Ripplon damping in superfluid ^{4}He at low temperatures

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The damping coefficient (imaginary part of the frequency) of ripplons in superfluid ^{4}He at low temperatures T is found to be $\tau_q^{-1} = 0.376(\hbar/\rho_0)(k_B T/\hbar\gamma)^{10/3}$ due to processes in which the acoustic ripplon is absorbed by a thermal ripplon. The range of wave vectors for which a direct experimental observation of the ripplon might be possible is discussed.

I. INTRODUCTION

The present experimental evidence for the existence of quantized capillary waves, or ripplons,[1] on the free surface of superfluid ^{4}He is based on measurements[2,3] of the low-temperature dependence of the surface tension $\sigma(T)$ and on the assumption of infinitely-long-lived ripplons. For small q the ripplon dispersion relation ω_q is given at $T=0$ by [4]

$$\omega_q = \gamma q^{3/2}, \tag{1}$$

where T is the temperature, $\gamma = (\sigma_0/\rho_0)^{1/2}$, $\sigma_0 \equiv \sigma(T=0)$, ρ_0 is the $T=0$ bulk mass density, and q is a surface (two-dimensional) wave vector. A theoretical calculation of the ripplon contribution to the dynamic structure function of superfluid ^{4}He has been performed [5] based on the assumption of infinitely-long-lived ripplons. No direct observation of ripplons has been reported. In order to assess the experimental feasibility of a direct observation of well-defined ripplons, we investigate in this paper the lifetime $\tau_q(T)$ of an acoustic ripplon using a quantum-hydrodynamic (QHD) approach. We find that the leading T dependence of the damping coefficient of acoustic ripplons, i.e., ripplons that satisfy $\hbar\omega_q \ll k_B T$, is

$$\frac{1}{\tau_q(T)} = 0.376\frac{\hbar}{\rho_0}\left(\frac{k_B T}{\hbar\gamma}\right)^{10/3} . \tag{2}$$

Note that the lifetime $\tau_q(T)$ of acoustic ripplons at fixed T is independent of q. The result (2) complements the small-q calculation of $\tau_q(0)$ recently obtained by Saam[6]:

$$1/\tau_q(0) = 0.162(\hbar/\rho_0)q^5 . \tag{3}$$

II. CALCULATION

We begin with the three-ripplon QHD interaction Hamiltonian H_1 given by Saam[6] and determine τ_q using a finite-temperature Green's-function technique. The present approach is similar to that used in QHD calculations of phonon damping[7,8] in bulk superfluid ^{4}He at low T. To lowest order in H_1, the self-energy diagrams have a one-loop structure.[8] The lifetime τ_q is related to the ripplon self-energy $\Sigma(q,\omega)$ by $\tau_q^{-1} = \mathrm{Im}\,\Sigma(q, \omega_q - i0^+)$. (For simplicity we set $\hbar = k_B = 1$ throughout this section.) In this way, τ_q^{-1} is found to be

$$\begin{aligned}\tau_q^{-1} = 2\pi\int\frac{d^2p}{(2\pi)^2}[2|\langle\vec q,\vec p\,'|H_1|\vec p\rangle|^2(f_{p'}-f_p)\delta(\omega_q+\omega_{p'}-\omega_p)\\ +|\langle\vec p,-\vec p\,'|H_1|\vec q\rangle|^2(1+f_{p'}+f_p)\delta(\omega_q-\omega_{p'}-\omega_p)\\ -|\langle 0|H_1|\vec q,\vec p,\vec p\,'\rangle|^2(1+f_{p'}+f_p)\delta(\omega_q-\omega_{p'}-\omega_p)],\end{aligned} \tag{4}$$

where $\vec p\,' = \vec p - \vec q$ and $f_p = (e^{\omega_p/T}-1)^{-1}$. The first matrix element of H_1 in (4) is given by[6]

$$\langle \vec{q},\vec{p}\,'|H_1|\vec{p}\rangle = \left[\frac{qp'p}{8\rho_0\omega_q\omega_{p'}\omega_p}\right]^{1/2} [\omega_p\omega_{p'}(\hat{p}\cdot\hat{p}+1)+\omega_q\omega_p(\hat{q}\cdot\hat{p}+1)+\omega_q\omega_{p'}(\hat{q}\cdot\hat{p}'-1)]\,. \tag{5}$$

The latter two matrix elements of H_1 in (4) are similar in form to (5). Here $\hat{p}\equiv\vec{p}/p$, etc.

