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Critical properties of a dilute Ising model in the percolation limit

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The critical behavior of the dilute Ising model on a Bethe lattice of coordination z is studied in the percolation limit ($T \to 0$) by means of a new moment-expansion procedure. Exact numerical results for the magnetization as a function of magnetic field at the critical concentration, $p = p_c$, are presented for $z = 3$ and $z = 4$. In particular, the critical exponent δ_p is expected to agree well for very low fields with the approximate value $\delta_p = 2$ obtained by Essam *et al.* The nature of deviations from the value $\delta_p = 2$ is discussed and their consequences are noted. The case $p \neq p_c$ is also discussed.

I. INTRODUCTION

The study of critical properties of quenched dilute ferromagnets in which magnetic (nonmagnetic) atoms occupy at random a fraction p $(1-p)$ of the sites of a regular lattice is of current interest. The model normally considered is that in which the magnetic atoms are represented by Ising spins and an exchange interaction J (>0) between the spins exists only for spin pairs occupying nearest-neighbor sites on the lattice. Here we shall be interested essentially in the temperature range $k_BT \ll J$, where percolation behavior is expected since in this limit the pure system ($p=1$) would be ferromagnetic. If one gradually decreases the fraction of magnetic atoms from $p=1$ a critical fraction p_c of magnetic sites will be reached below which the system no longer exhibits a transition to ferromagnetic order at any finite temperature. For $p=p_c$ the critical temperature is at 0 °K. The critical value p_c is the concentration below which there are no infinite clusters of nearest-neighbor magnetic atoms in the system, and is referred to as the percolation threshold.[1]

Detailed studies of critical behavior of the dilute Ising model in the percolation limit ($T \to 0$), as well as for $T \neq 0$ ($p \neq p_c$), have appeared recently for the case where the spins are arranged on a Bethe lattice.[2,3] It is generally believed that the critical behavior of real three-dimensional lattices will have at least some features in common with the Bethe lattice.[4] At finite temperature the transition to magnetic order is inhibited by thermal fluctuations and therefore it occurs at a value $p > p_c = (z-1)^{-1}$ (z is the lattice coordination number). For the Bethe lattice the critical temperature at a given concentration p or, alternatively, the critical concentration at a given temperature is defined by[5,6] $p \tanh(J/k_BT) = p_c$. The fact that the critical exponents near the finite temperature transition are found to be quite different from the corresponding exponents in the percolation limit raises the interesting question of the crossover between these two types of behavior.[3]

The present paper is devoted to a study of the magnetization in a nonzero magnetic field in the percolation limit. At the critical concentration $p=p_c$ ($T \to 0$) one expects the mean magnetization per occupied site $\langle m \rangle$ in an applied field h to vary as

$$\langle m \rangle \simeq A h^{1/\delta_P} \tag{1}$$

for sufficiently low fields. Here δ_P is a critical exponent and A is a constant depending on details of the system such as the lattice coordination number, etc. A relation of this type with the value $\delta_P = 2$ has been derived approximately by Essam *et al.*[2] for the Ising model on a Bethe lattice, starting from an exact expression for the magnetization as an average over the distribution of sizes of (finite) clusters of nearest-neighbor spins, isolated from the rest of the lattice by a boundary of nonmagnetic sites. From the discussion of Sec. II it appears, however, that the approximations made by Essam *et al.*[2] (see also Ref. 3) in obtaining their value $\delta_P = 2$ are not mathematically justified. As a result, the value $\delta_P = 2$ is *not* exact and could be at best a reasonable approximation only. In Sec. II we derive an exact moment expansion for the configuration-averaged magnetization for the dilute Ising model on a Bethe lattice, which we then study numerically for $z=3$ and $z=4$. The numerical results indicate, however, that the value $\delta_P = 2$ of Essam *et al.*[2] represents a good approximation for this critical exponent for extremely low fields, $h \lesssim 10^{-4}(2\beta)^{-1}$. The approximate numerical value of the prefactor A obtained by Young[3] is also consistent with our exact results. Finally, for $p \neq p_C$ our treatment readily yields the value 1 for the exponent γ_P which characterizes the divergence of the susceptibility[5,6] for $p \to p_c$ ($T=0$). Some final conclusions are drawn in Sec. III and further consequences of the deviation of δ_P from the value of 2 are pointed out.

