Renormalization-group approach for critical percolation behavior in two dimensions

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A renormalization-group approach using a scaling transformation in real space is applied to the critical behavior of two-dimensional percolation systems. In various approximations for the triangular-site lattice and the square-bond lattice, the location of the fixed point and the correlation-length exponent are calculated by determining the behavior of the probabilities under a scale transformation. The fixed points for all approximations for the two lattices are found to be in complete agreement with known exact results for the critical percolation probability.

INTRODUCTION

Critical percolation behavior has been studied recently by renormalization-group approaches.[1-5] We present a new method of renormalization-group approach for calculating the critical behavior of two-dimensional percolating systems using a scaling procedure in real space. The method is applied to the triangular-site lattice and the square-bond lattice and gives good results for the correlation-length index ν by a linearized transformation about the fixed point. The fixed points for these two lattices are in complete agreement with known exact results[6] for the critical percolation probability $p_c = \frac{1}{2}$.

If the elements (sites or bonds) in the original lattice are independently occupied with probability p, which in the case of the elements will scale into a transformed probability p' of a single element on the new lattice and will determine the scaling relationship of the transformation via

$$p' = R(p), \tag{1}$$

Then a fixed point p^* is determined by

$$p^* = R(p^*). \tag{2}$$

A nontrivial fixed point gives an approximation for the critical percolation probability p_c.

The linearized form R^L of the renormalization-group transformation near the fixed point has eigenvalues λ_i with $\lambda_1 > 1 > \lambda_2 \cdots$. The correlation length index ν is then given by

$$b^{1/\nu} = \lambda_1, \tag{3}$$

where b is the change in length scale.

Cluster approximations for the triangular site and for the square-bond lattices will be worked out for an original cluster group consisting of four, nine, and sixteen elements (sites or bonds). We discuss each approximation for the two lattices separately.

TRIANGULAR LATTICE

Four-cluster approximation

The original triangular lattice of Fig. 1(a) is scaled into the new lattice of Fig. 1(b) with a scale factor $b = 2$: the original cluster-group sites 1-4 (2×2 sites) scales into a single site I with new probability p'. The transformed probability p' is defined as the probability that the cluster group is conductive when it is sandwiched between two plane electrodes A and B made of perfect conductors. The combinations of paths contributing to the transformed probability distribution under the transformation are shown in

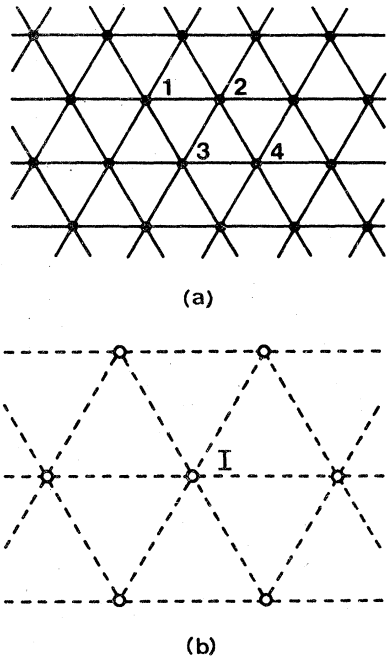

FIG. 1. Transformation of the triangular-site lattice: (a) original lattice; (b) the transformed lattice; with scale factor $b = 2$.

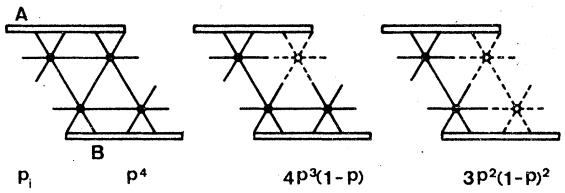

FIG. 2. Combinations of paths conducting between two plane electrodes A and B. Each black site with full lines and each white site with broken lines represent sites occupied with probability p and $1-p$, respectively.

Fig. 2. In that diagram, each conducting site has been connected with full lines; the white sites connected with broken lines represent insulating sites.

