetry properties of the Hamiltonian.

II. SYMMETRY OPERATIONS IN THE PARAMETER SPACE

We will be concentrating on the systems described by the Landau-Ginzburg-Wilson (LGW) Hamiltonians of the general form:

$$H[\psi] = -\frac{1}{2}\sum_{i=1}^{l}\int \vec{\nabla}\psi_i \cdot \vec{\nabla}\psi_i \, d^d x - \sum_{j=0}^{m} u_j \int I_j \, d^d x \,, \tag{1}$$

where $\psi_i \equiv \psi_i(\vec{x})$, $i=1,\ldots,l$, is an l-component field (order parameter) which spans physically irreducible (i.e., irreducible over the field of reals) representation D of the symmetry group of the system; $d^d x$ is the volume element in the d-dimensional configuration space; $I_0 = \sum_{i=1}^{l} \psi_i^2$ is the only D-invariant homogeneous polynomial quadratic in ψ; I_j, $j=1,\ldots,m$, are all quartic in ψ linearly independent D-invariant polynomials (we denote I_0^2 by I_1); u_j, $j=0,\ldots,m$ are real parameters, analytic functions of thermodynamic variables. These parameters form an $(m+1)$-dimensional parameter space Π for the system (1):

$$\Pi = \left\{ u;\ u \in R^{m+1},\ \lim_{I_0 \to +\infty} \sum_{j=0}^{m} u_j I_j = +\infty \right\}, \tag{2}$$

where R is a real line. We denote by u a point $(u_0, u_1, \ldots, u_m)$. Definition (2) represents a normalizability condition of the probability distribution $e^{H[\psi]}$: the partition function Z is defined as a functional integral,

$$Z = \int \mathfrak{D}\psi \, e^{H[\psi]} \,. \tag{3}$$

The partition function is a function of the point u in the space Π.

In the space Π, we also define RG transformations: an RG transformation $\mathfrak{R}$ is defined as a (nonlinear) transformation in Π,

$$\mathfrak{R}: u \to \bar{u} \equiv \mathfrak{R}u, \tag{4}$$

which connects two systems that are otherwise identical except for a scale change. Immediate consequences of the definition of $\mathfrak{R}$ are (a) $\mathfrak{R}$ must satisfy

$$\mathfrak{R}_s \mathfrak{R}_t = \mathfrak{R}_{st}, \quad \text{for all } s \text{ and } t \text{ from } [1,\infty], \tag{5}$$

where we used subscripts s, t, st to denote corresponding scale change factors; (b) $\mathfrak{R}$ cannot decrease the symmetry of the Hamiltonian; (c) an infinite number of RG transformations may increase the symmetry of the Hamiltonian; (d) $\mathfrak{R}$ cannot produce couplings in the, originally, decoupled Hamiltonian; (e) an infinite number of RG transformations may eliminate some of the couplings, originally present in the Hamiltonian. Any RG transformation $\mathfrak{R}$ must satisfy conditions (a)–(e).

It is of particular interest to find fixed points u^* of $\mathfrak{R}$,

$$\mathfrak{R}u^* = u^*, \tag{6}$$

critical exponents for the system are given by eigenvalues of the linearized from $\mathfrak{R}^*$ of $\mathfrak{R}$ near the fixed point. A matrix $\mathfrak{R}^*$ is defined as

$$\mathfrak{R}^* = \frac{\partial \mathfrak{R}u}{\partial u}\bigg|_{u=u^*}. \tag{7}$$

Hence, although definition of $\mathfrak{R}$ is not unique, in order for different $\mathfrak{R}$'s to describe the same physics, it is sufficient that corresponding $\mathfrak{R}^*$s have the same (relevant) eigenvalues.[8,9] Such $\mathfrak{R}$'s we call equivalent.

It was shown by Jona-Lasinio[7] that every diffeomorphism T, in Π, can produce RG transformation $\mathfrak{R}^T$ equivalent to the transformation $\mathfrak{R}$

$$\mathfrak{R}^T = T^{-1}\mathfrak{R}\, T. \tag{8}$$

We restrict T to linear nonsingular transformations on Π. Then, Eq. (8) expresses arbitrariness in the choice of the basic invariants I_j, $j=1,\ldots,m$. It is apparent from Eq. (8) that symmetry properties of $\mathfrak{R}$ are determined by these T's which commute with $\mathfrak{R}$. Such T's form a group, which we call G_T

$$G_T = \{T; \mathfrak{R}^T = \mathfrak{R}\}. \tag{9}$$

Before we analyze the consequences of Eq. (9), we will shown how G_T can be determined if we know the quartic part of the Hamiltonian (1).

