PHYSICAL REVIEW B VOLUME 18, NUMBER 3 1 AUGUST 1978

First-order transition induced by cubic anisotropy

Joseph Rudnick

Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106

(Received 28 April 1977)

A two-component spin system with cubic anisotropy sufficiently large to remove the renormalization-group fixed point is investigated using differential recursion relations. An explicit free energy is constructed and it is verified explicitly that the loss of a fixed point corresponds to a first-order transition to the system's ordered state. A tricritical point is located and scaling properties of the first-order transition are investigated close to it.

I. INTRODUCTION

A system of some interest in the current theory of critical phenomena is an n-component spin system with cubic anisotropy in the spin Hamiltonian. This kind of anisotropy represents the influence of cubic lattice structure on the localized moments of a magnetic insulator. Cubic anisotropy can also appear in the effective Hamiltonian of a lattice undergoing a structural phase transition.[1] Renormalization-group calculations[2-6] indicate that cubic anisotropy is "relevant" in a three-dimensional spin system with n, the number of spin components, greater than three.[6] That is, in such a system the existence of cubic anisotropy gives rise to different critical behavior from that of the isotropic n-component system.

In addition to changing the critical behavior of the system cubic anisotropy can lead to the loss of a renormalization-group fixed point.[2] This is commonly interpreted as an indication that the transition to the ordered state is first order rather than continuous. This interpretation has been recently verified for cubic anisotropy using a parquet-graph analysis.[7-9] It has also been borne out in other systems through different methods.[10,11] In this paper, a recursion-relation approach is applied to the first-order transition induced by cubic anisotropy. In part, it reproduces the results of the earlier parquet-graph approaches, including predictions about the trictitical behavior of the system and about universal amplitude ratios. However, since the recursion relation approach has proven a powerful tool in the analysis of complicated multicritical systems it seems worthwhile to publish a recursion-relation based analysis of this interesting model. In this context, it is interesting to note that Domany, Fisher, and Mukamel[12] have already extended the work contained here to show how sufficient quadratic anisotropy can restore the continuous transition that is removed by cubic anisotropy alone.

The system to be investigated here has a two-component spin field. Although the methods to be used can be readily applied to a system with more than two components the analysis for two components is somewhat simpler than for n components. The more general case is discussed by Domany *et al.*[12]

An outline of the paper is as follows. In Sec. II the Hamiltonian of the spin system is displayed, and mean-field predictions are reviewed. A mean-field phase diagram is constructed. In Sec. III a more complete treatment is applied to a system with extreme cubic anisotropy. This system undergoes a first-order transition to its ordered state in zero magnetic field. It is similar to some field-theoretical models considered by Coleman and Weinberg,[13] and the analysis in this section is quite close to theirs.

Section IV contains renormalization-group equations for a two-component system with cubic anistropy. The equations are "sharp-cutoff" renormalization-group equations, obtained as per the prescription of Wegner and Houghton[14] or, alternatively, by differentiation of appropriate diagrams.[15] They are for a $(4-\epsilon)$-dimensional system with ϵ small and are accurate to low order in ϵ only. The equations are solved and their fixed points are identified. We also identify the regions in which fixed points are lost because of the cubic anisotropy. In Sec. V the free energy of the $(4-\epsilon)$-dimensional system is constructed according to a method outlined by Rudnick and Nelson.[16] With the aid of this free energy it is possible to show that when there is no fixed point because of cubic anisotropy the phase transition of the spin system to its ordered state in zero magnetic field will be first order rather than continuous. Section IV concerns properties of the phase transition when it is nearly continuous. Tricitical scaling predictions are developed and discussed with the use of nonlinear scaling fields. This seems to be the first case in which these quantities have been constructed for such a complicated crossover problem.

II. FIELD-THEORETICAL HAMILTONIAN; MEAN-FIELD PREDICTIONS

Our two-component system has the following Landau-Ginzburg-Wilson effective Hamiltonian

$$\frac{\mathcal{H}_{\text{eff}}}{k_BT}=\sum_{\vec{q}}(r+q^2)[S_x(\vec{q})S_x(-\vec{q})+S_y(\vec{q})S_y(-\vec{q})]$$
$$+\frac{u}{N}\sum_{\vec{q}_1-\vec{q}_4}[S_x(\vec{q}_1)S_x(\vec{q}_2)S_x(\vec{q}_3)S_x(\vec{q}_4)+S_y(\vec{q}_1)S_y(\vec{q}_2)S_y(\vec{q}_3)S_y(\vec{q}_4)]\,\delta_{\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4}$$
$$+\frac{v}{N}\sum_{\vec{q}_1-\vec{q}_4}S_x(\vec{q}_1)S_x(\vec{q}_2)S_y(\vec{q}_3)S_y(\vec{q}_4)\delta_{\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4}$$
$$-\sqrt{N}\,\vec{h}\cdot\vec{S}(0)\,. \tag{2.1}$$

N is the number of q modes in the sytem and $\vec{S}(\vec{q})$ is the Fourier transform of the lattice spin field $\vec{S}_{\vec{n}}$, given by

$$\vec{S}(\vec{q})=\frac{1}{\sqrt{N}}\sum_{\vec{n}}e^{i\vec{q}\cdot\vec{R}_{\vec{n}}}\vec{S}_{\vec{n}}\,. \tag{2.2}$$

This is the generalized X, Y model considered in the original ϵ-expansion paper by Wilson and Fisher.[2]

The free energy of this spin system is

$$F=-k_BT\ln Z \tag{2.3}$$

with

$$Z=\int\cdots\int e^{-\mathcal{H}_{\text{eff}}/k_BT}\prod_{\vec{q}}d\vec{S}(\vec{q})\,. \tag{2.4}$$

The mean-field approximation neglects all the $\vec{S}(\vec{q})$'s except $\vec{S}(0)$ which is replaced by

$$\vec{M}=(1/\sqrt{N})\vec{S}(0)\,. \tag{2.5}$$

The mean-field Hamiltonian is

$$\mathcal{H}_m/k_BT=N[r(M_x^2+M_y^2)+u(M_x^4+M_y^4)+vM_x^2M_y^2-\vec{h}\cdot\vec{M}]\,, \tag{2.6}$$

and the partition function Z_m is

$$Z_m=\iint e^{-\mathcal{H}_m/k_BT}\,dM_x\,dM_y\,. \tag{2.7}$$

In the limit $N\to\infty$ the mean-field free energy is

$$F_m=\mathcal{H}_m(\vec{M}_0)\,, \tag{2.8}$$

where $\vec{M}_0$ minimizes $\mathcal{H}_m$.

We may define a "stability wedge" in the u-v plane. The wedge, shown shaded in Fig. 1, lies be-between the lines $u=0$ and $u=-\frac{1}{2}v$. If (u,v) lies within the wedge then the Hamiltonian (2.1) and the mean-field Hamiltonian (2.6) are stable. If it lies outside of the wedge they are unstable in that their partition functions are given by divergent integrals. The Hamiltonian can be stabilized by the addition of sixth- or higher-order terms in $\vec{S}(q)$ or $\vec{M}$. In the absence of such terms (u,v) must lie inside the stability wedge.

