^{3}He excitations in dilute mixtures with ^{4}He

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(Received 10 November 1977; revised manuscript received 28 April 1978)

The deviation of the ^{3}He quasiparticle spectrum from the Landau-Pomeranchuk parabolic form, seen in neutron scattering, is shown to be accounted for by including the process of roton emission by ^{3}He quasiparticles in a self-consistent calculation. The normal-fluid density and second-sound velocity (as functions of temperature and concentration of ^{3}He), calculated from this spectrum (which exhibits no ^{3}He roton minimum) with no adjustable parameters, are in agreement with experiment.

I. INTRODUCTION

The ^{3}He quasiparticle spectrum in dilute ^{3}He $-$ ^{4}He mixtures was postulated to have a "roton" dip[1-3] as in the case of the ^{4}He excitation spectrum following experimentally observed deviations[4,5] from the predictions of the parabolic Landau-Pomeranchuk (LP) $\epsilon_q = q^2/2m^*$ spectrum. (Factors of $\hbar$ and k_B will often be omitted from equations.) A fit to the normal-fluid density[6] gave the ^{3}He roton parameters as $\Delta_3 = 6.6$ K, $k_3 = 1.95$ Å^{-1}, $\mu_3 = 0.07 m_3$ (m_3 is the mass of the ^{3}He atom). This contention has been challenged recently.[7-9] A microscopic calculation of excitations in mixtures[8] shows negligible shift in the ^{4}He spectrum due to hybridization with the ^{3}He quasiparticle spectrum at small ^{3}He concentration (less than 5-mole%),[10] in agreement with Raman[11,12] and neutron[13,14] measurements, but no dips in the ^{3}He spectrum in second-order perturbation theory. Kummer *et al.*,[9] have measured the second-sound velocity in mixtures up to 1-mole% ^{3}He and conclude that a noticeable discrepancy exists between experiment and the prediction of the LP spectrum, which is increased by the addition of a roton dip to the ^{3}He spectrum.

Recent neutron measurements[14] indicate deviations from the LP spectrum around $q \sim 1.4$ Å^{-1} to lower energy values, but are inconclusive regarding the existence of a ^{3}He roton due to resolution problems beyond 1.7 Å^{-1}. It is the purpose of this paper to show that there is an alternative explanation of the neutron data, which can be reconciled with the normal-fluid density and second-sound results using *no adjustable parameters* and without involving a deep narrow roton minimum.[6]

The basic idea is the following—as the ^{3}He quasiparticle momentum increases, it approaches the threshold for emission of a ^{4}He roton. [At the threshold, obtained by drawing a parabola through the ^{4}He roton with an effective mass $(m^* + \mu_4)$, i.e., the sum of the ^{3}He quasiparticle and ^{4}He roton masses, the quasiparticle emits a roton and continues traveling with it.] The threshold causes a bendover of the ^{3}He spectrum in the same manner as the threshold for decay of a ^{4}He excitation into two rotons.[15] However, the phase space for decay is not as large (the imaginary part of the self-energy has a $(\epsilon - \epsilon_{th})^{1/2}$ behavior), and the bendover is less dramatic.

The vertex for this decay is the same as the vertex for virtual decay of a $q \sim 0$ ^{3}He quasiparticle into a ^{4}He roton and ^{3}He quasiparticle, which dominates the renormalization of the ^{3}He quasiparticle mass.[16] Thus, the renormalized mass (known from experiment) can be used to determine the vertex, and the ^{3}He spectrum can be calculated self-consistently, using a zero temperature Green's-function formalism.

The calculation shows a single quasiparticle pole in the ^{3}He Green's function below a roton quasiparticle continuum for q less than a critical wave vector q_c. As the pole approaches the continuum, it transfers spectral weight to the continuum and ceases to exist as a well-defined excitation beyond q_c. Instead, there is a broad peak in the spectral function $S(q, \omega)$ which may be thought of as a quasiparticle with finite lifetime even at zero temperature and concentration. However, the thermodynamic quantities at low temperature are not sensitive to minor modifications of the spectrum beyond q_c and so it suffices to accurately parametrize the quasiparticle spectrum below q_c (and reasonably beyond) to calculate the normal-fluid density and second-sound velocity to the accuracy of the present experiments.

II. THEORETICAL DETAILS

The Hamiltonian of the system can be written

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}, \tag{1}$$

where

$$\mathcal{H}_0 = \sum_{\vec{k}} \omega_k b^\dagger_{\vec{k}} b_{\vec{k}} + \sum_{q} \epsilon^0_q c^\dagger_{\vec{q}} c_{\vec{q}} \tag{2a}$$

is the noninteracting part ($b^\dagger$ and $c^\dagger$ create ^{4}He excitations and ^{3}He particles, respectively, while ω_k and $\epsilon_q^0 = q^2/2m_3$ are the corresponding energies) and

$$\mathcal{H}_{\rm int} = \sum_{\vec{k}\vec{q}} \gamma_k (b_{\vec{k}}^\dagger + b_{-\vec{k}}) c_{\vec{q}}^\dagger c_{\vec{q}+\vec{k}} \tag{2b}$$

is the interaction between ^{3}He particles and ^{4}He excitations.