First we note that the partition function (3) is invariant under a change of "gauge"

$$\psi \to V\psi, \tag{10}$$

where V is an orthogonal $l\times l$ matrix: $V \in O(l)$. However, under the action of V the Hamiltonian will be, in general, taken outside of Π. Therefore, we restrict our attention to those V's which transform Π into Π. I_0 remains invariant under all V and we need only consider those V's that transform invariants I_j, $j=1,\ldots,m$ into linear combinations of themselves. Call the group of such V's G_g:

$$G_g = \{V; V\in O(l),\ V\Pi \subseteq \Pi\}. \tag{11}$$

Therefore, by definition, every $V \in G_g$ will induce a linear transformation on Π. We prove in the Appendix that all linear transformations induced in such a way commute with the $\mathfrak{R}$ and thus they belong to a group G_T. G_T is, in effect, a linear representation of G_g. Particularly, subgroup D of G_g is represented by the identity in G_T. We can always choose invariants I_j, $j=1,\ldots,m$, in such a way to have G_T orthogonal and reduced.

The simplest way to obtain groups G_g and G_T is by the "brute force" method. Take a general $V(\theta_1,\ldots) \in O(l)$, which is suitably parametrized (parameters: $\theta_1,\ldots$). The condition

$$V(\theta_1,\ldots)\Pi \subseteq \Pi \tag{12}$$

will then give a system of equations for parameters $\theta_1,\ldots$. By solving this system we automatically determine G_g as well as G_T. A set of the solutions is not empty: there will always be solutions that correspond to $G_g = D$.

Let us denote by $\bar{D}$ a maximal group that leaves *each* I_j, $j=1,\ldots,m$, invariant. $\bar{D}$ must be a normal subgroup of G_g. Therefore, we can also obtain a group G_g as a normalizer of $\bar{D}$ in $O(l)$

$$G_g = \{V; V\in O(l),\ V^{-1}\bar{D}V = \bar{D}\}. \tag{13}$$

A group G_T is then obtained from the irreducible representations of G_g which appear in the reduction of the fourth symmetrized power of G_g. Only these irreducible representations which subduce identity representation of $\bar{D}$ are relevant. In this way a group G_T may be obtained in its reduced form.

In the similar fashion one may construct a group G_g (and G_T) by reduction (and subduction) of the fourth symmetrized power of the entire group $O(l)$.

The aforementioned methods for obtaining a group G_T will be illustrated in Sec. IV of this paper.

Now that we assume the group G_T is determined, we return to its definition (9). Consequences of this definition are very strong: the RG transformation $\mathfrak{R}u$ must be covariant under G_T,

$$\Re T u = T \Re u , \quad \text{for all } T \in G_T . \tag{14}$$

As a result, a flow line diagram generated in Π by the action of $\Re$ must have symmetry G_T. In particular: (i) a *set* of fixed points must be invariant under G_T; (ii) two fixed points related by G_T must have the same stability properties. Furthermore we can expand $\Re u$ as[15]

$$\Re u = \sum_{\nu} f_{\nu}(\mathcal{I}(u))\, c_{\nu}(u) , \tag{15}$$

where $\mathcal{I}(u)$ is a set of all algebraically independent invariants (i.e., integrity basis[16]) of G_T; $c_{\nu}(u)$ are all linearly independent (modulo invariant) covariant functions of G_T; f_{ν} are arbitrary functions of their arguments. In the cases that are most likely to occur (e.g., G_T is a point group), there is a finite set of covariants $c_{\nu}(u)$.[15]

In the special case where we are interested in the small u region, for example in the ϵ expansion,[17,18] we expand $\Re u$ in the powers of u. Thus $\Re u$ will be determined by several constants. Then by Eq. (15) we can reduce the number of these constants to those coming from expansions of functions f_{ν}. Further elimination of constants is imposed by the general conditions (a) to (e) of this section. Therefore we will be left with only a few constants to be determined. This means, in the ϵ expansion, that we need calculate fewer Feynman diagrams.

III. FIXED POINTS

For a given RG transformation $\Re$, it is important to find the set of all fixed points, and to determine their stability. The problem of finding all fixed points usually reduces to the solution of a system of $m+1$ nonlinear equations for parameters $u_j, j=0,\ldots,m$. In general, it is difficult to analytically find the set of solutions of these equations. We propose here a systematic procedure for finding the fixed points. By this procedure we reduce the number of independent parameters among u's. In some cases the number of independent parameters is reduced to only one.