II. MAGNETIZATION IN NONZERO FIELD

Consider a Bethe lattice for which the central site, denoted by 0, has z nearest neighbors denoted by i, $i=1,\ldots,z$, and each of the neighbors is connected to $z-1$ further nearest neighbors going outwards (i.e., neglecting the origin) and denoted by i_α, where $\alpha=1,\ldots,z-1$. Looking outwards into the network each of the sites i_α is again connected to $z-1$ nearest neighbors and so on. We introduce as usual a random variable ϵ_i equal to unity if there is a magnetic atom characterized by an Ising spin at site i and zero otherwise. The probability distribution $p_\epsilon(\epsilon_i)$ of ϵ_i is thus given by

$$p_\epsilon(\epsilon_i)=p\delta(\epsilon_i-1)+(1-p)\delta(\epsilon_i). \qquad (2)$$

In order to calculate the magnetization per occupied site averaged over the magnetic site configurations, we start from the general formulation for finite temperatures, given by Young.[3] Thus, following Young,[3] we introduce the probability F_0 that the spin at the origin is down for a given configuration of occupied sites divided by the probability that it is up for the same configuration. F_0 is simply given by

$$F_0=Z_\downarrow/Z_\uparrow, \qquad (3)$$

where Z_σ ($\sigma=\uparrow$ or $\downarrow$) is the partition function obtained by performing a trace over all spins except that at the origin whose direction σ is specified. Since for a Bethe lattice the traces over spins in the individual branches leaving from the origin are independent, the quantity $Z_\downarrow$ may be written in the form

$$Z_\downarrow=e^{-\beta h}\prod_{i=1}^{z}(Z_\downarrow^i e^{\beta\epsilon_i J}+Z_\uparrow^i e^{-\beta\epsilon_i J}), \qquad (4)$$

where $\beta=1/k_BT$ and h is the external magnetic field. The quantity Z_σ^i is the partition function for all the spins in the $z-1$ branches going outwards from i, but with the spin at site i fixed in direction. Using the notation $Z_\downarrow^i/Z_\uparrow^i=G_i$ one obtains from (3) and (4)

$$F_0=\lambda\prod_{i=1}^{z}R_i, \qquad (5)$$

where

$$R_i=(1+G_iv_i)/(v_i+G_i) \qquad (6)$$

and

$$v_i=e^{2\beta\epsilon_i J},\quad \lambda=e^{-2\beta h}, \qquad (7)$$

By repeating the argument leading to Eqs. (4) and (5) for a spin of specified direction at site i one gets

$$G_i=\lambda\prod_{\alpha=1}^{z-1}R_{i_\alpha}, \qquad (8)$$

where R_{i_α} is given by the analog of Eq. (6) for the site i_α. Thus Eqs. (6) and (8) form a closed system of (nonlinear) algebraic equations for the variables R_i for a fixed configuration of the disordered system. Using Eqs. (3) and (5) the magnetization at the central site assumed to be occupied is defined by

$$m_0=\Big(1-\lambda\prod_{i=1}^{z}R_i\Big)\Big/\Big(1+\lambda\prod_{i=1}^{z}R_i\Big), \qquad (9)$$

for a given configuration of the spin system. The mean (site-independent) magnetization $\langle m\rangle$ per occupied site is then obtained by averaging Eq. (9) over configurations.[7] Denoting by $p_R(R_i)$ the probability distribution of the site variable R_i and recalling that for a Bethe lattice the R_i are independent random variables we have

$$\langle m\rangle=\int\cdots\int\prod_{i=1}^{z}dR_i\,p_R(R_i)\times\Big(1-\lambda\prod_{i=1}^{z}R_i\Big)\Big/\Big(1+\lambda\prod_{i=1}^{z}R_i\Big). \qquad (10)$$

The problem now reduces to the determination of the probability distribution $p_R(R_i)$ as a function of magnetic field, and to the calculation of $\langle m\rangle$ by means of Eq. (10). Here this problem is solved exactly in the percolation limit.