One may obtain Eq. (2b) from a bare interaction between ^{3}He and ^{4}He particles; this gives[8]

$$\gamma_k \sim k^2(1-S_k^2)/S_k^{3/2}, \tag{3}$$

where S_k is the ^{4}He liquid-structure factor.[17] Thus, $\gamma_k \sim \sqrt{k}$ for small k, while it has a peak near the roton wave vector. At large k, $S_k \to 1$ and γ_k falls off rapidly.

The exact Green's function for the ^{3}He quasiparticle is given by Dyson's equation [see Fig. 1(a)]

$$G(\vec{q},\omega) = G_0(\vec{q},\omega) + G_0(\vec{q},\omega)\Sigma(\vec{q},\omega)G(\vec{q},\omega), \tag{4}$$

where

$$\Sigma(q,\omega) = i\int \frac{d^3k}{(2\pi)^3}\frac{d\omega'}{(2\pi)} \gamma_k \Gamma_{\vec{k}\vec{q}} D(\vec{k},\omega') G(\vec{q}-\vec{k},\omega-\omega'). \tag{5}$$

Here $\Gamma(\vec{k},\vec{q})$ is the proper renormalized vertex,[18] γ_k is the bare vertex, and the proper self-energy Σ has renormalized ^{3}He quasiparticle and phonon-roton propagators G and D.[19]

$$G_0(\vec{q},\omega) = \frac{\Theta(q-k_F)}{\omega-\epsilon_q^0+i\eta} + \frac{\Theta(k_F-q)}{\omega-\epsilon_q^0-i\eta} \tag{6a}$$

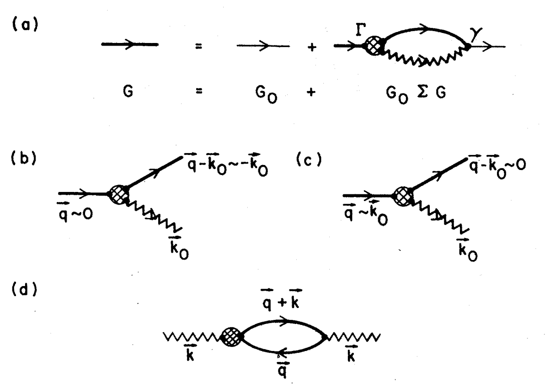

FIG. 1. (a) Dyson's equation; (b) and (c) vertices for $q\sim 0$ and $q \sim k_0$ quasiparticles, respectively; and (d) usual diagram for calculation of hybridization effects. Wavy and straight lines represent ^{4}He and ^{3}He propagators. Renormalized (unrenormalized) propagators are in bold (light).

and

$$D_0(\vec{k},\omega) = \frac{1}{\omega-\omega_k+i\eta} - \frac{1}{\omega+\omega_k-i\eta} \tag{6b}$$

are the unrenormalized (bare) propagators. The factor $\frac{1}{2}\omega_k$ which conventionally appears in D_0 has been incorporated into the definition of the vertices.

For small concentrations of ^{3}He, the renormalization of the ^{4}He spectrum is negligible (~0.2 K for molar concentration $x \sim 5\%$), so $D(\vec{k},\omega)$ may be approximated by $D_0(\vec{k},\omega)$ in the limit $x \to 0$. For the ^{3}He intermediate state [Eq. (5)] we take the form [Eq. (6a)] with ϵ_q^0 replaced by $\epsilon_q = q^2/2m^*$ for (approximate) self-consistency.[20] Within the accuracy of this calculation, it suffices to approximate the ^{4}He spectrum by

$$\omega_k = \begin{cases} ck(\text{phonons}), & k<k_1 = 1.3\ \text{Å}^{-1}, \\ \Delta + (k-k_0)^2/2\mu_4(\text{rotons}), & k_1<k<k_2 = 2.5\ \text{Å}^{-1} \end{cases} \tag{7}$$

where k_1 is the point of intersection of the phonon and roton branches, and k_2 the zero of γ_k after the peak in S_k. $c = 238$m/sec, $\Delta = 8.6$ K, $\mu_4 = 0.16m_4$.