The first step in the determination of the fixed points is to use condition (b) of the previous section. This means that we have to make a sequence of subspaces σ of Π,

$$\begin{aligned}\sigma_G^* = \{0\} \subset \sigma_H = \{(u_0, u_1, 0, \ldots)\} \\ -\sigma_G^* \subset \ldots \subset \sigma_{D'}, \subset \ldots \subset \sigma_{\bar{D}} \\ = \Pi - \bigcup_{D'} \sigma_{D'} ,\end{aligned} \tag{16}$$

which correspond to possible symmetries of the Hamiltonian

$$E(l) \supset \mathrm{O}(l) \supset \ldots \supset D' \supset \ldots \supset \bar{D} . \tag{17}$$

In other words, symmetry of the Hamiltonian is D' whenever $u \in \sigma_{D'}$. Therefore, for each D' and each finite scale change $\Re$:

$$\Re \sigma_{D'} = \sigma_{D'} . \tag{18}$$

We have isolated subspace σ_G^*,

$$\sigma_G^* = \{0\} , \tag{19}$$

which corresponds to the symmetry $E(l)$, which is a semidirect product of $\mathrm{O}(l)$ and full translation group $T(l)$ in l-dimensional space[19] [i.e. $E(l)$ is a l-dimensional Euclidian group]

$$E(l) = \mathrm{O}(l) \overset{s}{\otimes} T(l) . \tag{20}$$

Subspace σ_G^* we call the Gaussian fixed point.

We have also isolated subspace σ_H,

$$\sigma_H = \{(u_0, u_1, 0, \ldots)\} - \sigma_G^* , \tag{21}$$

which exists for every LGW Hamiltonian (1): invariants I_0 and I_1 are both invariant under $\mathrm{O}(l)$, moreover, they are the only possible invariants (up to fourth degree in ψ) of $\mathrm{O}(l)$. Therefore, subspace σ_H corresponds to the symmetry $\mathrm{O}(l)$. This subspace we call the Heisenberg subspace and the corresponding fixed points we call the Heisenberg fixed points.

The second step is to use condition (*d*) of Sec. II. Thus we find all subspaces $\sigma_{D'}^1, \sigma_{D'}^2, \ldots \subseteq \sigma_{D'}$, which decouple the Hamiltonians corresponding to each symmetry D'. Therefore for each D' and each $\mu = 1, 2, \ldots$ and each finite scale change $\Re$ we have

$$\Re \sigma_{D'}^{\mu} = \sigma_{D'}^{\mu} . \tag{22}$$

We can immediately determine a subspace σ_G of σ_H which decouples the Hamiltonian into a system of noninteracting harmonic oscilators

$$\sigma_G = \{(u_0, 0, \ldots)\} . \tag{23}$$

This subspace we call the Gaussian subspace.

Our conjecture is that most of the fixed points will be either points that decouple the Hamiltonian, or points related to these by the group G_T. Note that decoupled Hamiltonian may be related to the coupled one by the group G_T.

At the present, we cannot say much about the stability of fixed points. If we assume that a stable fixed point corresponds to the infinite correlation length (excluding possibility of zero correlation length[5]), then fixed points that correspond to the decoupled Hamiltonians, or Hamiltonians related by G_T to decoupled ones, must be unstable in the directions which involve couplings. Therefore, most of the fixed points should be unstable.

In the special case where the parameters of the Hamiltonian belong to the Gaussian subspace, the partition function (3) as well as the RG transformation may be obtained exactly.[18] There is only one fixed point in this case, the Gaussian fixed point $u^* = 0$, which is unstable. This fixed point must also be unstable in all other directions and in particular in the u_1 direction. Therefore, if there is only one Heisenberg fixed point, we conclude that it must be stable in the u_1 direction.

It is important to note that there is a possibility of finding a line of fixed points whenever the group G_T turns out be be a continuous group. For example, we find this to be the case in Sec. IV and in the case of four-dimensional R-point representation of the O_h^3 symmetry group.[14] In both cases the representations used are direct sums of a complex conjugate representations. We believe that, in general, appearance of marginal eigenoperators (i.e., lines of fixed points) may be attributed to the existance of the continuous symmetry group G_T.

In the following section we will illustrate application of the conclusions presented here. We will also make comparison with the ϵ-expansion results.