As long as $\vec{h}=0$ and (u,v) lies within the stability wedge the mean-field transition to the ordered state is continuous, and is described by mean-field exponents. The nature of the $\vec{h}=0$ ordering is determined by the region of the stability wedge in which (u,v) lies. If $-2u<v<2u$ then $\vec{M}$ will point along one of the diagonals (1,1), (1,−1), (−1,1) or (−1,−1). If $0<u<\frac{1}{2}v$, $\vec{M}$ will point along either the x or the y axis. If $v=2u$, $\vec{M}$ will have no preferred orientation.

The phase diagram in Fig. 2 displays the different kinds of ordering that can take place. The line along which $v=2u$ is also a line of first-order transitions in that as the system crosses it $\vec{M}$ changes discontinuously. There is no description in the figure of what might happen when (u,v) lies outside the stability wedge, except for the dashed first-order lines to indicate the possibility of a first-order transition in the presence of stabilizing terms in the Hamiltonian.

III. FIRST-ORDER TRANSITION INDUCED BY EXTREME CUBIC ANISOTROPY

Returning to the full Hamiltonain (2.1) we will consider the effect of extreme cubic anisotropy on the transition to the ordered state. Extreme cubic anisotropy here means that (u,v) lies just inside the stability wedge. The transition in this case turns out to be first order, in contrast to the prediction of mean-field theory.

To obtain this result we consider a free energy consisting of a sum over an infinite number of

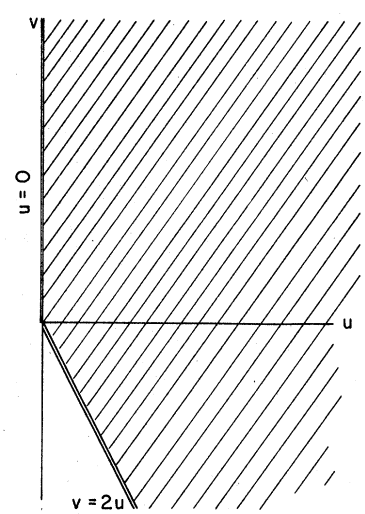

FIG. 1. Stability wedge for the Hamiltonian (2.1). If u and v lie in the shaded region the Hamiltonian is stable. Otherwise the integral on the right-hand side of the expression (2.4) for the partition function is divergent.

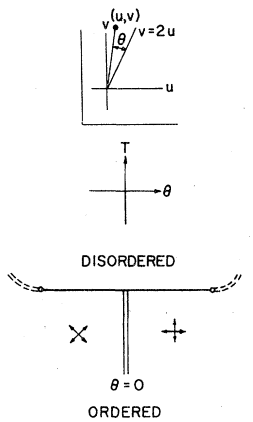

FIG. 2. Mean-field phase diagram for the two-component system with cubic anisotropy. The direction of increasing temperature T is shown and the angular variable θ is defined in the inset at the top. Whether the system orders along principal axes or diagonals is indicated in the ordered-phase portions of the diagram.

Feynman diagrams. The sum can be shown to be accurate for the free energy in the regions of interest to us. This will be established in part by verifying that the Ginzburg criterion[17] is satisfied in one of the regions. A complete demonstration requires a detailed analysis of all diagrams in all the regions of interest. It is not contained in this paper. A fuller discussion of this sort of system can be found in the paper of Coleman and Weinberg.[13]

The extremely anisotropic system has u and v satisfying the following conditions

$$u, v > 0 , \tag{3.1}$$

$$(u/v^2)\ln(v^2/2u) \ll 1 , \tag{3.2}$$

$$v \ll 1 . \tag{3.3}$$

Condition (3.1) ensures that (u,v) lies inside the stability wedge while condition (3.2) places it very near the $u=0$ boundary. Condition (3.3) insures that weak coupling approximations to be made here are asymptotically accurate.

Following the mean-field predictions we expect the system to order either along the x or the y axis. For simplicity we will set $M_y = 0$ and replace $S_x(0)$ by $\sqrt{N}\,M$. The Hamiltonian then takes the following form.

$$\begin{aligned}\frac{\mathcal{H}_{\text{eff}}}{k_B T} = {} & N(rM^2 + uM^4) + \sum_{\vec{q}\neq 0}(r + 6uM^2 + q^2)S_x(\vec{q})S_x(-\vec{q}) + \sum_{\vec{q}\neq 0}(r + vM^2 + q^2)S_y(-\vec{q})S_y(-\vec{q}) \\ & + \frac{4uM}{\sqrt{N}} \sum_{\vec{q}_1,\vec{q}_2,\vec{q}_3\neq 0} S_x(\vec{q}_1)S_x(\vec{q}_2)S_x(\vec{q}_3)\delta_{\vec{q}_1+\vec{q}_2+\vec{q}_3} + \frac{2vM}{\sqrt{N}} \sum_{\vec{q}_1,\vec{q}_2,\vec{q}_3\neq 0} S_x(\vec{q}_1)S_y(\vec{q}_2)S_y(\vec{q}_3)\delta_{\vec{q}_1+\vec{q}_2+\vec{q}_3} \\ & + \frac{u}{N} \sum_{\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4\neq 0} [S_x(\vec{q}_1)S_x(\vec{q}_2)S_x(\vec{q}_3)S_x(\vec{q}_4) + S_y(\vec{q}_1)S_y(\vec{q}_2)S_y(\vec{q}_3)S_y(\vec{q}_4)]\,\delta_{\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4} \\ & + \frac{v}{N} \sum_{\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4\neq 0} S_x(\vec{q}_1)S_x(\vec{q}_2)S_y(\vec{q}_3)S_y(\vec{q}_4)\delta_{\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4} .\end{aligned} \tag{3.4}$$

We have set $\vec{h} = 0$.

The leading-order contribution to the free energy is given by

$$F/k_B T = r + uM^4 + \mathfrak{F} , \tag{3.5}$$

where $\mathfrak{F}$ represents the diagrammatic sum displayed in Fig. 3. The double propagator lines in the diagrams are partially renormalized $\langle S_y S_y\rangle$ propagators satisfying the Dyson equation shown in Fig. 4. The double loop in Fig. 4 is a similarly renormalized $\langle S_x S_x\rangle$ propagator. This Dyson equation yields for the inverse propagators

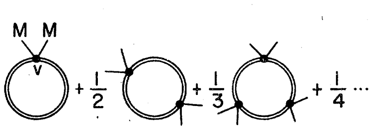

FIG. 3. Single-loop sum performed to find the free energy of a system with extreme cubic anisotropy.