We shall determine $p_R(R_i)$ self-consistently from Eq. (6) after rewriting the latter in the more convenient form[3]

$$R_i=1-\epsilon_i+\epsilon_i(1+vG_i)/(v+G_i),\quad v=e^{2\beta J}, \qquad (11)$$

based on the observation that ϵ_i only takes the values 0 and 1. In the percolation limit, $k_BT\ll J$, this expression simplifies to

$$R_i=1-\epsilon_i+\lambda\epsilon_i\prod_{\alpha=1}^{z-1}R_{i_\alpha}+O(v^{-1}). \qquad (12)$$

Since the variables R_i for different sites on the lattice are statistically at par the probability distributions $p_R(R_{i_\alpha})$ for the variables R_{i_α} must be such that when used in the right-hand side of Eq. (12) it leads to the same probability distribution for the variable R_i on the left-hand side of this equation. From this it follows that $p_R(R_i)$ satisfies the self-consistency condition:

$$p_R(R_i)=\int\cdots\int d\epsilon_i\prod_{\beta=1}^{z-1}dR_{i_\beta}\,p_\epsilon(\epsilon_i)p_R(R_{i_\beta})\,\delta\Big(R_i-1+\epsilon_i-\lambda\epsilon_i\prod_{\alpha=1}^{z-1}R_{i_\alpha}\Big). \qquad (13)$$

In the absence of the external field ($\lambda = 1$) this equation may be solved by inserting a simple binary distribution

$$p_R(R_i) = R\delta(R_i - 1) + (1 - R)\delta(R_i), \qquad (14)$$

which expresses $p_R(R_i)$ as a sum of a contribution from finite clusters (which do not contribute to the mean spontaneous magnetization) and a contribution from infinite clusters. By inserting (2) and (14) in Eq. (13) one finds that a solution is obtained for R satisfying the equation

$$R = 1 - p + pR^{z-1}, \qquad (15)$$

which, in fact, defines R as the probability that a particular branch emerging from the origin is finite, in accordance with Eq. (14). According to this interpretation of R, the spontaneous magnetization is just

$$\langle m\rangle = 1 - R^z, \qquad (16)$$

as may also be verified explicitly by substituting (14) in (10). This equation leads to a variation of $\langle m\rangle$ linear in $p - p_c$ for $p \to p_c$, and thus to a critical index $\beta_P = 1$.[2,3]

We now consider the solution of Eq. (13) for finite values of h ($\lambda \neq 1$). For $p \neq p_c$ the exact solution of Eq. (13) to lowest order in h is straightforward. Writing

$$p_R(R_i) = R\delta(R_i - 1) + (1 - R)\delta(R_i) + g(R_i), \qquad (17)$$

where $g(R_i)$ is linear in h to lowest order, we get

$$g(R_i) = \frac{2p\beta h R^{z-1}}{1 - pp_c^{-1}R^{z-2}}\,\delta'(R_i - 1) + O(h^2) \qquad (18)$$

and from Eq. (10)

$$\langle m\rangle = 1 - R^z + R^z\left(1 + \frac{zpR^{z-2}}{1 - pp_c^{-1}R^{z-2}}\right)\beta h + O(h^2). \qquad (19)$$

Thus for $p \neq p_c$, $\langle m\rangle$ varies linearly with h as discussed, for example, by Young.[3] In particular, Eq. (19) shows that the zero-field susceptibility, $\chi = (\partial/\partial h)\langle m\rangle|_{h=0}$, diverges for $p \to p_c$ with a critical exponent $\gamma_P = 1$.[5,6]

On the other hand, in the case $p = p_c$ (where $R = 1$) Eq. (18) breaks down and a solution of Eq. (13) in closed analytic form is not possible. However, an exact solution in terms of the successive moments of $p_R(R_i)$,

$$F_n = \int_{-\infty}^{\infty} dR_i\, R_i^n p_R(R_i), \qquad (20)$$

may be obtained quite easily. The range of integration in Eq. (20) reduces effectively to the interval $(0^-, \infty)$ since R_i is intrinsically positive. By substituting (2) into (13), multiplying both sides of the resulting equation by R_i^n, and integrating over R_i from $-\infty$ to ∞ we get

$$F_n = 1 - p + p\lambda^n F_n^{z-1}, \quad n \neq 0 \qquad (21)$$

which determines F_n in terms of n and of the parameters h and p. The expression of $p_R(R_i)$ in terms of the moments F_n is obtained in a standard way from the moment-generating function [Fourier transform of $p_R(R_i)$] and is given by

$$p_R(R_i) = \delta(R_i) + \sum_{n=1}^{\infty} \frac{(-1)^n F_n}{n!}\,\delta^{(n)}(R_i), \qquad (22)$$

where $\delta^{(n)}(R_i)$ denotes the nth-order derivative of the δ function. For $h = 0$ ($F_n = R$, $n \neq 0$) this equation reduces to Eq. (14) because of the identity

$$\delta(R_i - 1) = \sum_{n=0}^{\infty} (-1)^n (n!)^{-1}\delta^{(n)}(R_i).$$

For $h \neq 0$ and $p \neq p_c$ one verifies that the solution of Eq. (21) to linear order in h coincides with the nth moment of Eqs. (17) and (18).