The sum of the probabilities p_i of these graphs gives the transformed p' as

$$p' = \sum_i p_i = 3p^2 - 2p^3 , \tag{4}$$

where the fixed-point value of p is found to be

$$p^* = \tfrac{1}{2} . \tag{5}$$

This result for p^* is in complete agreement with the known exact value[6] for the critical percolation probability of the triangular-site lattice. Transformation (4) has an eigenvalue $\lambda_1 = \frac{3}{2}$. Thus the correlation length index ν is given by

$$\nu = 1.7095 . \tag{6}$$

Nine-cluster approximation

As in the four-cluster approximation, the probability p of 3×3 sites connections on the original lattice transforms into the probability p' for connection between two plane electrodes. Considering the possible connections and combining their probabilities using the exclusion-inclusion principle, we can obtain the probability of the single site on new lattices as

$$p' = 8p^3 - 6p^4 - 6p^5 + 12p^7 - 9p^8 + 2p^9 . \tag{7}$$

The fixed-point value is again $p^* = \frac{1}{2}$. This gives an eigenvalue $\lambda_1 = 1.9453$ and putting $b = 3$ in (3), we find $\nu = 1.6510$.

Sixteen-cluster approximation

The 16 sites are arranged in a diamond shape forming a 4×4 lattice. In a similar way, we can obtain the transformed probability p' from combinations of paths conducting between two plane electrodes. The result is

$$p' = 20p^4 - 12p^5 - 30p^6 - 20p^7 + 45p^8 + 100p^9 - 66p^{10} - 300p^{11} + 520p^{12} - 360p^{13} + 120p^{14} - 16p^{15} . \tag{8}$$

Since transformation (8), when linearized about its fixed point $p^* = \frac{1}{2}$, has the eigenvalue $\lambda_1 = 2.3486$, we obtain $\nu = 1.6236$.

SQUARE LATTICE

Similar calculations may be carried out for the square bond lattice using the corresponding transformations.

Four-cluster approximation

Consider the transformation of the square lattice by the scale factor $b = \sqrt{2}$ obtained by rotating the axes by $\frac{1}{4}\pi$. The transformation is illustrated in Fig. 3, in which two types of the basic scaling procedure are displayed. These illustrate the only procedures that scale into a new lattice without destroying the symmetry of lattice. Let p'_1 and p'_2 denote the transformed probabilities for the two basic illustrations shown in Figs. 3(a) and (b), respectively. The transformed p' is then defined from their arithmetic mean, namely,

$$p' = \tfrac{1}{2}(p'_1 + p'_2) = 3p^2 - 2p^3 . \tag{9}$$

This transformation is equivalent to Eq. (5) for the four-cluster approximation on the triangular lattice. Similarly we can obtain the fixed point $p^* = \frac{1}{2}$ which is known to be exact value[6] of the critical percolation probability for the square bond lattice. The eigenvalue of the scaling transformation (9) is $\lambda_1 = \frac{3}{2}$, from which we find $\nu = 0.8548$.

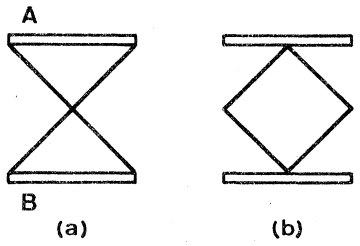

FIG. 3. Transformation of the square-bond lattice by the scale factor $b = \sqrt{2}$, showing the cluster group of four bonds sandwiched between two plane electrodes A and B. The two illustrations represent the only procedures that scale into the new lattice with preservation of the symmetry of the lattice. The transformed probability is defined from the arithmetic mean of the probabilities for these two scalings.

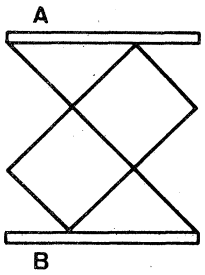

FIG. 4. Transformation for a cluster group of 3×3 bonds. The scale factor is $b=3/\sqrt{2}$.

Nine-cluster approximation

Only the simple system of nine bonds with probability p is illustrated in Fig. 4. From the connections of 3×3 bonds between two plane electrodes, the scaling transformation of the probability p' is obtained and the result is equivalent to Eq. (7). The fixed point value is $p^*=\frac{1}{2}$ for the square lattice. This gives an eigenvalue $\lambda_1=1.9453$, and, substituting $b=3/\sqrt{2}$ into (3), we find $\nu=1.1302$.

Sixteen-cluster approximation

A more intricate scaling transformation is possible using the original cluster group shown in Fig. 5. For this cluster approximation, there are also only two types of basic scaling procedure of the cluster group consisting of 16 bonds. The arithmetic mean of the transformed probabilities for those scalings gives the transformed probability

$$p'=21p^4-18p^5-16p^6-36p^7+54p^8+98p^9-66p^{10}-300p^{11}+520p^{12}-360p^{13}+120p^{14}-16p^{15}, \quad (10)$$

which has the fixed point $p^*=\frac{1}{2}$. From the corresponding eigenvalue $\lambda_1=2.3408$ and the scale factor $b=2\sqrt{2}$, we obtain the correlation length index $\nu=1.2225$.