$$\begin{aligned}\langle S_x(\vec{q})S_x(-\vec{q})\rangle^{-1} &= \langle S_y(\vec{q})S_y(-\vec{q})\rangle^{-1} \\ &= r + q^2 - \Sigma(q) \equiv r' + q^2 \\ &= r + dv\int_0^1 \frac{q'^{d-1}dq'}{r' + q'^2} + q^2\end{aligned} \tag{3.6}$$

or

$$\begin{aligned}r' = {} & r + \frac{vd}{d-2} - vr'd\int_0^\infty \frac{q'^{d-3}\,dq'}{r' + q'^2} \\ & + vr'd\int_1^\infty \frac{q'^{d-3}\,dq'}{r' + q'^2} .\end{aligned} \tag{3.7}$$

d on the right-hand side of (3.6) and (3.7) is the dimensionality of the system, taken to be between two and four. The q's are restricted to the interior of a spherical zone of radius one. Iterating (3.7) once we obtain the following expression for r':

$$\begin{aligned}r' \approx {} & r + \frac{vd}{d-2} - vd\left(r + \frac{vd}{d-2}\right)^{(d-2)/2}\int_0^\infty \frac{Q^{d-3}\,dQ}{1+Q^2} \\ & + vd\left(r + \frac{vd}{d-2}\right)\int_1^\infty \frac{q'^{d-3}\,dq'}{[r + vd/(d-2)] + q'^2} .\end{aligned} \tag{3.8}$$

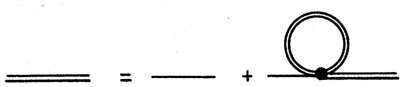

FIG. 4. Dyson equation (3.6), satisfied by the partially renormalized $\langle S_x S_x \rangle$ and $\langle S_y S_y \rangle$ in the system with extreme cubic anisotropy.

The right-hand side of (3.8) is well approximated by the first two terms if

$$\left(r+\frac{vd}{d-2}\right)^{(4-d)/2} \gg vd\int_0^\infty \frac{Q^{d-3}\,dQ}{1+Q^2}\,. \tag{3.9}$$

We thus require

$$r+vd/(d-2) \gg v^{2/(4-d)}\,. \tag{3.10}$$

If $r+vd/(d-2)\approx v$, (3.10) holds as long as (3.3) is satisfied and $4 \geqslant d > 2$.

The diagram sum for the free energy yields

$$\frac{F}{k_B T} = rM^2 + uM^4 + d\int_0^1 q^{d-1}\ln(r'+vM^2+q^2)\,dq - d\int_0^1 q^{d-1}\ln(r'+q^2)\,dq\,. \tag{3.11}$$

The integrals in (3.11) are over logarithms because of the combinatorial factors indicated in Fig. 3. When M is small, F can be expanded in a power series in terms of it. Equation (3.11) yields

$$\frac{F}{k_B T} = r'M^2 + \left(u - \tfrac{1}{2}dv^2\int_0^1 \frac{q^{d-1}\,dq}{(r'+q^2)^2}\right)M^4 + M^6\,\tfrac{1}{3}v^3 d\int_0^1 \frac{q^{d-1}\,dq}{(r'+q^2)^3} + \cdots\,. \tag{3.12}$$

If r' is of order v the coefficient of M^4 will be given by

$$u_4 = u - \tfrac{1}{2}d\alpha^{(d-4)/2}v^{d/2}\int_0^\infty \frac{Q^{d-1}\,dQ}{(1+Q^2)^2} + \tfrac{1}{2}dv^2\int_1^\infty \frac{q^{d-1}\,dq}{(\alpha v+q^2)^2}\,, \tag{3.13}$$

where we have set $r'=\alpha v$. When $d<4$ we can verify, using (3.2), that the coefficient of M^4 is less than zero. The coefficients of the quadratic and sixth-order terms are both greater than zero. The system looks like one that will have a first-order transition. However, it turns out that the finite value that M takes just below the transition is too large for a low-order expansion like (3.12) to be accurate.

To investigate the transition in detail we will take $r<0$, $r'=r+vd/(d-2)>0$ with r' of order v. We will now see what happens when M becomes very large while u is arbitrarily small. For very large M, we have

$$\frac{F}{k_B T} \approx rM^2+uM^4+d\int_0^1 \ln(vM^2)q^{d-1}\,dq - d\int_0^1 \ln(r'+q^2)q^{d-1}\,dq\,,$$
$$= rM^2+uM^4+\ln(vM^2) - d\int_0^1 \ln(r'+q^2)q^{d-1}\,dq\,. \tag{3.14}$$

If r is less than zero and u is arbitrarily small then for sufficiently large M the free energy can become large and negative. To see this we will solve for the large-M minimum of the free energy. The equation of state for large M is

$$\frac{1}{k_B T}\frac{\partial F}{\partial M} \approx 2rM + 4uM^3 + \frac{1}{M} = 0\,. \tag{3.15}$$

If $u/r^2 \ll 1$ the solution of (3.15) for the large-M minimum is given to a good approximation by

$$M^2 = |r|/2u\,. \tag{3.16}$$

The free energy is then

$$\frac{F}{k_B T} \approx -\frac{r^2}{4u} + \ln\left(\frac{v|r|}{2u}\right) - \int_0^1 \ln(r'+q^2)q^{d-1}\,dq\,. \tag{3.17}$$

As $u\to 0$ this free energy will become infinitely large and negative. At the same time because $r'>0$ the free energy has a local minimum $=0$ at $M=0$. The free energy has a double minimum and the large-M minimum can be much lower than the $M=0$ minimum. Note that this double minumum occurs in the region of validity of simple propagator renormalized diagrammatic perturbation theory, as defined by the Ginzburg criterion.[17]

To find the first-order transition we search for the r for which, with a given small u, the large-M minimum in the free energy equals the $M=0$ minimum. We require

$$rM^2+uM^4+\ln(vM^2) - d\int_0^1 \ln(r'+q^2)q^{d-1}\,dq = 0\,, \tag{3.18}$$

with M given by (3.16). This assumes in advance that $u/r^2 \ll 1$ at the first-order transition. Substituting (3.16) into (3.18) yields

$$-\frac{r^2}{4u} + \ln\left(\frac{v|r|}{2u}\right) - d\int_0^1 \ln(r'+q^2)q^{d-1}\,dq = 0\,. \tag{3.19}$$

This can be solved by iteration for r^2. Iterating twice,

$$r^2 \approx 2u\ln(v^2/2u) + 2u\ln[\ln(v^2/2u)]\,. \tag{3.20}$$

By (3.2) the second term on the right-hand side of (3.20) will be much smaller than the first one. Notice that if r is given by (3.20) then

$$\frac{r^2}{v^2} \approx \frac{2u}{v^2}\ln\left(\frac{v^2}{2u}\right) \ll 1\,, \tag{3.21}$$

the inequality following from (3.2). Thus $r' = r + dv/(d-2) \approx dv/(d-2)$ and the Ginzburg criterion is satisfied. Also, we have

$$u/r^2 \approx [\ln(v^2/2u)]^{-1} \ll 1\,, \tag{3.22}$$

and

$$vM^2 \approx \frac{v|r|}{2u} \approx \left[\frac{v^2}{2u}\ \ln\left(\frac{v^2}{2u}\right)\right]^{1/2} \gg 1\,. \tag{3.23}$$

The last inequality confirms the large-M approximation (3.14) to the free energy around the finite-M minimum.