Finally by inserting Eq. (22) into Eq. (10) one may express the magnetization directly in terms of the moments F_n. Using the relation

$$\frac{d^n}{dx^n}\left(\frac{1 - cx}{1 + cx}\right)\bigg|_{n=0} = 2c^n(-1)^n n! \quad (n \neq 0),$$

we obtain for $\langle m\rangle$ the exact moment expression

$$\langle m\rangle = 1 + 2\sum_{n=1}^{\infty} (-\lambda)^n F_n^z, \qquad (23)$$

which we shall use to compute $\langle m\rangle$ for $p = p_c = (z - 1)^{-1}$, from the solution of the Eqs. (21). Among the $z - 1$ solutions of Eq. (21) for a given z we must choose the one which satisfies $F_n \to R \to 0$ in the limit $p \to 1$.

In order to exhibit the variation of $\langle m\rangle$ as a function of h and z for $p = p_c$ we have performed explicit numerical calculations for $z = 3$ and $z = 4$. The acceptable solutions of (21) for $p = p_c$ are given by

$$F_n = \lambda^{-n}[1 - (1 - \lambda^n)^{1/2}], \quad z = 3 \qquad (24a)$$

$$F_n = (2/\sqrt{\lambda^n})\cos\{\tfrac{1}{3}[\cos^{-1}(-\sqrt{\lambda^n}) + 4\pi]\}, \quad z = 4. \qquad (24b)$$

For convergence reasons one is not allowed, in general, to expand Eqs. (23), (24a), and (24b) in powers of the magnetic field, keeping only the lowest-order term. We also note that for small values of $2\beta h$ the series (23) converges very slowly since, e.g., for $2\beta h = 10^{-3}$ one needs about 10^4 terms to get the correct value of the sum in Eq. (23) to four significant figures. The results for the mean magnetization (23) determined with the above accuracy are plotted in Fig. 1 for the range $0.001 \leqslant 2\beta h \leqslant 0.1$. Values of the magnetization $\langle m\rangle \equiv q_i$, $z = i$, in the range $10^{-4} \leqslant 2\beta h \leqslant 1.410^{-3}$ are listed in Table I. The latter values were obtained by including in the summation of Eq. (23) all terms $\geqslant 10^{-5}$ which, typically, involves summing 10^4 terms

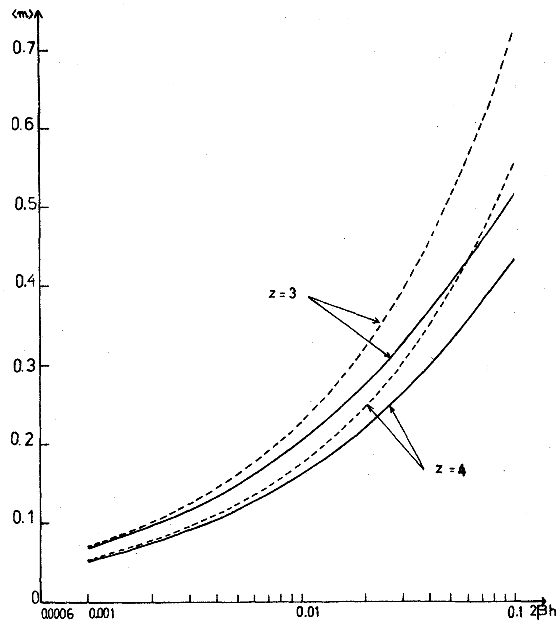

FIG. 1. Magnetization per occupied site for the dilute Ising model as a function of magnetic field h, at the percolation threshold ($p=p_c$). The full line represents the exact results and the broken line the approximate formula [Eq. (25)] of Essam *et al.*, Refs. 2 and 3.

for $2\beta h=10^{-3}$ and 2×10^5 terms for $2\beta h=10^{-4}$.