IV. EXAMPLE: SYSTEM WITH C_4 SYMMETRY

As an example we will discuss a system wich C_4 symmetry, desribed by a two-component field

$$\psi_1 \equiv X ,\quad \psi_2 \equiv Y . \tag{24}$$

We take these fields to span a physically irreducible representation D,

$$D = E \otimes E^* , \tag{25}$$

of the group C_4. E and E^* are complex conjugate, one-dimensional, irreducible representations of C_4, as given in Ref. 20.

The representation D has three linearly independent quartic invariants[21]:

$$\begin{aligned} I_1 &= I_0^2 = (X^2 + Y^2)^2 , \\ I_2 &= \tfrac{3}{8} X^2 Y^2 - \tfrac{1}{16}(X^4 + Y^4) , \\ I_3 &= XY(X^2 - Y^2) . \end{aligned} \tag{26}$$

We have chosen this particular set of invariants such that we obtain a group G_T in orthogonal, reduced form. Included here as a special case (when $u_3 = 0$ in the Hamiltonian) is the example of Ref. 10.

In order to obtain a group G_T we will first apply a "brute-force" method. A general gauge transformation, $V \in O(2)$, is either

$$V_+(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi) , \tag{27}$$

or

$$V_-(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V_+(\theta) , \quad \theta \in [0, 2\pi) . \tag{28}$$

It is immediately clear that a gauge transformation

$$V_-(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is an element of G_g and that it induces transformation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{29}$$

of G_T. On the other hand, invariants I_2 and I_3 transform under $V_+(\theta)$ as

$$\begin{aligned} V_+(\theta) I_2 &= \cos(4\theta) I_2 - \sin(4\theta) I_3 , \\ V_+(\theta) I_3 &= \sin(4\theta) I_2 + \cos(4\theta) I_3 . \end{aligned} \tag{30}$$

Therefore, for any $V \in O(2)$ invariants I_2 and I_3 are transformed into linear combinations of themselves. Thus, the group G_g is the entire group $O(2)$:

$$G_g = O(2) . \tag{31}$$

From Eqs. (29) and (30) we see that elements of the group G_T are

$$T_+(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(4\theta) & \sin(4\theta) \\ 0 & 0 & -\sin(4\theta) & \cos(4\theta) \end{bmatrix} \tag{32}$$

and

$$T_-(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} T_+(\theta) . \tag{33}$$

Thus, we find from Eqs. (32) and (33) that G_T is given as

$$G_T = 2Z_+^0 \oplus Z^4 , \tag{34}$$

where Z_+^0 is the identity representation of $O(2)$ and Z^4 is a two-dimensional irreducible representation of $O(2)$ as given in Ref. 22.

We note that we could find a group $G_g = O(2)$ by the second procedure described in the text. Namely, invariants I_0, I_2, I_3 form the integrity basis of the group C_4, and therefore C_4 is a maximal group which leaves all I's invariant. On the other hand, C_4 is a *normal* subgroup of $O(2)$.

Therefore, we conclude $G_g = O(2)$.

The group G_T is continuous. When restricted to the subspace spanned by u_2 and u_3, G_T is identical to O(2). As a consequence the RG transformation must have axial symmetry (axis: $u_2 = u_3 = 0$).

Now that we know a group G_T, we can further analyze symmetry properties of the RG transformation. The Gaussian and Heisenberg subspaces where discussed in the previous section, giving Gaussian and Heisenberg fixed points

$$u_G^* = (0,0,0,0) ,$$
$$u_H^* = (u_0^H, u_1^H, 0, 0) . \quad (35)$$

The reminder of Π,

$$\sigma_D = \Pi - \sigma_H - \sigma_G^* , \quad (36)$$

is the last important subspace in the chain (16). σ_D contains a subspace σ_I,

$$\sigma_I = \{(u_0, u_1 - \tfrac{16}{3} u_1, 0)\} , \quad (37)$$

in which the Hamiltonian is decoupled into two "Φ^4" Hamiltonians. This subspace we call an Ising subspace and corresponding fixed points Ising fixed points. Clearly, by the action of G_T, these fixed points will produce lines (circles) of fixed points. Therefore, each such fixed point will possess a marginal eigenoperator. The fact that the Gaussian fixed point is unstable and the assumption that there is only one Ising fixed point, leads to the conclusion that the Ising fixed point is stable along the line given by (37). On the basis of conclusions of Sec. III it is unstable in other directions. Thus, if we assume that there is a crossover line from the Ising to the Heisenberg behavior, we conclude that the Heisenberg fixed point must be stable. Our conclusions are summarized in Fig. 1.

