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Exactly soluble model of interacting classical spins in one dimension with random interactions

A. K. Rajagopal*

IBM, Thomas J. Watson Research Center, Yorktown Heights, New York 10598

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An exact calculation of the thermodynamic properties of a linear chain of classical spins with nearest-neighbor bilinear and biquadratic isotropic random exchange interactions is presented. Both open and closed chains are discussed. Only bond disorder is being considered here. It is shown that in contrast to the uniform system, "dipolar disorder points" as well as "quadrupolar disorder points" with qualitative and quantitative differences from the uniform case occur in this system. This is because the competing dipolar and quadrupolar interactions are further enriched by the random signs of these interactions. It is interesting to point out that much of the analysis can be made by means of a study of the eigenvalues of the "transfer kernel." The method can also be applied to obtain exact results for a linear chain of spins of arbitrary dimensionality interacting through an arbitrary isotropic nearest-neighbor Hamiltonian. In the case of the open chain, the Edwards-Anderson expedient is found to be exact in all the cases considered.

I. INTRODUCTION

Thorpe and Blume[1] studied the statistical mechanics of a linear chain of classical spins interacting with the Hamiltonian

$$H = J_1 \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \sum_i (\vec{S}_i \cdot \vec{S}_{i+1})^2 . \tag{1}$$

Here $\vec{S}_i$ is the three-dimensional unit vector at the ith site in the chain. They found the following results: (i) At $T=0$, this system can be either ordered or disordered depending on the relative magnitudes and signs of J_1 and J_2. (ii) There can be no "dipolar disorder point" since the nearest-neighbor dipolar correlation function has always the sign of $-J_1$. And, (iii) only "quadrupolar disorder points" can exist; there are *no such disorder points when J_2 is negative*. For a small range of temperatures two disorder points can occur.

In this paper we study the same system but with the Hamiltonian

$$H = \sum_i J_i^D \vec{S}_i \cdot \vec{S}_{i+1} + \sum_i J_i^Q (\vec{S}_i \cdot \vec{S}_{i+1})^2 , \tag{2}$$

where J_k^D, J_k^Q are considered random, and as such only the bond disorder is considered here.

The analysis presented by Thorpe and Blume[1] can be applied in this case also because of the special feature of the Hamiltonian (2), namely that the eigenfunctions of the "transfer kernel" *do not depend* on the random variables, J_k^D and J_k^Q. This feature is absent in the quantum case as well as in the spin-half one-dimensional Ising chain with random exchange coupling, as is well known.[2] We may also point out that the "transfer kernel" method can be used both for open and for closed chain systems.[3] We will develop the theory of both these cases in parallel. It is shown that in the large N limit, (thermodynamic limit) an argument involving the "largest eigenvalue" of the transfer kernel can be made in the random system also. For the "open chain," only the largest eigenvalue appears automatically. We calculate the average of the free energy, dipolar and quadrupolar correlation functions, entropy and specific heat in Sec. II. In Sec. III, we specify the probability distribution function and draw some general conclusions about the physical properties of this random system. In Sec. IV we summarize our results as well as make a few concluding remarks.

II. AVERAGE FREE ENERGY AND CORRELATION FUNCTIONS

We first note the following identity[1]

$$\begin{aligned} \exp[-\beta J_i^D \vec{S}_i \cdot \vec{S}_{i+1} - \beta J_i^Q (\vec{S}_i \cdot \vec{S}_{i+1})^2] &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} 4\pi \lambda_l^i (J_i^D, J_i^Q) Y_{lm}(\vec{S}_i) Y_{lm}^*(\vec{S}_{i+1}) \\ &\equiv T_i(\vec{S}_i, \vec{S}_{i+1}) , \end{aligned} \tag{3}$$

where $\{Y_{lm}(\vec{S})\}$ are the usual spherical harmonics and

$$\lambda_l^i (J_i^D, J_i^Q) = \frac{1}{2} \int_{-1}^{+1} e^{-\beta J_i^D x - \beta J_i^Q x^2} P_l(x)\, dx \tag{4}$$

with $P_l(x)$ being the Legendre polynomial of degree l. The notations and the various properties of Y_{lm}, etc., are the same as in Ref. 1. Here β is related to the temperature via $1/k_B T$ with k_B, the Boltzmann constant. This is the basic "transfer kernel" for our problem.