We have treated the various approximations for the critical behavior of two-dimensional percolating systems for two lattices by a renormalization-group approach. The results for the eigenvalue λ_1, the fixed point p^*, and the correlation length index ν of all the transformations are summarized in Table I.

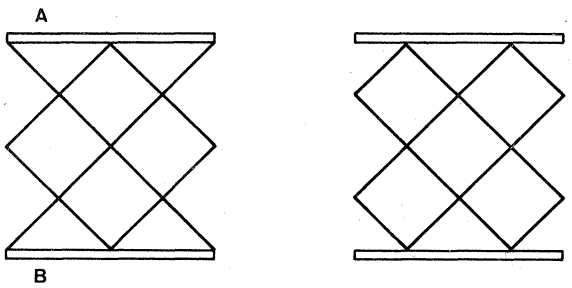

FIG. 5. Transformation of the square-bond lattice with $b=2\sqrt{2}$. Two basic scaling procedures for the cluster group consisting of 16 bonds are illustrated.

TABLE I. Results of cluster approximations.

No. of elements	Scale factor	Eigenvalue	Correlation length index	Fixed point
		Triangular-site lattice		
4	2	$\frac{3}{2}$	1.7095	$\frac{1}{2}$
9	3	1.9453	1.6510	$\frac{1}{2}$
16	4	2.3486	1.6236	$\frac{1}{2}$
		Square-bond lattice		
4	$\sqrt{2}$	$\frac{3}{2}$	0.8548	$\frac{1}{2}$
9	$3/\sqrt{2}$	1.9453	1.1302	$\frac{1}{2}$
16	$2\sqrt{2}$	2.3408	1.2225	$\frac{1}{2}$

The most remarkable feature is that the nontrivial fixed points of all these approximations are in complete agreement with known exact results[6] for the critical percolation probability p_c of the triangular site and the square bond lattices. Note that the fixed points for both lattices have the same value $\frac{1}{2}$ for all three cases of the present cluster approximation. This suggests that this renormalization-group transformation will always give the same fixed point even if the cluster group becomes still bigger.

The approximate critical exponent for the triangular lattice becomes smaller as the scale factor b becomes larger. In contrast, the approximate critical exponent of square lattice becomes larger as the scale factor becomes larger. The increase and decrease of the critical-exponent estimates are nearly linear in b^{-2} within these cluster approximations. If we assume the linear dependence of ν on b^{-2} for large b, we can estimate a value of the correlation length index for $b\to\infty$; this yields $\nu=1.595$ for the triangular lattice and $\nu=1.348$ for the square lattice. The result for the critical exponent ν of square lattice for the 4×4 cluster approximation is in good agreement with the value 1.34 ± 0.02.[7] The estimated value $\nu=1.348$ for $b\to\infty$ is also in excellent agreement with this value; for the triangular site lattice these are in fair agreement with the result[8-10] 1.34 ± 0.05 obtained by the scaling relation.

Any bond-percolation problem is equivalent to a site problem on the corresponding covering lattice.[2,11] Since the present scaling transformations for the triangular site and the square bond

lattices have almost the same symmetry and eigenvalues and the same fixed points, some critical percolation behavior of the square bond problem can be described in terms of the triangular site problem. Similarly, we can expect to analyze the site problem on the kagomé lattice by writing it as a bond problem on the honeycomb lattice, using an adequate scaling transformation in real space.

The present renormalization-group approach to the site and bond problems has been shown to give the exact results for the critical percolation probability. We now describe a method for constructing adequate clusters and preserving the symmetry of the lattice. In all the transformations, the elements (sites or bonds) which form the original cluster group are arranged in diamond shape with side length equal to the scale factor. Generally, the elements must be constructed in convex polygon shape with side length equal to the scale factor and having symmetry with respect to diagonals. Suppose that the original cluster group based on such a polygon is sandwiched between two plane electrodes made of perfect conductors. Then the transformed probability is determined as the probability that the cluster group is conductive. We will note that the original cluster group with elements of sites or bonds rescales into the new lattice without destroying the symmetry of lattice. Thus the renormalization-group transformation should be so scaled that the transformed parameter has the symmetry for the lattice. Finally, we mention that the present procedure for employing the renormalization-group approach should be very promising for the study of other critical percolation behavior.

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