These conclusions we have just presented can be compared with the ϵ-expansion results.

The most general form for *quadratic* $\mathfrak{R}$ which satisfies the symmetry condition (15) and (a) to (e) of Sec. II is

$$\bar{u} = s^\alpha [u_1 + \beta(1 - s^\alpha) u_1^2 + \gamma(1 - s^\alpha)(u_2^2 + u_3^2)] ,$$
$$\bar{u} = s^\alpha \{u_2 + [\beta + (\tfrac{16}{3})^2 \gamma](1 - s^\alpha) u_1 u_2\} , \quad (38)$$
$$\bar{u} = s^\alpha \{u_3 + [\beta + (\tfrac{16}{3})^2 \gamma](1 - s^\alpha) u_1 u_3\} ,$$

where α, β, γ are functions of ϵ (i.e., of the configuration space dimensionality $d = 4 - \epsilon$) only; s is the scale-change factor. We have also assumed that u_0 does not couple to $\bar{u}_i$, $i = 1, 2, 3$, enabling us to extract the $i = 1, 2, 3$ part of $\mathfrak{R}$. From Eq. (38), it is apparent that only three constants need to be obtained by the ϵ expansion. It suffices to consider contributions shown in Fig. 2. Contribution from the first diagram of Fig. 2 is one obtained

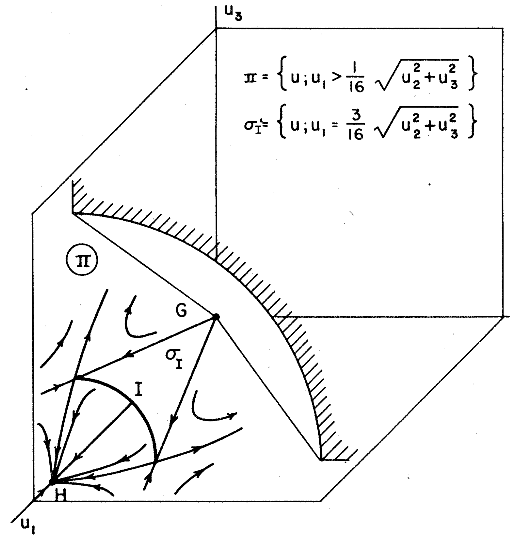

FIG. 1. RG flow diagram for the two-dimensional representation of the C_4 symmetry; the parameter space Π is the interior of the shaded cone; only the flow lines in the u_1-u_2 and u_1-u_3 planes are shown; $u_2 = u_3 = 0$ is the symmetry axis; G and H are the Gaussian and Heisenberg fixed points, respectively; I is the line of the Ising fixed points; σ_I is the Ising line [Eq. (37)] and σ_I^* is the cone generated from σ_I by G_T.

from a simple scale change argument, giving α to the lowest order in ϵ

$$\alpha \approx \epsilon . \quad (39)$$

Because of the axial symmetry in Π, it is sufficient to analyze the $u_3 = 0$ plane. Then we compare Eq. (38) with the ϵ-expansion calculation of Refs. 11 and 17. Their calculation is in agreement with (38) yielding

$$\beta = -40(K_4/\epsilon) ,$$
$$\gamma = -(\tfrac{9}{32})(K_4/\epsilon) , \quad (40)$$

where K_4 is the area of a three dimensional unit sphere. The fixed points are obtained in the subspaces σ_G, σ_H, and σ_I by solving corresponding quadratic equations for *one* unknown. These fixed points and their stability, as calculated in Refs. 11 and 17, correspond to those of Fig. 1, which we have obtained on the basis of our analysis.

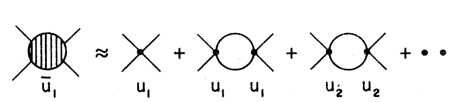

FIG. 2. Feynman diagrams needed to determine α, β, and γ of Eq. (38); only the contributions shown explicitly need to be considered; for example, contribution $u_3 u_3$ need not be considered.

V. DISCUSSION

In the previous sections, we have discussed the symmetry properties of RG transformations. We have also proposed a systematic procedure which allows us to find most of the fixed points. We concluded that most of the fixed points must be unstable. This agrees with present ϵ-expansion calculations which show that very few stable fixed points exist. This is particularly apparent in the case when the number of components of the field exceeds 3 (i.e., $l>3$).[11-14] ϵ expansion has also shown that the components of u^* happen to be rational.[11-14] This is an interesting observation, which we believe, will be understood on the basis of symmetry arguments.