The first term in (4) corresponds to an absorption process in which a ripplon $\vec{q}$ is absorbed by a thermal ripplon $\vec{p}\,'$ ($\omega_{p'}\sim T$) resulting in a thermal ripplon $\vec{p}$ and another process in which $\vec{p}\,'$ and $\vec{p}$ are interchanged. The second term in (4) represents a decay process in which a ripplon $\vec{q}$ decays into two ripplons $\vec{p}$ and $-\vec{p}\,'$. The third term in (4) corresponds to an annihilation process in which three ripplons $\vec{q},\vec{p},\vec{p}\,'$ annihilate. At $T=0$ only the decay process is possible; this process was considered by Saam[6] to obtain (3). At $T\neq 0$ and for acoustic ripplons ($\omega_q << T$) the decay and annihilation terms can be neglected since the relevant δ functions cannot be satisfied and only the first term in (4) need be considered.

The first term in (4) can be divided into two integrals: $\tau_q^{-1}=I_1-I_2$, where I_1 and I_2 denote the integrals proportional to $f_{p'}$ and f_p respectively. We evaluate explicitly I_2 which is given by

$$I_2=\frac{\pi}{4\rho_0}\int\frac{d^2p}{(2\pi)^2}\frac{qpp'}{\omega_q\omega_p\omega_{p'}}[\omega_p\omega_{p'}(\nu+1)+\omega_p\omega_q(\mu+1)+\omega_{p'}\omega_q(\eta+1)]^2 f_p\delta(\omega_q+\omega_{p'}+\omega_p)\,, \tag{6}$$

where $\nu\equiv\hat{p}\cdot\hat{p}'$, $\mu\equiv\hat{p}\cdot\hat{q}$, and $\eta\equiv\hat{p}'\cdot\hat{q}$. The change of variables $x=\omega_p$, $y=\omega_{p'}$, $z=\omega_q$ is introduced and (6) is rewritten

$$I_2=\frac{1}{36\pi\rho_0 z}\gamma^{-10/3}\int_z^x\frac{dx}{x^{2/3}}f(x)\int_{y_{\min}}^{y_{\max}}\frac{dy}{(1-\mu^2)^{1/2}}[xy(\nu+1)+xz(\mu+1)+yz(\eta-1)]^2\delta(z+y-x)\,, \tag{7}$$

with $f(x)=(e^{x/T}-1)^{-1}$. The δ function in (7) implies that $y=x-z$, which can be shown to lie between $y_{\max}$ and $y_{\min}$. For acoustic ripplons $z<<x$ and the x integrand can be expanded in terms of z/x. To lowest order in z/x, $\nu=1,\mu=\eta=\frac{2}{3}(z/x)^{1/3}$ and the lower limit of the x integration can be set equal to zero. Thus to $O(z/x)$, I_2 becomes

$$I_2=\frac{1}{9\pi\rho_0 z}\gamma^{-10/3}\int_0^\infty dx\,x^{10/3}f(x)\left[1+\frac{2}{9}\left(\frac{z}{x}\right)^{2/3}-2\left(\frac{z}{x}\right)+O\left(\left(\frac{z}{x}\right)^{4/3}\right)\right].$$