A consideration of the form [Eq. (3)] for γ_k suggests the approximation of one pair of vertices $\gamma_{\rm ph}(k)$ and $\Gamma_{\rm ph}(k) \sim \sqrt{k}$ for phonons, and another γ_r and Γ_r (constants) for rotons. If the ratio of the renormalized vertices $[\Gamma_{\rm ph}(k)/\Gamma_r]$ is of the order of the ratio of the unrenormalized ones $[\gamma_{\rm ph}(k)/\gamma_r]$, most of the ^{3}He mass renormalization turns out to be due to rotons in the intermediate state, so the exact magnitude of the phonon vertex is not crucial. We shall work out results for the two cases,

$$\Gamma_{\rm ph} = 0 \tag{8a}$$

and

$$\Gamma_{\rm ph}(k)/\Gamma_r = \gamma_{\rm ph}(k)/\gamma_r = 2.4k^{1/2}(k \text{ in Å}^{-1}). \tag{8b}$$

In the above discussion, it has been possible to suppress the q dependence (distinct from the dependence on k, the momentum transfer) of Γ, because the q dependence of the self-energy is found to be dominated by rotons in the intermediate state. Thus the vertex Γ for small q, which determines most of the mass renormalization [Fig. 1(b)], and $q \sim k_0$ [Fig. 1(c)], which determines the bendover, are the same[21]; in the intermediate q region, the approximation is less accurate but the results should still be reasonable.

With these approximations, the ^{3}He self-energy in the dilute zero-temperature ($T, \epsilon_F \to 0$) limit becomes the sum of phonon and roton terms,

$$\Sigma(q,\epsilon)=\Sigma_{\rm ph}(q,\epsilon)+\Sigma_{\rm roton}(q,\epsilon)\,, \tag{9}$$

with

$$\Sigma_{\rm ph}(q,\epsilon)=\frac{\gamma_{\rm ph}(q)\Gamma_{\rm ph}(q)}{4\pi^2 q}\int_0^{k_1}k^3\,dk \times\int_{-1}^{+1}\frac{d\cos\theta}{\epsilon-(\vec{q}-\vec{k})^2/2m^*-ck+i\eta}\,, \tag{10a}$$

and

$$\Sigma_{\rm roton}(q,\epsilon)=\frac{\gamma_r\Gamma_r}{4\pi^2}\int_{k_1}^{k_2}k^2\,dk\int_{-1}^{+1}\frac{d\cos\theta}{\epsilon-(\vec{q}-\vec{k})^2/2m^*-\Delta-(k-k_0)^2/2\mu_4+i\eta}\,, \tag{10b}$$

where θ is the angle between $\vec{q}$ and $\vec{k}$, which may be evaluated exactly. We have used $\gamma_{\rm ph}(q)\Gamma_{\rm ph}(k) \sim k$; thus the prefactor $[\gamma_{\rm ph}(q)\Gamma_{\rm ph}(q)/4\pi^2 q]$ in Eq. (10a) is a constant. The imaginary part due to phonons $\Sigma^I_{\rm ph}(q,\epsilon_q)$ vanishes for $q<m^*c$ ($\sim$2.7 Å^{-1}), while that due to rotons becomes nonzero above a threshold energy:

$$\epsilon_{\rm th}(q)=\Delta+(q-k_0)^2/2(m^*+\mu_4)\,, \tag{11}$$

$$\Sigma^I_{\rm roton}(q,\epsilon)=\begin{cases}0, & \epsilon\le\epsilon_{\rm th}\\ -\dfrac{\gamma_r\Gamma_r}{4\pi^2}\,\dfrac{\sqrt{m^*\mu_4}}{(1+\mu_4/m^*)^2}\left(\dfrac{k_0}{q}+\dfrac{\mu_4}{m^*}\right) \times\sqrt{2(m^*+\mu_4)(\epsilon-\epsilon_{\rm th})}\,, & \epsilon>\epsilon_{\rm th}\,.\end{cases} \tag{12}$$

The real part of the self-energy Σ^R is given by

$$\Sigma^R=\Sigma^R_{\rm ph}+\Sigma^R_{\rm roton}\,, \tag{13}$$

with

$$\begin{aligned}\Sigma^R_{\rm ph}(q,\epsilon)=&\left(\frac{\gamma_{\rm ph}(q)\Gamma_{\rm ph}(q)}{4\pi^2 q}\right)\frac{m^*}{q}\Bigg[\tfrac{1}{3}\Bigg(k_1^3\ln\left|\frac{m^*c+\frac12 k_1-q}{m^*c+\frac12 k_1+q}\right|+8(m^*c-q)^3\ln\left|\frac{m^*c+\frac12 k_1-q}{m^*c-q}\right|8(m^*c+q)^3\ln\left|\frac{m^*c+\frac12 k_1+q}{m^*c+q}\right|\\ &-2qk_1^2+16m^*cqk_1\ \ +4m^*\left(\epsilon-\frac{q^2}{2m^*}\right)\Bigg((m^*c-q)\ln\left|\frac{m^*c+\frac12 k_1-q}{m^*c-q}\right|\\ &-(m^*c+q)\ln\left|\frac{m^*c+\frac12 k_1+q}{m^*c+q}\right|\Bigg)+O\left(\left(\frac{\epsilon-q^2}{2m^*}\right)^2\right)\Bigg],\end{aligned} \tag{14a}$$