IV. RENORMALIZATION-GROUP RECURSION RELATIONS AND THEIR SOLUTIONS

An investigation of the behavior of a system whose cubic anistropy is not extreme requires a more elaborate analysis. We will begin with a solution of the renormalization-group equations for a (4-ϵ)-dimensional system. This solution will yield the renormalized coefficients $r(l)$, $u(l)$, and $v(l)$ of a "block-spin" system with a Brillouin-zone radius $q_b = e^{-l}$. The equations are those of the sharp-cutoff renormalization group and may be obtained either via the method of Wegner and Houghton[14] or diagrammatically via an extension of the technique outlined in an earlier paper.[15]

The equations are

$$\frac{dr(l)}{dl} = 2r(l) + \frac{3du(l)}{1+r(l)} + \frac{d}{2}\ \frac{v(l)}{1+r(l)}\,, \tag{4.1}$$

$$\frac{du(l)}{dl} = (4-d)u(l) - \frac{qdu(l)^2}{[1+r(l)]^2} - \frac{dv(l)^2}{4[1+r(l)]^2}\,, \tag{4.2}$$

$$\frac{dv(l)}{dl} = (4-d)v(l) - \frac{6du(l)v(l)}{[1+r(l)]^2} - \frac{2dv(l)^2}{[1+r(l)]^2}\,. \tag{4.3}$$

To the accuracy in ϵ desired here d can be set equal to 4 except when 4 is subtracted from it. Furthermore, we can neglect the difference between $[1+r(l)]^2$ and 1 in (4.2) and (4.3). Finally we extract from (4.1) the equation describing the initial propagation of $r(l)$ from the critical hypersurface. The equations become

$$\frac{dt(l)}{dl} = 2t(l) - 12u(l)t(l) - 2v(l)t(l)\,, \tag{4.4}$$

$$\frac{du(l)}{dl} = \epsilon u(l) - 36u(l)^2 - v(l)^2\,, \tag{4.5}$$

$$\frac{dv(l)}{dl} = \epsilon v(l) - 24u(l)v(l) - 8v(l)^2\,. \tag{4.6}$$

$t(l)$ in (4.4) is equal to $r(l) - r_c(l) \approx r(l) + 6u(l) + v(l)$.

Equations (4.5) and (4.6) describe to leading order in $[\epsilon, u(l), v(l)]$ the propagation of $u(l)$ and $v(l)$ along the critical hypersurface. They can be solved by making the substitution

$$v(l) = V(l)e^{\epsilon l}\,, \tag{4.7}$$

$$u(l) = U(l)e^{\epsilon l}\,, \tag{4.8}$$

$$e^{\epsilon l} = x\,. \tag{4.9}$$

The following equations for $U(x)$ and $V(x)$ result.

$$\epsilon\frac{dU}{dx} = -36U^2 - V^2\,, \tag{4.10}$$

$$\epsilon\frac{dV}{dx} = -24UV - 8V^2\,. \tag{4.11}$$

Writing $U = Vf(V)$ we obtain, dividing (4.10) by (4.11)

$$\frac{dU}{dV} = f + V\frac{df}{dV} = \frac{36f^2+1}{24f+8} \tag{4.12}$$

or

$$V\frac{df}{dV} = \frac{(6f-1)(2f-1)}{24f+8}\,. \tag{4.13}$$

Integrating (4.13) we obtain the following solution for V in terms of f.

$$V(f) = v\left(\frac{2f-1}{2f_0-1}\right)^5 \left(\frac{6f_0-1}{6f-1}\right)^3\,, \tag{4.14}$$

where $f_0 = f(x=1)$ and $v = V(x=1) = v(l=0)$. An expression for x in terms of f follows from (4.10). We have

$$\epsilon\frac{dU}{dx} = \epsilon\frac{d}{dx}(Vf) = \epsilon\left(\frac{dU}{dx}f + V\frac{df}{dx}\right)$$
$$= -V^2(36f^2+1)\,. \tag{4.15}$$

Using (4.13) we obtain

$$\epsilon\left(\frac{dV}{dx} + V\frac{df}{dx}\right) = \epsilon\left(f\frac{dV}{df} + V\right)\frac{df}{dx}$$
$$= \frac{\epsilon V(36f^2+1)}{(6f-1)(2f-1)}\ \frac{df}{dx}$$
$$= V^2(36f^2+1)\,, \tag{4.16}$$

or

$$-\frac{dx}{\epsilon} = \frac{df}{V(6f-1)(2f-1)}$$
$$= \frac{1}{v}\ \frac{(2f_0-1)^5}{(6f_0-1)^3}\ \frac{(6f-1)^2}{(2f-1)^6}\,df\,. \tag{4.17}$$

This yields

$$\frac{x-1}{\epsilon} = \frac{e^{\epsilon l}-1}{\epsilon}$$
$$= \frac{1}{10v}\ \frac{(2f_0-1)^5}{(6f_0-1)^3}\,[X(f) - X(f_0)]\,, \tag{4.18}$$

where

$$X(f) = (60f^2 - 30f + 4)/(2f-1)^5\,. \tag{4.19}$$

We now have parametric solutions for U, V, and

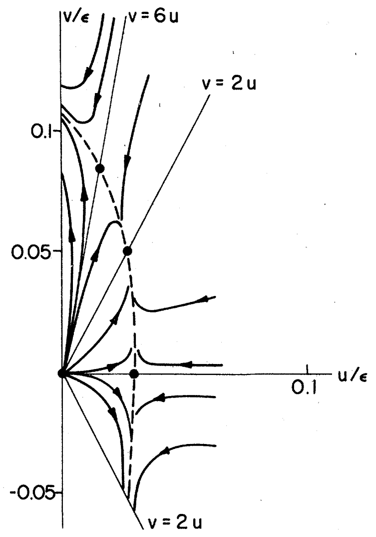

FIG. 5. Trajectory plot of $u(l)$ vs $v(l)$ obtained by solving the renormalization-group equations (4.5) and (4.6). The direction of increasing l is indicated by arrowheads, and fixed points by solid circles. The principal trajectories connecting the fixed points are indicated by a dashed curve.

x in terms of f. These can be used to obtain trajectory plots of $u(l)$ vs $v(l)$. Such a plot is displayed in Fig. 5. Arrowheads on the trajectories indicate the direction of increasing l. In addition to the line $u=\frac{1}{2}v$ the lines $v=0$ and $v=6u$ have a special significance in the plot. On those two lines the system decouples into two noninteracting single-component systems. When $v=6u$ the decoupling is achieved by transforming to the spin fields

$$S_1=(1/\sqrt{2})\,(S_x+S_y)\,, \qquad (4.20)$$

$$S_2=(1/\sqrt{2})(S_x-S_y)\,. \qquad (4.21)$$

Because the decouplings along those lines as well as the rotational symmetry along the line $v=2u$ are preserved in the renormalization-group transformations a $[u(l),v(l)]$ that starts on one of the three lines stays there.

As indicated on the plot there are four fixed points of Eqs. (4.5) and (4.6). The fixed points on the lines $v=6u$ and $v=0$ are Ising-like, as is to be expected. The fixed point on the line $v=2u$ is $X-Y$ like and there is the Gaussian fixed point at $u(l)=v(l)=0$. The relative stability of the fixed points is in agreement with the results of Wilson and Fisher and others.[2-6] Outside of the wedge bounded by the lines $v=0$ and $v=6u$ no $O(\epsilon)$ point is approached and $[u(l),v(l)]$ propagates off the stability wedge. The system itself does not become unstable because higher-order terms are generated by the renormalization procedure. Those higher-order terms must be taken into account in an analysis of the properties of the spin system when (u,v) lies outside of the smaller wedge.