We first observe that for a *fixed* set of parameters $\{J_k^D, J_k^Q\}$, in view of the properties of Legendre polynomials, we have

$$|\lambda_l^k/\lambda_0^k| < 1 \text{ for } l = 1, 2, 3, \ldots . \tag{5}$$

In other words λ_0^k is the largest of the quantities, $\{\lambda_l^k\}$. Several properties of the λ_l^k follow at once from the knowledge of $P_l(x)$:

$$\text{(a)}\quad \lambda_0^k(J_k^D, J_k^Q) = \int_0^1 e^{-\beta J_k^Q x^2} \cosh(\beta J_k^D x)\, dx > 0 .$$

$$\text{(b)}\quad \lambda_1^k(J_k^D, J_k^Q) = -\int_0^1 e^{-\beta J_k^Q x^2} x \sinh(\beta J_k^D x)\, dx , \tag{6}$$

and so

$$(\operatorname{sgn} J_k^D)\, \lambda_1^k(J_k^D, J_k^Q) \leq 0 . \tag{7}$$

$$\text{(c)}\quad \lambda_l^k(J_k^D, J_k^Q) = (-1)^l \lambda_l^k(-J_k^D, J_k^Q), \text{ for all } l . \tag{8}$$

The above properties hold for all k. We use Eq. (3) and the orthonormality properties of $\{Y_{lm}\}$ in calculating the various physical quantities of interest. We discuss the open and closed chains under separate subsections.

A. Open chain

In Eq. (2), i runs from 1 to $N-1$ so that the first member of the chain is $i = 1$ and the last one is $i = N-1$. The partition function is then

$$Z_N^{\text{open}} = \int \frac{d\vec{S}_1}{4\pi} \cdots \int \frac{d\vec{S}_N}{4\pi}\, T_1(\vec{S}_1, \vec{S}_2) \cdots T_{N-1}(\vec{S}_{N-1}, \vec{S}_N)$$

and using Eq. (3) and the orthonomality properties of Y_{lm}, we at once obtain the result

$$Z_N^{\text{open}} = \prod_{k=1}^{N-1} \lambda_0^k(J_k^D, J_k^Q) . \tag{9}$$

The following correlation functions are of special interest to us in the discussions to follow. They are calculated using the same tricks as indicated above. The thermodynamic average is denoted by the usual angular brackets.

$$\langle \vec{S}_i \cdot \vec{S}_{i+r} \rangle^{\text{open}} = \prod_{k=i-1}^{i+r} \frac{\lambda_1^k(J_k^D, J_k^Q)}{\lambda_0^k(J_k^D, J_k^Q)} , \tag{10}$$

$$\left\langle \frac{3}{2}(\vec{S}_i \cdot \vec{S}_{i+r})^2 - \frac{1}{2} \right\rangle^{\text{open}} = \prod_{k=i-1}^{i+r} \frac{\lambda_2^k(J_k^D, J_k^Q)}{\lambda_0^k(J_k^D, J_k^Q)} . \tag{11}$$

B. Closed chain

In Eq. (2), i runs from 1 to N *with the condition*

$$\vec{S}_{N+1} \equiv \vec{S}_1 , \tag{12}$$

and we also place the periodic condition on the coupling strengths. Then,

$$Z_N^{\text{closed}} = \int \frac{d\vec{S}_1}{4\pi} \cdots \int \frac{d\vec{S}_N}{4\pi}\, T_1(\vec{S}_1, \vec{S}_2) \cdots T_N(\vec{S}_N, \vec{S}_1) .$$

The condition (12) then leads us to the more complicated expression

$$Z_N^{\text{closed}} = \sum_{l=0}^{\infty} (2l+1) \prod_{k=1}^{N} \lambda_l^k(J_k^D, J_k^Q) . \tag{13}$$