$$\begin{aligned}\Sigma^R_{\rm roton}(q,\epsilon)=&\left(\frac{\gamma_r\Gamma_r}{4\pi^2}\right)\frac{m^*}{q}\Bigg[(\tfrac12 k_2^2+\mu\Omega)\ln\left|\frac{k_2^2-2\mu c_1k_2+2\mu\Omega}{k_2^2-2\mu c_2k_2+2\mu\Omega}\right|-(\tfrac12 k_1^2+\mu\Omega)\ln\left|\frac{k_1^2-2\mu c_1k_1+2\mu\Omega}{k_1^2-2\mu c_2k_1+2\mu\Omega}\right|\\ &-\mu^2c_1^2\ln\left|\frac{k_2^2-2\mu c_1k_2+2\mu\Omega}{k_1^2-2\mu c_1k_1+2\mu\Omega}\right|+\mu^2c_2^2\ln\left|\frac{k_2^2-2\mu c_2k_2+2\mu\Omega}{k_1^2-2\mu c_2k_1+2\mu\Omega}\right|+\frac{2\mu q}{m^*}(k_2-k_1)\\ &+2\mu c_1(2\mu\Omega-\mu^2c_1^2)^{1/2}\left(\tan^{-1}\frac{k_2-\mu c_1}{(2\mu\Omega-\mu^2c_1^2)^{1/2}}-\tan^{-1}\frac{k_1-\mu c_1}{(2\mu\Omega-\mu^2c_1^2)^{1/2}}\right)\\ &-2\mu c_2(2\mu\Omega-\mu^2c_2^2)^{1/2}\left(\tan^{-1}\frac{k_2-\mu c_2}{(2\mu\Omega-\mu^2c_2^2)^{1/2}}-\tan^{-1}\frac{k_1-\mu c_2}{(2\mu\Omega-\mu^2c_2^2)^{1/2}}\right)\Bigg]\,,\end{aligned} \tag{14b}$$

where $c_1=k_0/\mu_4-q/m^*$, $c_2=k_0/\mu_4+q/m^*$, $\mu=\mu_4 m^*/(\mu_4+m^*)$, and $\Omega=\Delta+k_0^2/2\mu_4-\epsilon+q^2/2m^*$. In Eq. (14a), the self-energy has been truncated at the lowest-order term in deviations from the parabolic form which is accurate to the same degree as replacement of the intermediate ^{3}He state by the LP form. Inclusion of the $O((\epsilon-q^2/2m^*)^2)$ term leads to changes which are much smaller than the differences between the results for the two approximations in Eq. (8). The factor inside the square root, in the last term in Eq. (14b), is proportional to $(\epsilon_{\rm th}-\epsilon)$, and thus $\Sigma^R_{\rm roton}$ has a $\sqrt{\epsilon_{\rm th}-\epsilon}$ term, as expected from the result for the imaginary part [Eq. (12)]. [Also note that ϵ as it appears on the right-hand side of Eqs. (14) is measured from the renormalized zero of energy.]

For wave vectors q less than a certain value q_c [such that the renormalized quasiparticle energy determined self-consistently from Eq. (15) lies below $\epsilon_{\rm th}(q)$ defined in Eq. (11)], the renormalized Green's function has a pole at an energy given by

$$\epsilon_q=q^2/2m_3+\Sigma^R(q,\epsilon_q)-\Sigma^R(0,0)\,, \tag{15}$$

where an uninteresting constant shift has been removed. As expected, as q approaches q_c, the term $\sqrt{\epsilon_{\rm th}(q)-\epsilon_q}$ in $\Sigma^R_{\rm roton}$ arising from the emission of rotons by ^{3}He quasiparticles causes a bendover of the single quasiparticle pole in the Green's function. However, the (finite) bendover term cannot sustain a pole below $\epsilon_{th}(q)$ beyond a critical wave vector q_c, and the single quasiparticle pole merges into the roton-quasiparticle continuum, developing a finite line width even at zero temperature and concentration. Using the $\sqrt{\epsilon_{th}-\epsilon}$ nature of the bendover term in Σ^R, the quasiparticle pole may be shown to approach $\epsilon_{th}(q)$ tangentially, with a spectral weight $Z_q \sim (1+d\Sigma/d\epsilon)^{-1}_{\epsilon=\epsilon_q}$ decreasing linearly to zero as $q \to q_c$.