Similarly to $O(z/x)$, I_1 is given by

$$I_1=\frac{1}{9\pi\rho_0 z}\gamma^{-10/3}\int_0^\infty dx\,x^{10/3}f(x)\left[1+\frac{2}{9}\left(\frac{z}{x}\right)^{2/3}+\frac{4}{3}\left(\frac{z}{x}\right)+0\left(\left(\frac{z}{x}\right)^{4/3}\right)\right].$$

Since $\tau_q^{-1}=I_1-I_2$, we see that the z^{-1} and $z^{-1/3}$ terms cancel and that the leading term in τ_q^{-1} is given by a z-independent term:

$$\tau_q^{-1}=\frac{10}{27\pi\rho_0}\gamma^{-10/3}\int_0^\infty dx\,x^{7/3}f(x)\left[1+0\left(\left(\frac{z}{x}\right)^{1/3}\right)\right]$$

$$=\frac{10}{27\pi}\Gamma(\tfrac{10}{3})\zeta(\tfrac{10}{3})\frac{1}{\rho_0}\left(\frac{T}{\gamma}\right)^{10/3}\left[1+0\left(\left(\frac{\omega_q}{T}\right)^{1/3}\right)\right]. \tag{8}$$

If we substitute into (8), $\Gamma(\frac{10}{3})=\frac{28}{27}(2.679)$ and $\zeta(\frac{10}{3})=1.148$ and reintroduce $\hbar$ and k_B, we obtain the result quoted into (2). The first correction to the leading behavior of τ_q^{-1} given in (2) is $O((\hbar\omega_q/k_BT)^{1/3})$ which would give to $\tau_{q_{-1}}$ a correction term $\propto q^{1/2}T^3$. This correction arises from the z/x expansion of the integrals in the absorption term in (4) and is within the validity of QHD.

III. DISCUSSION

The acoustic ripplon damping can be reexpressed in terms of the attentuation $\alpha_q\equiv Imq$. We write $\omega_q+i\tau_q^{-1}=\gamma(\mathrm{Re}q+i\alpha_q)^{3/2}$, assume $\alpha_q<<\mathrm{Re}q\approx q$, and find

$$\alpha_q=\frac{}{\tau_q}=0.25\frac{\hbar}{\rho_0\gamma q^{1/2}}\left(\frac{k_BT}{\hbar\gamma}\right)^{10/3}, \tag{9}$$

where $v_q=d\omega_q/dq=(\frac{3}{2})\gamma q^{1/2}$ is the ripplon group velocity. Note that in contrast to the case of bulk phonons, the q dependence of the ripplon attenuation α_q differs from the q dependence of the ripplon damping τ_q^{-1}.

We follow the discussion of Pines and Noziéres[10] for bulk phonons and consider the qualitative behavior of the damping of ripplons as a function of frequency. The function $\arg q = \mathrm{Im}q/\mathrm{Re}q$ is a convenient measure of the effectiveness of the damping. The collisionless regime for which $\arg q \sim 1/\omega\tau_q$ can be divided into two parts corresponding to the different collisions mechanics. Saam[6] has already considered the decay regime, $\hbar\omega >> k_B T$, in which a ripplon decays into a pair of ripplons and finds that

$$\arg q = 0.11(\hbar/\rho_0)\gamma^{-10/3}\omega^{7/3}.$$

For $\hbar\omega << k_B T$, we are in the thermal regime in which an acoustic ripplon is absorbed by a thermal ripplon. From (9) we find in this limit that

$$\arg q = 0.25(\hbar/\rho_0)(k_B T/\hbar\gamma)^{10/3}\omega^{-1} .$$

As the frequency is reduced we pass from the collisionless to the hydrodynamic regime. We follow Landau and Lifshitz[10] and find that in the limit of low frequencies,

$$\arg q = (4\eta/3\rho_0)\gamma^{-4/3}\omega^{1/3} , \tag{10}$$

where η is the (bulk) shear viscosity. We see that for low-ω ripplons, $\arg q$ approaches zero with infinite slope, in contrast to the case of sound in a Fermi liquid[9] for which $\arg q \sim \omega$ for small ω.