The correlations functions can also be similarly calculated, and they are

$$\langle \vec{S}_i \cdot \vec{S}_{i+r} \rangle^{\text{closed}} = \frac{\sum_{l=0}^{\infty} (2l+1) \prod_{k=1}^{N} \lambda_l^k \prod_{k_1=i-1}^{i+r} \frac{\lambda_{l+1}^{k_1}}{\lambda_l^{k_1}}}{\sum_{l=0}^{\infty} (2l+1) \prod_{k=1}^{N} \lambda_l^k} \tag{14}$$

and

$$\langle \tfrac{3}{2}(\vec{S}_i \cdot \vec{S}_{i+r})^2 - \tfrac{1}{2} \rangle^{\text{closed}} = \frac{\sum_{l=0}^{\infty} (2l+1) \prod_{k=1}^{N} \lambda_l^k \prod_{k_1=i-1}^{i+r} \frac{\lambda_{l+2}^{k_1}}{\lambda_l^{k_1}}}{\sum_{l=0}^{\infty} (2l+1) \prod_{k=1}^{N} \lambda_l^k} . \tag{15}$$

The free energy is given by

$$-\beta F = \ln Z$$

and we need the average of F and the various correlation functions over the random distribution of $\{J_k^D, J_k^Q\}$ in order to discuss the physical properties of the system. We denote this averaging by enclosing the function being averaged within a second set of angular brackets, $\langle \cdots \rangle$.

We *assume* $\{J_k^D, J_k^Q\}$ are independent random variables and only a distribution law specified by the function $P(J^D, J^Q)$. In this way, we have kept the possible statistical correlation in J^D, J^Q, for purposes of generality. We do not envisage here a correlation among the parameters of different sites, J_k^D with J_{k+1}^D for example, as this would make the calculation more complex, even if interesting.[2] In the open chain case, this averaging procedure leads us naturally to the following expressions:

$$-\beta \langle F_N^{\text{open}} \rangle = (N-1) \int \int_{-\infty}^{\infty} P(J^D, J^Q)\, dJ^D \times dJ^Q \ln \lambda_0(J^D, J^Q) , \tag{16}$$

$$\langle\langle \vec{S}_i \cdot \vec{S}_{i+r} \rangle^{\text{open}} \rangle = \left(\int \int_{-\infty}^{\infty} P(J^D, J^Q)\, dJ^D dJ^Q \times \frac{\lambda_1(J^D, J^Q)}{\lambda_0(J^D, J^Q)} \right)^{|r|} , \tag{17}$$

and

$$\left\langle\left\langle\frac{3}{2}(\vec{S}_i\cdot\vec{S}_{i+r})^2-\frac{1}{2}\right\rangle^{\text{open}}\right\rangle = \left[\iint_{-\infty}^{\infty} P(J^D,J^Q)\,dJ^D\,dJ^Q\,\frac{\lambda_2(J^D,J^Q)}{\lambda_0(J^D,J^Q)}\right]^{|r|}. \tag{18}$$

We are thus naturally led to consider certain averages of the eigenvalues of the "transfer kernel" over the distribution of the random coupling parameters. The similarity in the formulation for the pure system case is thus demonstrated. From Eq. (16), we may note that the Edwards-Anderson trick[4] is exact in the open chain case. Explicitly, we have quite generally, the result

$$\lim_{n\to 0}\left(\frac{\langle (Z^{\text{open}})^n-1\rangle}{n}\right)=\lim_{n\to 0}\left(\frac{\left\langle\prod_{i=1}^{N}(\lambda_0^i)^n\right\rangle-1}{n}\right)$$

$$=\lim_{n\to 0}\left[\frac{\left\langle\exp n\sum_{i=1}^{N}(\ln\lambda_0^i)\right\rangle-1}{n}\right]$$

$$\cong\lim_{n\to 0}\left[\frac{\exp n\sum_{i=1}^{N}\langle\ln\lambda_0^i\rangle-1}{n}\right]$$

$$=\sum_{i=1}^{N}\langle\ln\lambda_0^i\rangle\,,$$

in the thermodynamic limit.