We end this section with the derivation of some results that will be useful in the scaling analysis of Sec. VI. First is the solution of (4.4) for $t(l)$. We have

$$t(l)=te^{2l}\exp\left(-\int_0^l[12u(l')+2v(l')]\,dl'\right)\,. \qquad (4.22)$$

The integral in the exponential can be evaluated as follows:

$$\begin{aligned}\int_0^l[12u(l')+2v(l')]\,dl' &=\int_0^l[12v(l')x(l')+2V(l')x(l')]\,dl',\\ &=\int_0^l V(l')[12f(l')+2]x(l')\,dl',\\ &=\int_0^l V(l')[12f(l')+2]x(l')\frac{dl'}{dx(l')}\,\frac{dx(l')}{df(l')}\,df(l'),\\ &=-\int_{f_0}^{f(l)}\frac{12f+2}{(6f-1)(2f-1)}\,df,\\ &=-2\ln\left|\frac{2f(l)-1}{2f_0-1}\right|+\ln\left|\frac{6f(l)-1}{6f_0-1}\right|\,. \end{aligned}\qquad (4.23)$$

We have used the definitions (4.7), (4.8), and (4.9) of U, V, and x, and (4.17) for the derivative dx/df. We have for $t(l)$

$$t(l)=te^{2l}\left|\frac{2f(l)-1}{2f_0-1}\right|^2\left|\frac{6f_0-1}{6f(l)-1}\right|\,. \qquad (4.24)$$

We can similarly obtain the following result:

$$\begin{aligned}\int_0^l t(l')^2e^{-l'd}\,dl' &=\int_0^l t(l')^2e^{-4l'+\epsilon c'}\,dl'\\ &=-\frac{t^2}{2v}\left|\frac{2f_0-1}{6f_0-1}\right|\\ &\quad\times\left(\frac{1}{2f_0-1}-\frac{1}{2f(l)-1}\right).\end{aligned}\qquad (4.25)$$

This is a form that will enter into the free energy in the next section.

Finally, we will look at the behavior of $[u(l),v(l)]$ when it is asymptotically close to the fixed point on the line $v=6u$. To the order of accuracy of (4.5) and (4.6) the fixed point is $(\frac{1}{72}\epsilon,\frac{1}{12}\epsilon)$. If we write $u=\frac{1}{72}+\Delta_u$ and $v=\frac{1}{12}\epsilon+\Delta_v$ the following equations result to first order in Δ_u and Δ_v

$$\frac{d\Delta_u}{dl}=-\tfrac{1}{6}\Delta_v\,, \qquad (4.26)$$

$$\frac{d\Delta_v}{dl}=-\tfrac{2}{3}\epsilon\Delta_v-2\epsilon\Delta_u\,. \qquad (4.27)$$

The eigenvalues and associated vectors are

$$\lambda_1 = -\epsilon\ , \quad \Delta_v = 6\Delta_u\ ; \tag{4.28}$$

$$\lambda_2 = \epsilon/3\ , \quad \Delta_v = -2\Delta_u\ . \tag{4.29}$$

The asymptotic approach to the fixed point along the line $v = 6u$ is as $e^{-\epsilon l}$ and the asymptotic departure from it and away from the line is as $e^{\epsilon l/3}$.

V. ANISOTROPY-INDUCED FIRST-ORDER TRANSITION IN 4 - ϵ DIMENSIONS

To continue the analysis of a $(4-\epsilon)$ dimensional system with cubic ansisotropy we consider its free energy, generated from the recursion relations of the previous section by a technique outlined by Nelson and the author.[16] Briefly, the technique consists of integrating the recursion relations to an appropriately chosen l^* and then applying fluctuation corrected Landau theory to the renormalized Hamiltonian of the block spin system. The expression for the free energy is in terms of l^*, $v(l^*)$, $u(l^*)$ and $t(l^*)$. When no fixed point on the $u(l)$-$v(l)$ plane is approached this expression will predict a first-order zero-field transition to the ordered state.

It will be assumed that $v > 6u$. The region $v < 0$ can be mapped into this region by making the transformations (4.20), (4.21) so the discussion here covers both kinds of cubic anistropy.

Anticipating that the system will order either along the x or the y axis we set M_y equal to zero and M_x equal to M. To the two lowest orders in $[\epsilon, u(l^*), v(l^*)]$ the free energy is given by

$$\begin{aligned}\frac{1}{k_B T} F(t(l^*), u(l^*), v(l^*), M) &= -2\int_0^{l^*} t(l)^2 e^{-ld}\, dl \\ &+ e^{-l^*d}\{t(l^*)M(l^*)^2 + u(l^*)M(l^*)^4 \\ &\quad + \tfrac{1}{2} f[t(l^*) + 6u(l^*)M(l^*)^2] \\ &\quad + \tfrac{1}{2} f[t(l^*) + v(l^*)M(l^*)^2]\}\ , \end{aligned} \tag{5.1}$$

where

$$M(l^*) = M e^{(d-2)l^*/2} \tag{5.2}$$

and

$$f(x) = x^2[\ln(x) - \tfrac{1}{2}]\ . \tag{5.3}$$

The integral on the right-hand side of (5.1) and the first two terms in curly brackets are the leading-order contributions to the free energy. They are of order $[\epsilon, u(l^*), v(l^*)]^{-1}$. The remaining terms are of order $[\epsilon, u(l^*), v(l^*)]^0$. When $u(l^*)$ and $v(l^*)$ are set equal to a pair of non-Gaussian fixed point values this expression is accurate to zeroth order in ϵ.

It is important that the free energy be independent of the exact choice of l^* at least to its order of accuracy. That is because if we were to perform an exact integration of the full set of recursion relations and evaluate all the fluctuation corrections to the Landau theory the choice of l^* would have no effect on the final result. The lack of a dependence on l^* of the right-hand side of (5.3) is established to zeroth order in $[\epsilon, u(l^*), v(l^*)]$ in the Appendix.

A preliminary step in our choice of l^* is to note that the diagram sum performed in the derivation of the expression (5.1) is just the kind of single-loop sum performed in Sec. III.[18] This implies that $[u(l^*), v(l^*)]$ should be near the edge of the stability wedge. l^* will be chosen here so that $u(l^*) = 0$. Because of the approximate independence of l^* of the free energy the same results would obtain if l^* were chosen so that $u(l^*) \approx O(\epsilon^2, [v(l^*)]^2)$. Our particular choice of l^* has the advantage of simplifying intermediate steps.