The above behavior of $\arg q$ implies that the "window" for well-defined ripplons ($\arg q << 1$) should occur at intermediate frequencies (or wave vectors) between the high-frequency ripplon decay side and the low-frequency viscous damping side. We estimate the range of the window for well-defined ripplons by requiring $\omega_q\tau_q(T) >> 1$ with $\tau_q(T)$ due to absorption processes and by requiring $\omega_q\tau_q(0) >> 1$ with $\tau_q(0)$ due to the decay process. If we use (2),(3), and substitute[2,3] $\rho_0 = 0.146$ g/cm^3 and $\sigma_0 = 0.37$ erg/cm^2, we find that ripplons are well defined for

$$q_t < q < q_d , \tag{11}$$

where (T is in degrees Kelvin)

$$q_t \approx 0.09 T^{20/9} \text{ Å}^{-1} \tag{12}$$

and

$$q_d \approx 1 \text{ Å}^{-1} . \tag{13}$$

At $T = 1$ K, $q_t = 0.09$ Å^{-1}; at $T = 0.1$ K, $q_t = 0.0005$ Å^{-1}. From (11) and (12) we see that at a given T, long-wavelength ripplons[11] are not well defined. Note that the same requirement, i.e., $\omega_q\tau_q(T) >> 1$, for bulk phonons would imply that long-wavelength phonons are well-defined since[7] $\omega_q \propto q$ and $\tau_q^{-1} \propto qT^4$ for small q.

The estimate of the range given in (11) is made somewhat uncertain by the extension of the spectrum (1) to shorter wavelengths and by the fact that our result (2) for τ_q is restricted to acoustic ripplons for which $\hbar\omega_q << k_B T$ or

$$q << q_a \approx (\tfrac{1}{12} T^{2/3}) \text{ Å}^{-1} .$$

The latter restriction can be removed by numerically evaluating the integrals in (4) for arbitrary $\hbar\omega_q/k_B T$ (but $\omega_q\tau_q >> 1$). It is possible that the bulk phonons make an important contribution to the ripplon damping since in the range (11) the frequencies of the bulk phonon and the ripplon are comparable.

Our conclusion is that the ripplons in superfluid ^{4}He at fixed temperatures are not well defined at long wavelengths, but there is a window of "intermediate" values of q for which direct experimental observation of ripplons might be possible. Problems suggested for future work include an investigation of the contribution of bulk phonons to ripplon damping and a study of the contribution of ripplon lifetime effects[12] to the T-dependent surface tension.

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[1]K. R. Atkins, Can. J. Phys. 31, 1165 (1953).

[2]K. R. Atkins and Y. Narahara, Phys. Ref. 138, A437 (1965).

[3]D. O. Edwards, J. R. Eckart, and F. M. Gasparini, Phys Rev. A 9, 2070 (1974).

[4]Arguments have been given to indicate that the coefficient of the $q^{3/2}$ term in (1) is not given by the surface tension. See G. F. Bertsch, Phys. Rev. A 9, 819 (1974).

[5]D. L. Bartley and W. E. Massey, Phys. Rev. B 15, 151 (1977).

[6]W. F. Saam, Phys. Rev. A 8, 1918 (1973). The three-ripplon QHD Hamiltonian H_1 is given in Eq. (12).

[7]C. J. Pethick and D. ter Haar, Physica 32, 1905 (1966).

[8]J. Jäckle, Z. Phys. 231, 362 (1970). The one-loop self-energy diagrams are given in Fig. 3.

[9]D. Pines and P. Noziéres, *The Theory of Quantum Liquids* (Benjamin, New York, 1966), pp. 70–72.

[10]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, Mass., 1959), p. 240.

[11]We are referring to ripplons in the collisionless limit. Since $\arg q \propto \omega^{1/3}$ for collision-dominated ripplons, the window for well-defined ripplons in this limit is small.

[12]V. K. Wong and H. Gould, Phys. Rev. B 14, 3961 (1976).