The discussion of the closed chain case is somewhat complicated as is clear from the outset. In the thermodynamic limit, $N\to\infty$, the distinction between open and closed chain systems must disappear, and it indeed does so in the present case. From Eq. (13) we have

$$-\beta\langle F_N^{\text{closed}}\rangle = N\int_{-\infty}^{\infty}\!\!\int P(J^D,J^Q)\,dJ^D\,dJ^Q\ln\lambda_0(J^D,J^Q) + \left\langle\ln\left(1+\sum_{l=1}^{\infty}(2l+1)\prod_{k=1}^{N}\frac{\lambda_l^k(J_k^D,J_k^Q)}{\lambda_0^k(J_k^D,J_k^Q)}\right)\right\rangle \tag{19}$$

One must now use Eq. (5) to essentially drop the second term from further consideration, since averaging over the distribution of $\{J_k^D,J_k^Q\}$ will not spoil the inequality in any significant way. Exactly similar arguments must be used in computing the averages of the correlation function, Eqs. (14) and (15). We are thus led to consider the expressions for the open chain case only. We shall drop the nomenclature of "open" and "closed" henceforth with the understanding that we have taken the thermodynamic limit.

The "disorder points" will be discussed by studying the nature of the correlation functions, Eqs. (17), (18) as a function of temperature and parameters specifying the distribution function P.

The disorder points occur because of the competing interactions among the spins in this system. They occur as oscillations in the spin-spin correlations, Eqs. (17) and (18), in contrast to their usual exponential decay. In the random system, it is clear from the expressions derived above that these spin correlations being further averaged over the random variables open up interesting possibilities. In particular, in the uniform case, the signs and magnitudes of J^D,J^Q played important roles in determining the disorder points. In the random system one may allow for both signs of interactions with differing strengths, thus incorporating the various possible conditions for occurrence of disorder points becomes relatively easy; one has a further competition introduced into the system viz. ferro- and antiferromagnetic interactions along the chain. We discuss these in Sec. III.

The average energy of the system is also of interest and it is given by

$$\frac{\langle\langle H\rangle\rangle}{N}=\int_{-\infty}^{\infty}\!\!\int P(J^D,J^Q)\,dJ^D\,dJ^Q \times\left[J^D\frac{\lambda_1(J^D,J^Q)}{\lambda_0(J^D,J^Q)} + \frac{1}{3}J^Q\left(2\frac{\lambda_2(J^D,J^Q)}{\lambda_0(J^D,J^Q)}+1\right)\right]. \tag{20}$$

From (16) and the properties of the Legendre polynomials, we observe

$$\frac{\langle\langle H\rangle\rangle}{N}=\frac{\partial}{\partial\beta}\frac{\beta\langle F_N\rangle}{N}\,.$$

The entropy of the system is given by

$$S=\frac{\langle\langle H\rangle-F\rangle}{NT}=k_B\langle\ln\lambda_0\rangle+\frac{1}{T}\left\langle J_D\frac{\lambda_1}{\lambda_0}\right\rangle + \frac{1}{3T}\left\langle J^Q\left(2\frac{\lambda_2}{\lambda_0}+1\right)\right. , \tag{21}$$

and the specific heat is

$$\frac{C}{k_B}=\beta^2\left[\left\langle J^D\left(\frac{\lambda_1}{\lambda_0^2}\frac{\partial\lambda_0}{\partial\beta}-\frac{1}{\lambda_0}\frac{\partial\lambda_1}{\partial\beta}\right)\right\rangle + \frac{2}{3}\left\langle J^Q\left(\frac{\lambda_2}{\lambda_0}\frac{\partial\lambda_0}{\partial\beta}-\frac{1}{\lambda_0}\frac{\partial\lambda_2}{\partial\beta}\right)\right\rangle\right]. \tag{22}$$

Performing the indicated differentiations and using the properties of the Legendre polynomials, the specific heat is found to be

$$\frac{C}{k_B}=\beta^2\left[\left\langle J^{D2}\left(\frac{2}{3}\frac{\lambda_2}{\lambda_0}-\frac{\lambda_1^2}{\lambda_0^2}+\frac{1}{3}\right)\right\rangle + 2\left\langle J^D J^Q\left(\frac{\lambda_3}{\lambda_0}-\frac{2}{3}\frac{\lambda_1\lambda_2}{\lambda_0^2}+\frac{4}{15}\frac{\lambda_1}{\lambda_0}\right)\right\rangle + \left\langle J^{Q2}\left(\frac{8}{35}\frac{\lambda_4}{\lambda_0}-\frac{4}{9}\frac{\lambda_2^2}{\lambda_0^2}+\frac{26}{63}\frac{\lambda_2}{\lambda_0}-\frac{62}{315}\right)\right\rangle\right]. \tag{23}$$

In Sec. III we shall discuss the consequences of these calculations by taking a special but representative form for P and contrast the results with the uniform case considered in Ref. 1.