We have constructed a group G_T as an orthogonal group which arises from the orthogonal transformations on ψ. However, for systems with high symmetry, the group G_T obtained may just be the identity transformation. In such cases (e.g., cubic symmetry $l=3$)[11] one has to look for more general group G_T.

A question remains as to what happens with the symmetry properties of the RG transformation if we include higher-order invariants in the Hamiltonian (1). First we note that Π can be separated into subspaces which correspond to the degrees of the invariants. Such subspaces are not mixed by the group G_T. Therefore, the group G_T restricted to the original subspace is unchanged. In added subspace, it may occur, if $\overline{D}=D$, that the corresponding group G_g will remain the same. However, if $\overline{D}\neq D$, we may find that the new G_g is the group D itself. Then, again, we have to look for a group G_T more general then one discussed in this paper. Such a group may arise from transformations on ψ more complicated than simple orthogonal transformations.

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APPENDIX

In order to prove that transformations T, generated by gauge transformations V [Eq. (12)] belong to a group G_T, we have to show

$$(\Re VH)[\psi]=(V\Re H)[\psi]\,,\ \text{for all } V\in G_g\,, \tag{A1}$$

where we define

$$(VH)[\psi]\equiv H[V\psi]\,. \tag{A2}$$

First we consider the case of Wilson's approach of integrating out short-wavelength degrees of freedom.[5,17,18] The field ψ is defined in $\vec{k}$ space (Fourier transform), and $\Re$ is defined as

$$(\Re H)[\psi']=\ln\left(\int \mathfrak{D}\psi_> \, e^{H[\psi]}\right)\Bigg|_{\psi'(\vec{k})=\zeta\psi_<(s\vec{k})}\,, \tag{A3}$$

where $\psi_{\gtrless}$ is a field component which depends on such $\vec{k}$ that $|\vec{k}|\gtrless s^{-1}k_0$, k_0 being the cutoff wave number; s is a scale change factor and ζ is the field-scale change factor. Therefore, from (A3) and (A2) we find

$$(\Re VH)[\psi']=\ln\left(\int \mathfrak{D}\psi_> \, e^{H[V\psi]}\right)\Bigg|_{\psi'(\vec{k})=\zeta\psi_<(s\vec{k})}\,. \tag{A4}$$

After a change of variables $\psi\rightarrow V^{-1}\psi$, taking into account $|\det V|=1$, the left-hand side of Eq. (A4) becomes

$$\ln\left(\int \mathfrak{D}\psi_> e^{H[\psi]}\right)\Bigg|_{V\psi'(\vec{k})=\zeta\psi_<(s\vec{k})}\,. \tag{A5}$$

If we use Eq. (A3) again, (A5) becomes

$$(\Re H)[V\psi']=(V\Re H)[\psi']\,, \tag{A6}$$

where the last equality follows from definition (A2). Comparison of (A4) with (A6) concludes the proof.

The definition of the $\Re$ which is closely related to the original Kadanoff's definition of block spins can be written as[6]:

$$(\Re H)[\psi']=\ln\left[\int \mathfrak{D}\psi \prod_c \delta\left(\psi'-\int_{\Omega_c}\psi\right) e^{H[\psi]}\right]\,, \tag{A7}$$

where Ω_c is a small volume around a point c. This definition of the configuration space RG transformation is particularly useful in case when the field is defined on the discrete lattice.[23] We prove (A1) using the definition (A7) as well.

From (A7) and (A2) we find

$$(\Re VH)[\psi']=\ln\left[\int \mathfrak{D}\psi \prod_c \delta\left(V\psi'-\int_{\Omega_c}\psi\right) e^{H[\psi]}\right]\,. \tag{A8}$$

where we have changed variables $\psi\rightarrow V^{-1}\psi$, taking into account $\delta(V^{-1}\psi)=|\det V|\,\delta(\psi)$ and $|\det V|=1$. By comparison with definition (A7), the left-hand side of (A8) becomes

$$(\Re H)[V\psi']=(V\Re H)[\psi']\,, \tag{A9}$$

where the last equality follows from the definition (A2). Therefore, we have also proved (A1) for the $\Re$ defined by (A7).

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[1]See, for example, *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1976), Vol. 6; and *Renormalization Group in Critical Phenomena and Quantum Field Theory*, edited by M. S. Green and J. D. Gunton (Temple University, Philadelphia, 1973).

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