The above results indicate that shifts in the ^{4}He spectrum near the roton minimum at larger x, due to hybridization with the ^{3}He quasiparticle branch [Fig. 1(d)], cannot simply be calculated by using $q^2/2m^*$ or even ϵ_q in the intermediate ^{3}He state, but must properly take into account finite lifetime effects of the ^{3}He intermediate state. Thus, the shift of spectral weight to higher energies would tend to cancel level repulsion from the quasiparticle pole.

Before going on to comparison with experiment, we note that for finite concentrations x, the second term on the right-hand side of Eq. (6a) should be used instead of the first for quasiparticles with $q<k_F$. This leads to, in addition to a small fractional increase of the Fermi velocity of order $k_F^3 \propto x^2$, a rather weak singularity at the Fermi surface (for $T=0$) of the form $(1-q/k_F)^3 \ln|1-q/k_F|$ of the same order of magnitude. However, this would be washed away at finite temperature and probably not be noticeable at least for the weaker solutions.

III. RESULTS AND COMPARISON WITH EXPERIMENT

For a given ratio of phonon and roton coupling constants [Eqs. (8a) and (8b)], the renormalized spectrum depends on only one parameter—the product $\gamma_r\Gamma_r$. Determining that from the experimental renormalization of the ^{3}He mass ($m_3 \to 2.3m_3$) separately in each case [Eqs. (8a) and (8b)], the ^{3}He quasiparticle spectrum is obtained by solving the transcendental equation [Eq. (15)] on a computer. The results are shown in Fig. 2 for the two cases $\Gamma_{ph}=0$ and $\Gamma_{ph}/\gamma_{ph}=\Gamma_r/\gamma_r$, along with the neutron results. The magnitude of the bendover calculated is seen to be well within the experimental results, while the pure parabolic spectrum is not. Comparison is limited by the resolution of the neutron data and by the width $4\sqrt{\epsilon_F\epsilon_q}$ of the quasiparticle hole continuum which would be present at finite ^{3}He concentration, even in an ideal neutron experiment.

In order to evaluate the normal-fluid density and second-sound velocity in the dilute mixtures,

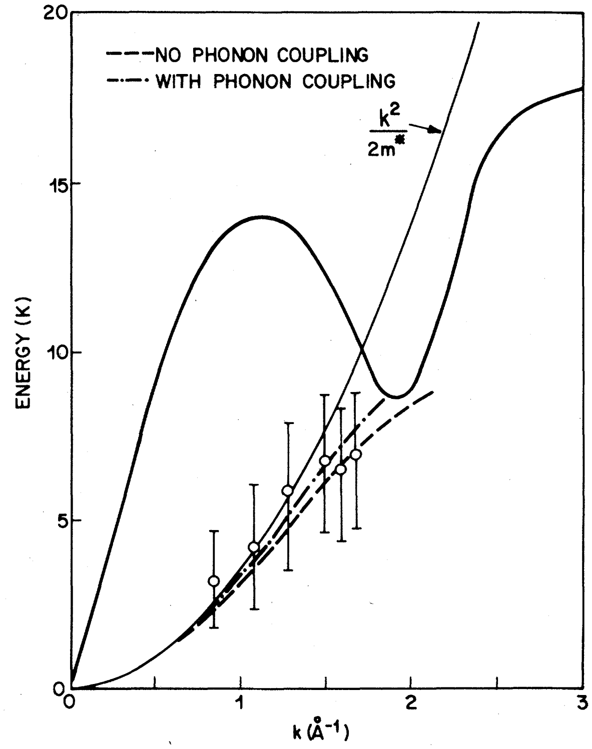

FIG. 2. Dashed and dot-dashed curves are the ^{3}He quasiparticle spectrum calculated using approximation discussed in text. Also shown are the LP parabolic curve and the spectrum of pure ^{4}He. The open circles are neutron results on $x=6$-mole% solution.

we parametrize the ^{3}He quasiparticle spectrum by

$$\epsilon_q = \begin{cases} (q^2/2m^*)(1-q^2/Q^2), & q<q_c \\ (q^2/2m^*)(1-q_c^2/Q^2), & q>q_c \end{cases}, \tag{16}$$

where q_c is the point where the spectrum intersects $\epsilon_{th}(q)$, and Q determines the deviation of the spectrum from the parabolic form below q_c. The expression [Eq. (16)] (for $q<q_c$) constitutes the first two terms in an expansion of ϵ_q in powers of q^2 and is found to be a surprisingly accurate approximation of the deviation of calculated spectrum over the relevant range of momenta (0.5 Å$^{-1}<q<1.7$ Å^{-1}).[22] The form for $q>q_c$ (chosen to coincide at q_c) is also a reasonable approximation to the damped mode above q_c, and results are not sensitive to minor variations of the form selected. Besides, the one-parameter (Q) form appears to be entirely adequate within the accuracy of the present theory (cf. the differences in the spectrum calculated for the two cases in Fig. 2), and the scatter and accuracy of the present experimental data. Fitting the two cases in Fig. 2 gives $Q=3.2$ and 3.8 Å^{-1}.