III. MODEL RANDOM SYSTEM

We choose the probability distribution function P in the following form:

$$P(J^D,J^Q)=[A_1\,\delta(J^D-J_1)+(1-A_1)\,\delta(J^D+J_1')]\times[A_2\,\delta(J^Q-J_2)+(1-A_2)\,\delta(J^Q+J_2')]\,. \tag{24}$$

This is normalized to unity explicity. It gives the uniform system as a special case when we take $A_1=1=A_2$. This function has the feature that it has both positive and negative signs of J^D and J^Q built into it with appropriate weights. In this way we have allowed for both ferro- and antiferromagnetic interactions in the system. The constants A_1,A_2,J_1,J_1',J_2, and J_2' characterize the random system. We now have: (we use the properties of A,B,C of Sec. II in deducing the properties below)

$$\frac{-\beta\langle F_N\rangle}{N}=A_1A_2\ln\lambda_0(J_1,J_2)+A_1(1-A_2)\ln\lambda_0(J_1,-J_2')+(1-A_1)A_2\ln\lambda_0(J_1',J_2)+(1-A_1)(1-A_2)\ln\lambda_0(J_1',-J_2')\,, \tag{25}$$

$$\langle\langle\vec{S}_i\cdot\vec{S}_{i+1}\rangle\rangle=A_1A_2\frac{\lambda_1(J_1,J_2)}{\lambda_0(J_1,J_2)}+A_1(1-A_2)\frac{\lambda_1(J_1,-J_2')}{\lambda_0(J_1,-J_2')}-(1-A_1)A_2\frac{\lambda_1(J_1',J_2)}{\lambda_0(J_1',J_2)}-(1-A_1)(1-A_2)\frac{\lambda_1(J_1',-J_2')}{\lambda_0(J_1',-J_2')}\,. \tag{26}$$

And,

$$\left\langle\left\langle\frac{3}{2}(\vec{S}_i\cdot\vec{S}_{i+1})^2-\frac{1}{2}\right\rangle\right\rangle=A_1A_2\frac{\lambda_2(J_1,J_2)}{\lambda_0(J_1,J_2)}+A_1(1-A_2)\frac{\lambda_2(J_1,-J_2')}{\lambda_0(J_1,-J_2')}+(1-A_1)A_2\frac{\lambda_2(J_1',J_2)}{\lambda_0(J_1',J_2)}+(1-A_1)(1-A_2)\frac{\lambda_2(J_1',-J_2')}{\lambda_0(J_1',-J_2')}\,. \tag{27}$$

We shall now consider special situations:

Case (A): Taking $A_1=1,A_2=1$; or $A_1=0,A_2=0$; or $A_1=1,A_2=0$; or $A_1=0,A_2=1$; do not give ressults different from those already discussed by Thorpe and Blume.[1]

Case (B): If we take a symmetrical distribution of the coupling strengths, i.e., $J_1'=J_1$ and $J_2'=J_2$, then we obtain

$$\langle\langle\vec{S}_i\cdot\vec{S}_{i+1}\rangle\rangle_B=(2A_1-1)\left(A_2\frac{\lambda_1(J_1,J_2)}{\lambda_0(J_1,J_2)}+(1-A_2)\frac{\lambda_1(J_1,-J_2)}{\lambda_0(J_1,-J_2)}\right) \tag{28}$$

and

$$\langle\langle\tfrac{3}{2}(\vec{S}_i\cdot\vec{S}_{i+1})^2-\tfrac{1}{2}\rangle\rangle_B=A_2\frac{\lambda_2(J_1,J_2)}{\lambda_0(J_1,J_2)}+(1-A_2)\frac{\lambda_2(J_1,-J_2)}{\lambda_0(J_1,-J_2)}\,. \tag{29}$$