The free energy is now

$$\begin{aligned}\frac{F}{k_B T} &= -2\int_0^{l^*} t(l)^2 e^{-ld}\, dl \\ &+ e^{-l^*d}\{t(l^*)M(l^*)^2 + \tfrac{1}{2} f[t(l^*)] \\ &\quad + \tfrac{1}{2} f[t(l^*) + v(l^*)M(l^*)^2]\}\ . \end{aligned} \tag{5.4}$$

If $t = 0$ so that $t(l^*) = 0$ the free energy is given by

$$\tfrac{1}{2} e^{-l^*d} M(l^*)^4 v(l^*)^2 \{\ln[v(l^*)M(l^*)^2] - \tfrac{1}{2}\}\ . \tag{5.5}$$

As a function of M it has a nonanalytic maximum at $M = 0$ and a minimum at $M^2 = [v(l^*)]^{-1} e^{(2-d)l^*}$. When $t(l^*)$ is positive the coefficient of M^2 for small M is

$$e^{-2l^*}[t(l^*) + \tfrac{1}{2} v(l^*) t(l^*) \ln t(l^*)]\ . \tag{5.6}$$

If $t(l^*)$ is of order $v(l^*)^2$ the coefficient of M^2 is positive so $M = 0$ is a local minimum. On the other hand the finite M minimum has substantially the same value, less than the $M = O$ minimum. Thus we have the same kind of double minimum as in the case of extreme anisotropy.

To find the first-order transition we guess that $t(l^*)$ will be proportional to $v(l^*)$ and that $M(l^*)$ will be proportional to $[v(l^*)]^{-1}$. Writing $t(l^*) = bv(l^*)$ and $M^2(l^*) = c[v(l^*)]^{-1}$ we require that the free energy at the finite-M minimum equal the free energy at the $M = 0$ minimum. Those two conditions take the following forms.

(i) Free-energy equality:

$$bc + \tfrac{1}{2} f[bv(l^*) + c] = \tfrac{1}{2} f[bv(l^*)]\ . \tag{5.7}$$

(ii) Minimization condition at finite M:

$$2bc + 2c[bv(l^*) + c]\ \ln[bv(l^*) + c] = 0\ . \tag{5.8}$$

Using (5.3) for $f(x)$ we obtain the following equation from (5.7) and (5.8).

$$\tfrac{1}{2}bc-\tfrac{1}{4}c^2-\tfrac{1}{4}bv(l^*)[2b+2c+bv(l^*)]=\tfrac{1}{2}f[bv(l^*)]. \tag{5.9}$$

This yields $b=\frac{1}{2}c+O(v(l^*))$. Substituting this result back into (5.8) and extracting terms of leading order in $v(l^*)$ yields

$$c^2+2c^2\ln(c)=0, \tag{5.10}$$

or

$$c=e^{-1/2}. \tag{5.11}$$

At the first-order transition

$$t(l^*)=\tfrac{1}{2}e^{-1/2}v(l^*), \tag{5.12}$$

$$\begin{aligned}\Delta M&=M_{\text{ordered}}-M_{\text{disordered}}\\&=e^{-(d-2)l^*/2}[v(l^*)]^{-1/2}e^{-1/4}+O([v(l^*)]^0). \end{aligned}\tag{5.13}$$

Thus there is a first-order transition at positive t. The analysis leading to this result could not be carried out if $[u(l^*),v(l^*)]$ did not propagate to the edge of the stability wedge. When $u(l^*)$ and $v(l^*)$ approach fixed points inside the wedge the transition is continuous. The line $u=6u$ is a line of tricritical points.

Finally, as Fig. 5 indicates, if $[u,v]$ is near the line $v=6u$ but in the unstable region then when $u(l)=O\ v(l)$ will be of order ϵ. The expressions (5.12) and (5.13) for $t(l^*)$ and ΔM at the first-order transition become the lowest-order terms in ϵ expansions for those quantities.

VI. TRICRITICAL SCALING PREDICTIONS

According to the tricritical scaling hypothesis[19] the singular part of the free energy of a system close to its tricritical point can be written in the form

$$F(t,\Delta,h)=g_t^{d/y_{tc}}\,\Phi(\mathrm{sgn}(g_t),g_\Delta|g_t|^{-\phi},g_h|g_t|^{-x_{tc}/y_{tc}}) \tag{6.1}$$

y_{tc} is the Kadanoff temperature-scaling parameter for the tricritical point $=d/(2-\alpha_{tc})$, where α_{tc} is the tricritical specific heat index. ϕ is the tricritical crossover exponent of Riedel[19] and x_{tc} is the tricritical magnetic field scaling parameter of Kadanoff.[20] g_t is a nonlinear scaling field[21] which, when small, is proportional to the tricritical reduced temperature if $\Delta=0$ and to the critical reduced temperature, $[T-T_c(\Delta)]/T_c(\Delta)$, if $\Delta<0$. g_Δ is the nonlinear scaling field corresponding to the tricritical parameter Δ and g_h is the nonlinear scaling parameter corresponding to the magnetic field. A tricritical phase diagram in the g_t-g_Δ plane is displayed in Fig. 6.

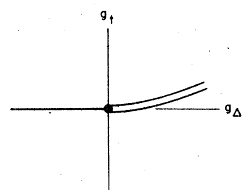

FIG. 6. Tricritical phase diagram in the g_t-g_Δ plane where g_t and g_Δ are *nonlinear* scaling fields (Ref. 21).

Our concern is with the first-order side of the tricritical point, when Δ and g_Δ are both positive. The first-order transition will occur when $g_t\propto g_\Delta^{1/\phi}$ and the discontinuity in $M\propto-\partial F/\partial g_h$ is proportional to $g_\Delta^{(d-x_{tc})/\phi y_{tc}}$. On the first-order side g_t, when small, is commonly taken to be proportional to the temperature reduced with respect to a linear extrapolation of the critical line. However, it has been pointed out that spurious corrections to scaling can be generated if ϕ is much smaller than one unless a correct, probably curved, extrapolation of the critical line is made.[22] Since here $\phi=\frac{1}{6}\epsilon$ to first order in ϵ these considerations must be taken into account.

The nonlinear scaling fields have the property that under renormalization-group transformations[21]

$$g_t(l)=g_t(0)e^{y_{tc}l}\equiv g_te^{y_{tc}l}, \tag{6.2}$$

$$g_\Delta(l)=g_\Delta e^{zl}, \tag{6.3}$$

$$g_h(l)=g_he^{x_{tc}l}, \tag{6.4}$$

these relations holding for arbitrarily large l. The exponent z describes propagation away from the unstable tricritical fixed point. According to (4.29)

$$z=\tfrac{1}{3}\epsilon+O(\epsilon^2). \tag{6.5}$$

Since $\phi=z/y$ and $x=2+O(\epsilon)$ the tricritical crossover exponent is given by

$$\phi=\tfrac{1}{6}\epsilon+O(\epsilon^2). \tag{6.6}$$

That means that the first-order transition takes place when $g_t\propto g_\Delta^{6/\epsilon+\epsilon^0}$ and that the discontinuity in M across the transition is proportional to $g_t^{3/\epsilon+O(\epsilon^0)}$, where to obtain the latter result we have used the fact that $d-x=1+O(\epsilon)$ in $(4-\epsilon)$ dimensions.

The remainder of this section is a verification of the above tricritical scaling predictions using the results of the renormalization-group analysis of the anisotropic system. The first step will be to develop expressions for the nonlinear scaling fields g_t and g_Δ. h will be set equal to zero. Because g_h is odd in h it also equals zero.