The normal-fluid density and specific heat due to the ^{3}He quasiparticles are given in terms of the usual phase space integrals[23]

$$\rho_{n3}=\tfrac{1}{3}\int q^2\left(-\frac{\partial n}{\partial\epsilon}\right)d\tau_q \tag{17}$$

and

$$C_v\equiv T\left(\frac{dS}{dT}\right)_N=\left(\frac{dE}{dT}\right)_N, \tag{18}$$

where

$$E=\int \epsilon n\, d\tau_q \tag{19a}$$

and

$$N=\int n\, d\tau_q. \tag{19b}$$

Here $n=[\exp(\epsilon-\mu)/T+1]^{-1}$ is the Fermi distribution and $\epsilon(q)$ is the excitation spectrum. For the Landau-Pomeranchuk form $\epsilon(q)=q^2/2m^*$, the results are

$$\rho_{n3}=m^*n_3 \tag{20a}$$

and

$$C_v=\tfrac{3}{2}n_3k_B\,,\quad (T\gg\epsilon_F)\,, \tag{20b}$$

where n_3 is the number of ^{3}He atoms per unit volume and ϵ_F is the Fermi energy.

For a spectrum with a ^{3}He roton dip

$$\epsilon(q)=\Delta_3+(q-k_3)^2/2\mu_3\,, \tag{21}$$

in addition to the parabolic form, one obtains [to $O(e^{-\Delta_3/T})$]

$$\rho_{n3}\simeq m^*n_3\left[1+\alpha\left(\frac{k_3^2}{2m^*T}-\frac{3}{2}\right)\right],\quad \epsilon_F\lesssim T\ll\Delta_3\,, \tag{22a}$$

$$C_{v3}\simeq\tfrac{3}{2}n_3k_B\left[1+\alpha\left(\frac{\Delta_3^2}{T^2}-\frac{2\Delta_3}{T}\right)\right],\quad \epsilon_F\ll T\ll\Delta_3\,, \tag{22b}$$

where

$$\alpha=(2\hbar^2k_3^2/3m^*k_BT)\sqrt{\mu_3/m^*}\,e^{-\Delta_3/T}. \tag{23}$$

In obtaining the above equations, the ^{3}He quasiparticle number has been conserved, and therefore the "roton" contributions to Eqs. (22) are not the same as for ^{4}He. In addition, while the expression for specific heat [Eq. (22b)] is true only for $T\gg\epsilon_F$, the result for normal-fluid density [Eq. (22a)] is very good even at low temperatures because Eq. (20a) is exact for the LP spectrum for all temperatures, and so errors in Eq. (22a) are $O(e^{-\Delta_3/\epsilon_F})$.

For the spectrum [Eq. (16)], one may work out expressions by expanding the spectrum about a parabolic form in order to evaluate thermal integrals. This amounts to a power series expansion in T/T_0, where $T_0=Q^2/2m^*$; however, because of the spectrum given by the first part of Eq. (16) bends over at large q, one cannot let q_c go to infinity and the analytical result is not particularly advantageous with finite q_c. Therefore, the normal-fluid density and specific heat have been evaluated numerically using Eqs. (17)–(19), for the spectrum given by Eq. (16).

The second-sound velocity for dilute mixtures is given by[23]:

$$u_{\rm II}^2=\frac{\rho_sT}{\rho_n}\left[\frac{(\sigma_4+k_Bx/m_4)^2}{C}+\frac{k_Bx}{m_4}\right], \tag{24}$$

where ρ_n and ρ_s are the total normal-fluid and superfluid densities, σ_4 is the ^{4}He entropy, and C is the total specific heat per unit mass of the solution, and x is the molar concentration of ^{3}He.

Turning now to the normal-fluid density, we fit the data[4] for $x=11\%$ using the normal-fluid density for ^{4}He from Bendt *et al.*[24] for all three spectra [Eq. (16), LP, and LP plus Eq. (21)]. In each case, m^* is determined from the data at the lower temperatures. The data are seen to be fit as well (Fig. 3) by the spectrum (16) with $Q=3.4$ Å^{-1} and $m^*=2.1m_3$ as the three-parameter roton fit ($\Delta_3=6.6K$, $k_3=1.95$ Å^{-1}, $\mu_3=0.07m_3$) with $m^*=2.25m_3$. The Landau-Pomeranchuk spectrum, however, underestimates the normal-fluid density at the highest temperatures by about 20% of the zero-temperature ρ_{n3}.