The disorder points occur when $\langle\langle\vec{S}_i\cdot\vec{S}_{i+1}\rangle\rangle$ or when $\langle\langle\tfrac{3}{2}(\vec{S}_i\cdot\vec{S}_{i+1})^2-\tfrac{1}{2}\rangle\rangle$ change sign, as a function of temperature. In view of Eq. (8), we note that *we cannot have* "dipolar disorder" as in *Case* (A) but "quadrupolar disorder" can exist as in *Case* (A). Since there is no "quadrupolar disorder" for negative J_2, in the uniform case, we may deduce from Eq. (29) that we *will* have quadrupolar disorder for certain values of J_2 and A_2. By letting A_2 be nonzero, we have allowed for some negative J_2 to occur in the system. This is a new feature. Another point, also stemming from the result of the uniform case, is that there can be two disorder points for a small range of temperature which may or may not appear now depending on the value of A_2. Thus, we find a qualitative as well as quantitative change in the nature of the results even for a symmetrical distribution of the parameters.

Case (C): Take $A_2=1$, but keep the asymmetry in the distribution of J^D. Then

$$\langle\langle S_i\cdot S_{i+1}\rangle\rangle_C=A_1\frac{\lambda_1(J_1,J_2)}{\lambda_0(J_1,J_2)}-(1-A_1)\frac{\lambda_1(J_1',J_2)}{\lambda_0(J_1',J_2)} \tag{30}$$

and

$$\langle\langle \tfrac{3}{2}(\vec{S}_i \circ \vec{S}_{i+1})^2 - \tfrac{1}{2}\rangle\rangle_C = A_1 \frac{\lambda_2(J_1, J_2)}{\lambda_0(J_1, J_2)} + (1 - A_1)\frac{\lambda_2(J_1', J_2)}{\lambda_0(J_1', J_2)} \quad . \tag{31}$$

The case with $A_2 = 0$ is similar.

Here we have a new situation, because we can now have both dipolar and quadrupolar disorder points. This is because the random distribution of J^D allows an unsymmetrical disposition of ferro- and antiferromagnetic interactions along the chain. When $A_1 = 0$ or $A_1 = 1$, we have no dipolar disorder but as A_1 becomes fractional lying between 0 and 1, this disorder begins to develop. In this case, we find a qualitatively different nature of the results compared to cases (*A*) and (*B*).

Case (*D*): If we keep A_1 and A_2 arbitrary as well as maintain the asymmetry, we obtain results similar to cases (*B*) and (*C*) but which will differ from them only quantitatively.

Let us denote $\vec{S}_i \cdot \vec{S}_{i+1}$ by $\cos\theta_{i,i+1}$. Then the disorder points occur whenever

$$\langle\langle \cos\theta_{i,i+1}\rangle\rangle = 0 \quad \text{(dipolar)} \ ,$$

or

$$\langle\langle \cos^2\theta_{i,i+1}\rangle\rangle = \tfrac{1}{3} \quad \text{(quadrupolar)} \ .$$

In this form, the ease with which the disorder points occur in the random case may be physically more transparent. In the uniform (nonrandom) situation, the thermodynamic average alone determined how the spins align or misalign with respect to one another at a given temperature. The new aspect of further averaging over the randomness allows for more possibilities of spin correlations of the type we are looking for. The quadrupole interactions tend to disalign the antiferromagnetic arrangment (when J_2 is negative) which gives $\langle\cos\theta_{i,i+1}\rangle$, a negative value; by allowing for both ferro- and antiferromagnetic interactions to exist in the random case, the possibility of dipolar disorder point in spite of the quadrupolar interactions becomes evident. For the quadrupolar disorder point to occur, we require that on the average the spins must orient themselves in all directions in an equally likely fashion so as to give $\langle\langle\cos^2\theta_{i,i+1}\rangle\rangle = \frac{1}{3}$. For certain strengths of J_2, J_2' then we may achieve this disposition of spins in the random case in contrast to the uniform situation.

We have thus shown that in the model system considered here we can have a rich variety of disorder points in contrast to the case discussed by Thorpe and Blume.[1] By changing the structure of P from that given by Eq. (24) we do not expect any drastic changes in the properties of the system. This can be achieved by making the delta functions spread out with preassigned widths, etc. We shall not discuss this any further.