We start by noting that (4.18) which gives $x=e^{\epsilon l}$ in terms of $f(l)=u(l)/v(l)$ can be written in the following form:

$$x=\frac{\epsilon}{10v}\,\frac{(2f_0-1)^5}{(6f_0-1)^3}\left(\left\{\tfrac{3}{4}[f(l)]^2-\tfrac{3}{2}f(l)+\tfrac{17}{16}\right\}\frac{(6f(l)-1)^3}{(2f(l)-1)^5}-\left[\tfrac{3}{4}f_0^2-\tfrac{3}{2}f_0+\tfrac{17}{16}\right]\frac{(6f_0-1)^3}{(2f_0-1)^5}\right)+1. \quad (6.7)$$

Substituting this into the relation $v(l)=xV(l)$ with $V(l)$ given by (4.4) we obtain

$$v(l)-\tfrac{1}{10}\epsilon\left[\tfrac{3}{4}f(l)^2-\tfrac{3}{2}f(l)+\tfrac{17}{16}\right]=\left[v-\tfrac{1}{10}\epsilon\left(\tfrac{3}{4}f_0^2-\tfrac{3}{2}f_0+\tfrac{17}{16}\right)\right]\times\left(\frac{2f(l)-1}{2f_0-1}\right)^5\left(\frac{6f_0-1}{6f(l)-1}\right)^3. \quad (6.8)$$

Using (6.8) to eliminate the factor

$$\left(\frac{2f_0-1}{2f(l)-1}\right)^5\left(\frac{6f(l)-1}{6f_0-1}\right)^3$$

in (6.7), yields

$$1-\frac{\epsilon}{10v(l)}\left\{\tfrac{3}{4}[f(l)]^2-\tfrac{3}{2}f(l)+\tfrac{17}{16}\right\}=e^{-\epsilon l}\left(1-\frac{\epsilon}{10v}\left(\tfrac{3}{4}f_0^2-\tfrac{3}{2}f_0+\tfrac{17}{16}\right)\right). \quad (6.9)$$

The curve

$$v=\frac{\epsilon}{10}\left[\frac{3}{4}\left(\frac{u}{v}\right)^2-\frac{3}{2}\frac{u}{v}+\frac{17}{16}\right] \quad (6.10)$$

is a principal trajectory on the u-v plane in that it passes between the fixed points on the Ising line $v=6u$ and the x-y line $v=2u$. All trajectories that start in the region $0<u<\frac{1}{6}v$ approach this curve as they propagate towards the $u=0$ edge of the stability wedge. This principal trajectory is indicated by a dashed curve in Fig. 5. The principal trajectory joining the fixed point on the x-y line with the fixed point on the $v=0$ Ising line is

$$u=\frac{\epsilon}{10}\frac{u}{v}\left[60\left(\frac{u}{v}\right)^2-30\frac{u}{v}+4\right]\Big/\left(6\frac{u}{v}-1\right)^3. \quad (6.11)$$

It is also indicated by a dotted curve in Fig. 5.

The left-hand side of (6.9) with $f(l)$ replaced by $u(l)/v(l)$ is an irrelevant nonlinear scaling field. It can be expanded as a power series in $[u(l)-\frac{1}{72}\epsilon]$ and $[v(l)-\frac{1}{12}\epsilon]$, the differences between $u(l)$ and $v(l)$ and their respective fixed point values on the $v=6u$ line. It will prove convenient later on to set this scaling field equal to zero.

If, now, we multiply (6.9) by $v(l)$ and replace the left-hand side of the resulting equation by the right-hand side of (6.8) we obtain, after some rearrangement,

$$\frac{[2f(l)-1]^5}{[6f(l)-1]^3}\frac{1}{v(l)}=\frac{1}{v}\frac{(2f_0-1)^5}{(6f_0-1)^3}e^{-\epsilon l}. \quad (6.12)$$

The nonlinear scaling field $g_\Delta(l)$ can be obtained from this equation. Taking the inverse cube root and multiplying both sides by $(128/81\epsilon)^{1/3}$ so that g_Δ close to the tricritical fixed point stays finite as $\epsilon\to 0$,

$$\left(\frac{128}{81\epsilon}\right)^{1/3}[v(l)]^{1/3}\left(6\frac{u(l)}{u(l)}-1\right)\left(2\frac{u(l)}{v(l)}-1\right)^{-5/3}\equiv g_\Delta(l)=g_\Delta e^{(\epsilon/3)l}. \quad (6.13)$$

When this nonlinear field is small it is proportional to a linear combination of $[u(l)-\frac{1}{72}]$ and $[v(l)-\frac{1}{12}\epsilon]$ and it can be expanded in a power series in them about the fixed point on the $v=6u$ line.

To derive an expression for the nonlinear scaling field g_t we rearrange (4.24) to obtain

$$t(l)\frac{|6f(l)-1|}{|2f(l)-1|^2}=t\frac{|6f_0-1|}{|2f_0-1|^2}e^{2l}. \quad (6.14)$$

Multiplying by the cube root of (6.12), we obtain

$$g_t(l)\equiv t(l)\left(\frac{18}{\epsilon}|2u(l)-v(l)|\right)^{-1/3}=t\left(\frac{18}{\epsilon}|2u-v|\right)^{-1/3}e^{(2-\epsilon/3)l}=g_te^{(2-\epsilon/3)l}=g_te^{y_{tc}l}. \quad (6.15)$$

The extra factor $(\frac{1}{18}\epsilon)^{1/3}$ has been inserted so that $g_t\to t$ at the tricritical fixed point.

In principle it is possible to express $t(l)$, $u(l)$, and $v(l)$ in terms of $g_t(l)$, $g_\Delta(l)$, and the irrelevant nonlinear scaling field in (6.9). The construction of explicit expressions is difficult in this case and will not be attempted. Fortunately, it turns out not to be necessary for our present purposes.

We will, instead, start by placing the system on the principal trajectory (6.10). Then the scaling fields g_Δ and g_t are given by

$$g_\Delta(l)=\left(\tfrac{128}{810}\left\{\tfrac{3}{4}[f(l)]^2-\tfrac{3}{2}f(l)+\tfrac{17}{16}\right\}\right)^{1/3}\times[1-6f(l)][1-2f(l)]^{-5/3} \quad (6.16)$$

and

$$g_t(l)=t(l)\left(\tfrac{9}{5}\left\{\tfrac{3}{4}[f(l)]^2-\tfrac{3}{2}f(l)+\tfrac{17}{16}\right\}\right)^{-1/3}\times[1-2f(l)]^{-1/3}. \quad (6.17)$$

The free energy in the previous section was evaluated at the l^* for which $f(l^*)=0$. At that l^*

$$g_\Delta(l^*)=g_\Delta e^{(\epsilon/3)l^*}=\left(\tfrac{64}{405}\right)^{1/3}, \quad (6.18)$$

where the far right-hand side was obtained by setting $f(l)=0$ in (6.16). Thus

$$l^*=(3/\epsilon)\ln\left[\left(\tfrac{64}{405}\right)^{1/3}/g_\Delta\right]. \quad (6.19)$$

To find g_t and ΔM at the transition we need to know $v(l^*)$. From the fact that the system is on the principal trajectory (6.10) and $f(l^*)=0$ we obtain

$$v(l^*) = 17\epsilon/160 . \quad (6.20)$$

Then, from (5.12), (5.15), and (6.19)

$$g_t = \tfrac{1}{2}e^{-1/2}(17\epsilon/160)(\tfrac{153}{80})^{-1/3}]g_\Delta(\tfrac{64}{405})^{-1/3}]^{6/\epsilon+O(\epsilon^0)} \quad (6.21)$$

at the first-order transition. Similarly, from (5.13), (6.19), and (6.20)

$$\Delta M = e^{-1/4}(17\epsilon/160)^{-1/2}[g_\Delta(\tfrac{64}{405})^{-1/3}]^{3/\epsilon+O(\epsilon^0)} . \quad (6.22)$$

The scaling predictions following (6.6) are borne out.