In Fig. 4(a), we compare the second-sound velocity determined experimentally[9] for the $x=0.71\%$ solution with the predictions [from Eq. (24)] of the three spectra—parabolic ($m^*=2.4m_3$), parabolic plus roton ($m^*=2.4m_3$, roton parameters as above), and Eq. (16) with $Q=3.4$ Å^{-1} and $m^*=2.25m_3$ using pure ^{4}He data again from Ref. 24. The renormalized masses are again determined by fitting the data at the lowest temperatures and for each spectrum are somewhat higher (by 0.15 m_3) than the previous case. This is not surprising because m^* in general depends on the concentration [through the concentration dependence of $\Sigma^R(q,\epsilon)$ which we have neglected in the present calculation, and also through ^{3}He – ^{3}He interaction omitted in the present model] and a decrease of m^* with increasing concentration has been found experimentally, too.[6,26] (In addition, any inaccuracy in the determination of x gets lumped into a renormalization of m^* for ρ_{n3}.) As can be seen, Eq. (16) yields a better fit than the parabolic plus roton spectrum, which in turn is in better agreement with experiment than the pure parabolic spectrum.[27] Figure 4(b) shows the fits using Eq. (16) with the same parameters as Fig. 4(a), for different concentrations. The small deviations

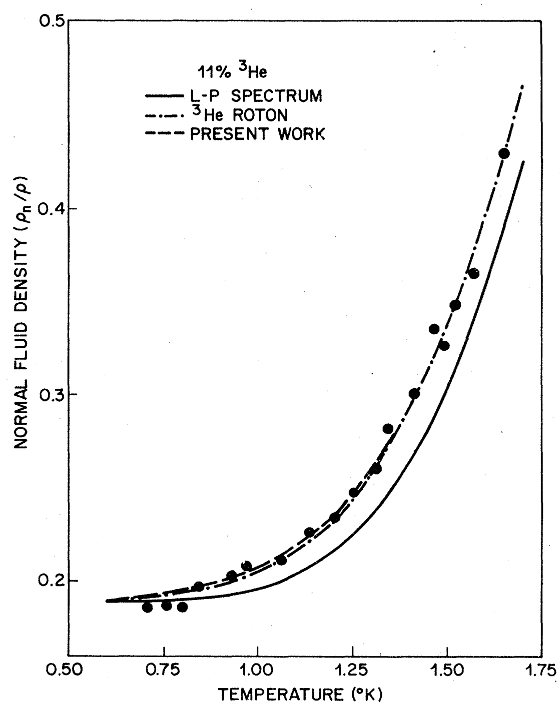

FIG. 3. Total normal-fluid density for $x = 11$-mole% (data of Ref. 4) along with calculations for various forms of the ^{3}He quasiparticle spectrum. *Where the calculation for Eq. (16) is not shown, it agrees with the roton spectrum result to within the width of the lines.*

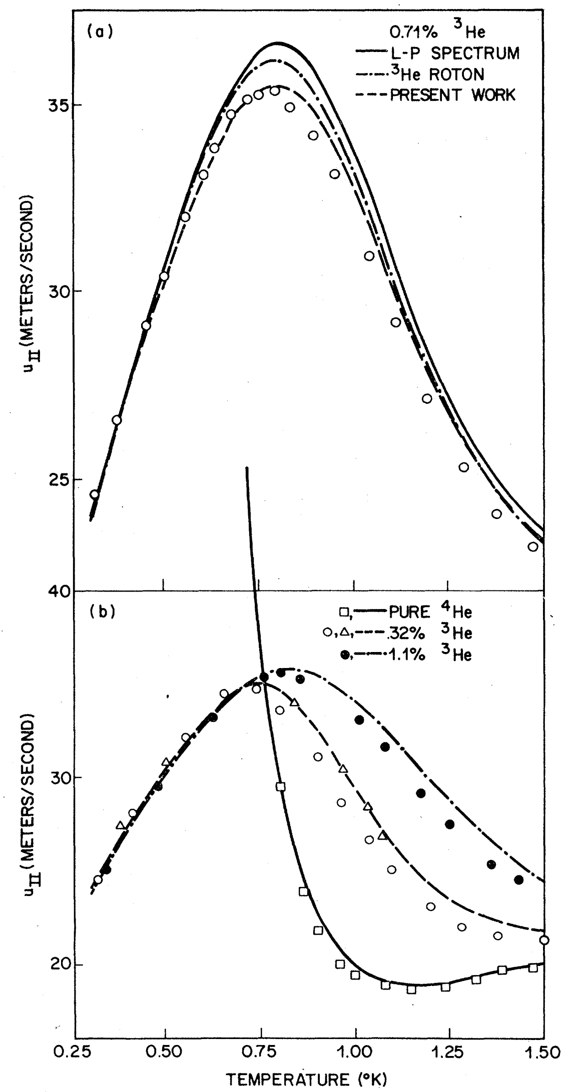

FIG. 4. Second-sound velocity obtained from Eq. (24) (a) in $x = 0.71$-mole% solution using different forms of the quasiparticle spectrum; (b) for different concentrations using the spectrum (9). Data are from Ref. 9 (except triangles from Ref. 25).

above 1 K are not experimentally significant and do not scale with x (as would be expected for errors in the ^{3}He spectrum or shift of the ^{4}He spectrum due to ^{3}He). We also note that previous second-sound velocity measurements below 0.6 K have been fitted there[26] using Eq. (16) with $Q = 4.0 \pm 0.8$ Å^{-1}.