From Eqs. (21) and (23), it may be noted that as one crosses the "disorder points" one does not obtain a dramatic change in the thermodynamic quantities. This is because they contain other terms besides those involving the disorder points. This observation shows that disorder points are perhaps best studied by a direct examination of the spin correlations themselves. One such experiment is the study of neutron scattering which would give us information about the dipolar disorder points, as will be shown in Sec. IV.

IV. SUMMARY AND CONCLUDING REMARKS

We have shown that a system of classical spins interacting isotropically with a Heisenberg and biquadratic exchange between nearest neighbors but with random interaction strengths can be solved exactly in one dimension. The discussion can be carried out in terms of the averages over the ratio of eigenvalues of the transfer kernel to its largest eigenvalue, $\langle(\lambda_l/\lambda_0)\rangle$, over the distribution of the random variable. This point may be of general theoretical interest in the discussion of these and other random systems. One major feature of our analysis is that we may have both dipolar and quadrupolar disorder points in the random case in contrast to the uniform system. This is because in the random case, the competing dipolar and quadrupolar interactions are further enriched by the possibility of having random signs of these interactions as well. It may be noted in passing that we can compute all the properties of the system when the quadrupolar interaction is absent. Fisher[5] was the first to discuss this uniform system of continuous spin, nearest neighbor Heisenberg Hamiltonian. The random case has been discussed independently by Thorpe[6] and by Tonegawa *et al.*[7]

We may calculate the average spin-susceptibility of our system by including a small magnetic field term in the Hamiltonian, Eq. (2) and calculating the *average* free energy to second order in the field. In the thermodynamic limit then, we obtain in the usual notation,

$$\langle\chi_{N\to\infty}(T)\rangle = \frac{Ng^2\mu_B^2\beta}{12}\,\frac{1+u}{1-u} \ , \tag{32}$$

where

$$u = \langle\langle\vec{S}_{i+1}\cdot\vec{S}_i\rangle\rangle = \langle\lambda_1/\lambda_0\rangle = \text{Eq. (26)} \tag{33}$$

for our model. In fact, the momentum-dependent static susceptibility is

$$\langle \chi_{N\to\infty}(q,T)\rangle = \frac{Ng^2\mu_B^2\beta}{12}\;\frac{1-u^2}{1-2u\cos qa+u^2}\;, \qquad (34)$$

where a is the distance between nearest neighbors, and q is the wave vector. This quantity is of interest in discussing the neutron scattering from such random systems.

If we consider an anisotropic Heisenberg model, with continuous spins, the trick used in this paper will not work because the eigenfunctions of the "transfer kernel" will depend on the exchange, coupling.[8] We then obtain

$$Z_N^{\text{open}} = \prod_{k=1}^{N-1} \left[(0k\,|\,0k+1)\lambda_0^k\right]\;, \qquad (35)$$

where $(0k\,|\,0k+1)$ is the overlap integral of the wave functions between nearest neighbors.

The method empoyed here works also for d-dimensional spins with the interactions of the form given by Eq. (2),[9] as well as arbitrary isotropic nearest-neighbor interactions,[10,11] given by

$$H = \sum_i f_i(\vec{S}_i\cdot\vec{S}_{i+1})\;. \qquad (36)$$

Here f_i is an arbitrary function, and $\vec{S}_i$ is a d-dimensional spin vector with the constraint $|\vec{S}_i|^2 = \Lambda$. In the case of the open chain, the Edwards-Anderson trick is found to be exact. For the Heisenberg model with $\Lambda = d$, in the limit $d\to\infty$, we obtain the results for the randomized version of the Berlin-Kac model discussed by Stanley.[9]

From Eq. (34) we note that the dipolar disorder point may be experimentally observed via a neutron scattering setup. We may also draw the attention to the reader to the review article of Steiner *et al.*[12] where a detailed account of theoretical and experimental studies on one-dimensional magnetic systems is given. In this article, among the new directions for investigation, the study of disordered chains was suggested. The present article is a contribution towards this area of research.

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*On sabbatical leave from Louisiana State Univ., Baton Rouge, La. 70803.

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