As a final result we present lowest order in ϵ calculations for the specific heat and susceptibility rations, C_+/C_- and χ_+/χ_- where C_+ is the specific heat at $h=0$ just above the first-order transition C_- is the specific heat just below. χ_+ and χ_- are similarly defined for the isothermal ordering susceptibility $=(\partial M/\partial h)_t|_{h=0}$. These ratios are readily obtained from the free-energy expression in Sec. V. The ratios are

$$C_+/C_- = 2v(l^*) + O([v(l^*)]^0) \to (17\epsilon/80) + O(\epsilon^0) , \quad (6.23)$$

$$\chi_+/\chi_- = \tfrac{1}{2} + O(\epsilon) . \quad (6.24)$$

APPENDIX: l^* INDEPENDENCE OF THE FREE ENERGY

The l^* independence of expression (5.1) is demonstrated by taking its derivative with respect to l^* and verifying that the derivative equals zero to order $[\epsilon, u(l^*), v(l^*)]^0$. The leading-order contribution to the free energy is

$$-2\int_0^{l^*} t(l)^2 e^{-ld}\,dl + e^{-l^*d}[t(l^*)M(l^*)^2 + u(l^*)M(l^*)^4] . \quad (A1)$$

These terms are of order $[\epsilon, u(l^*), v(l^*)]^{-1}$. In the case of the trajectory integral it is because $t(l)^2e^{-ld}$ varies slowly with l for small $[\epsilon, u(l), v(l)]$. For the remainder of the expression it is the large value of $M(l^*)$ that causes this leading-order behavior. See (5.13).

The l^* derivative of (A1) is of order $[\epsilon, u(l^*), v(l^*)]^0$. Using (4.4), (4.5), and (5.2) we obtain for the derivative

$$-2t(l^*)^2e^{-l^*d} + e^{-l^*d}\left(\frac{dt(l^*)}{dl^*}M(l^*)^2 - 2M(l^*)^2t(l^*) + \frac{du(l^*)}{dl^*}M(l^*)^4 + (d-4)u(l^*)M(l^*)^4\right)$$
$$= -2t(l^*)^2e^{-l^*d} + e^{-l^*d}[-12u(l^*)t(l^*)M(l^*)^2 - 2v(l^*)t(l^*)M(l^*)^2 - 36u(l^*)^2M(l^*)^4 - v(l^*)^2M(l^*)^4] . \quad (A2)$$

The leading-order contribution to the l^* derivative of the remaining terms in (5.1) comes from the derivative of the logarithms. To leading order in $[\epsilon, u(l^*), v(l^*)]$

$$\frac{d}{dl^*}\ln[t(l^*) + 6u(l^*)M(l^*)^2] = 2 + O([\epsilon, u(l^*), v(l^*)])$$

and

$$\frac{d}{dl^*}[\tfrac{1}{2}e^{-l^*d}f(t(l^*) + 6u(l^*)M(l^*)^2) + \tfrac{1}{2}e^{-l^*d}f(t(l^*) + v(l^*)M(l^*)^2)]$$
$$\approx e^{-l^*d}\{[t(l^*) + 6u(l^*)M(l^*)^2]^2 + [t(l^*) + v(l^*)M(l^*)]^2\}$$
$$= e^{-l^*d}[2t(l^*)^2 + 6u(l^*)t(l^*)M(l^*)^2 + 2v(l^*)t(l^*)M(l^*)^2 + 36u(l^*)^2M(l^*)^4 + v(l^*)^2M(l^*)^4] . \quad (A3)$$

Adding (A3) to (A2) we obtain zero to order $[\epsilon, u(l^*), v(l^*)]^0$.

[1]P. Bak and D. Mukamel, Phys. Rev. B 13, 5086 (1976) and previous work cited therein.

[2]K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).

[3]F. J. Wegner, Phys. Rev. B 6, 1891 (1972).

[4]R. A. Cowley and A. D. Bruce, J. Phys. C 6, L191 (1973).

[5]A. Aharony, Phys. Rev. B 8, 4270 (1973).

[6]I. J. Ketley and D. J. Wallace, J. Phys. A 6, 1667 (1973).

[7]I. F. Lyuksyutov and V. Pokrovskii, Pis'ma Zh. Eksp. Teor. Fiz. 21, 22 (1975) [JETP-Lett. 21, 9 (1975)].

[8]I. F. Lyuksyutov, Phys. Lett. A 56, 135 (1976).

[9]T. Natterman and S. Trimper, J. Phys. A 8, 2000 (1975); T. Natterman (unpublished).

[10]D. J. Wallace, J. Phys. C 6, 1390 (1973) (and Erratum).

[11]J. Rudnick, D. J. Bergman, and Y. Imry, Phys. Lett. A 46, 449 (1974).

[12]E. Domany, D. Mukamel, and M. E. Fisher, Phys. Rev. B 15, 5432 (1977).

[13] S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).

[14] F. J. Wegner and A. Houghton, Phys. Rev. A 8, 401 (1973).

[15] J. Rudnick, Phys. Rev. B 11, 363 (1975).

[16] J. Rudnick, and D. R. Nelson, Phys. Rev. B 13, 2208 (1976). For applications of the method to a multicritical system see D. R. Nelson and E. Domany, Phys. Rev. B 13, 236 (1976); E. Domany, D. R. Nelson, and M. E. Fisher *ibid.* 15, 3493 (1977); and E. Domany and M. E. Fisher, *ibid.* 15, 3510 (1977).

[17] D. J. Amit, J. Phys. C 7, 3369 (1974).

[18] This point is made explicitly in the context of the Callan-Symanzik approach by E. Brézin, J. C. LeGuillou, and J. Zinn-Justin, Phys. Rev. D 8, 2418 (1973).

[19] E. K. Riedel, Phys. Rev. Lett. 28, 675 (1972).

[20] L. Kadanoff *et al.*, Rev. Mod. Phys. 395, (1967).

[21] F. J. Wegner, Phys. Rev. B 5, 4529 (1972).

[22] D. J. Bergman, Y. Imry, and O. Entin Wohlman, Solid State Commun. 13, 1665 (1973).