IV. CONCLUSIONS

The present work has shown that the deviation of the ^{3}He spectrum in dilute ^{3}He $-$ ^{4}He mixtures from the Landau-Pomeranchuk parabolic form, seen in neutron scattering, is in agreement with a theoretical estimate of the bendover due to the decay channel $^3\mathrm{He} \to {}^3\mathrm{He} + \text{rotons}$ using a rather simple picture of interaction between ^{3}He and ^{4}He. Further, by parametrizing the theoretically calculated spectrum in terms of a single bendover parameter Q, which seems to be adequate to within the accuracy of both experiment and theory at present, the calculated normal-fluid density and second-sound velocity (as functions of temperature and concentration of ^{3}He) are found to be in agreement with experiment, with essentially no adjustable parameters.

No *a priori* claim can, of course, be made about quantitative accuracy of the theoretical calculation in view of the original simplifying assumptions about the vertices. However, the essential physics and qualitative nature of the bendover are, we believe, correct and, in view of the remarkable agreement with experiment, a quantitative first approximation. Further refinement of this model at this stage, though, would be little more than a parameter game because of lack of any experimental evidence requiring that and because of the somewhat intractable nature of calculations in a strongly coupled system like the ^{3}He $-$ ^{4}He mixtures. On the other hand, both the absence of any pronounced ^{3}He roton minimum in the calculation, and the amenability of present experimental data to explanation in terms of a spectrum without the minimum, provide incentive for further, more accurate, experimental work on, say, normal-fluid density and specific heat, to determine the quantitative details of the deviation of the ^{3}He quasiparticle spectrum in mixtures from the parabolic form.

Note added in proof. Recently, after this work was submitted for publication, Greywall has done high-precision specific-heat measurements[28] on dilute ^{3}He-^{4}He mixtures ($x = 0.004$–1.0 mole %). His results confirm that there are substantial deviations from the results of the LP parabolic spectrum (which increase with temperature, extrapolating to about 15% at 1 K). The data can be fit in terms of a roton only with unreasonable, concentration-dependent parameters ($\Delta_3 \sim 2.5$–3.5 K, $\mu_3 \sim 10^{-3}$–$10^{-4} m_3$). The deviations are in approximate quantitative agreement with the results based on the spectrum (16) with the present parameters, above 0.3 K. The data do, however, show a small kneelike feature (magnitude ~ 1%) in the specific heat versus temperature plot around 0.2 K, which is not present in the specific heat computed from the parametrized form in Eq. (16).

ACKNOWLEDGMENTS

This work was started while the author was at the Aspen Center for Physics. Helpful discussions with C. M. Varma, R. C. Dynes, V. Narayanamurti, and P. C. Hohenberg are gratefully acknowledged.

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[16]This result actually follows from the calculation, assuming that the ratio of the renormalized vertices for the phonon and roton is not much greater than that for the unrenormalized ones.

[17]This depends on identifying the ^{4}He excitations as density fluctuations, in the spirit of Feynman [Phys. Rev. 94, 262 (1954)] and Bogoliubov-Zubarev {Zh. Eksp. Teor. Fiz. 28, 129 (1955) [Sov. Phys. JETP 1, 83 (1955)]}, so higher-order interactions are left out.

[18]Vertex corrections are not of order of the Fermi momentum k_F, as stated in Ref. 8, and cannot be assumed to be negligible. Thus Eq. (3) should be used only for qualitative trends as regards $\Gamma(\vec{k},\vec{q})$.

[19]A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971), pp. 402–405.

[20]Actually, a constant should be added to $q^2/2m^*$, due to an overall shift of the spectrum, but since only energy differences enter the calculation of $\Sigma(q, \epsilon_q)$, the constant drops out and may be suppressed to begin with.

[21]This assumes that the q dependence of the vertex at small q is dominated by that of the self-energy. There is no *a priori* justification of this, except that using bare vertices gives 80% of the mass renormalization at small q (Ref. 8), and the approximation $\Gamma(\vec{k},\vec{q}) = \Gamma(k) \neq \gamma_k$ should do better. Besides the agreement with experiment could be viewed as *a posteriori* justification.

[22]It does slightly overestimate the deviation at small q and underestimate that at larger q. This would imply somewhat lesser deviation from the LP form at lower temperatures and larger at higher temperatures for the calculated spectrum than that given by Eq. (16).

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