We wish to compare the results of Fig. 1 and Table I with an approximate expression for the magnetization at $p=p_c$ obtained by Essam *et al.*[2] and by Young[3] and given by

$$\langle m\rangle \simeq \frac{2\mu z}{[(z-1)(z-2)]^{1/2}}\sqrt{\beta h}\,, \tag{25}$$

where $\mu=0.7602$ (Ref. 3) and the critical exponent δ_P has the value $\delta_P=2$. It is instructive to show how Eq. (25) may be obtained in the framework of the present treatment. Going back to Eq. (10) we first expand the denominator in the integrand in the form of an infinite power series in the quantity $\frac{1}{2}(1-\lambda\Pi_{i=1}^{z}R_i)$, which satisfies the condition $0\leq\frac{1}{2}(1-\lambda\Pi_{i=1}^{z}R_i)\leq1$ for any given configuration of the system. Next we insert Newton's binomial expansion for the quantities $(1-\lambda\Pi_{i=1}^{z}R_i)^n$ in the above series and use Eq. (22) for $p_R(R_i)$ to perform the integrations. In this way we obtain the exact expression

$$\langle m\rangle=\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{1}{2^n}(-1)^{r+1}\binom{n}{r}(1-\lambda^rF_r^z)\,, \tag{26}$$

which should be compared with the simpler form (23). On the other hand, recalling that for $h=0$ one has $F_r=R=1$ for $p=p_c$, we may obtain the low-order moments (such that $2r\beta h\ll1$) at nonzero field, for $p=p_c$, by iterating Eq. (21) to lowest order in $2r\beta h$ thus approximating λ^r by $1-2r\beta h$. This gives

$$F_r=1-\frac{2\sqrt{r\beta h}}{[(z-1)(z-2)]^{1/2}}[1+O(2r\beta h)]\,. \tag{27}$$

If one now arbitrarily uses Eq. (27) for *all* moments occurring in Eq. (26) one readily recovers Eq. (25) with μ defined by a numerical series identical to that given by Young.[3] However, since Eq. (27) is incorrect for the high-order moments in Eq. (26)—the iteration of F_r around unity and the replacement of the infinite series for λ^r by its first two terms are wrong for $r>(2\beta h)^{-1}$—the expression (25) for $\langle m\rangle$ is at best only approxi-

TABLE I. Exact values for the magnetization per occupied site $\langle m\rangle\equiv q_i$, the critical exponent $\delta_P\equiv\delta_i$, and the ratio q_3/q_4 at the percolation threshold ($p=p_c$), for Bethe lattices of coordination number $z=i$, at very low magnetic fields. Each critical exponent value δ_i is associated with two different values of the field parameter $2\beta h$ as explained in the text.

$2\beta h$ \ $2\beta h$		0.0014	0.001	0.0006	0.0003	0.0001
	q_3	0.081 912	0.069 668	0.054 387	0.038 760	0.022 558
	$\frac{q_3}{q_4}$ \ q_4	0.063 775	0.054 146	0.042 180	0.029 994	0.017 418
0.0014	1.284		$\delta_3=2.0782$ $\delta_4=2.0558$			
0.001	1.287			$\delta_3=2.0630$ $\delta_4=2.0454$		
0.0006	1.289				$\delta_3=2.0464$ $\delta_4=2.0329$	
0.0003	1.292					$\delta_3=2.0296$ $\delta_4=2.0214$
0.0001	1.295					

mate. In particular, the above discussion and some further remarks below indicate that the actual exponent δ_P differs from 2. We note, incidentally, that for $p=p_c$ the high-order moments ($2r\beta h \gg 1$) differ quite drastically from the low-order ones, Eq. (27), since they tend asymptotically to the value $1-p_c$ [Eq. (21)].

For comparison with our exact results the approximate expression (25) is also plotted in Fig. 1. It is seen that Eq. (25) agrees quite well with the exact results for very small fields ($2\beta h \sim 10^{-3}$), while large deviations both in the magnitude and in the field dependence of $\langle m \rangle$ appear already in the range $2\beta h \sim 0.1$. In connection with Eq. (25) we may remark that, due to the fact that it has not been justified mathematically, its range of validity within the domain $2\beta h \ll 1$ could only be found from a comparison with our exact numerical results and, furthermore, the existence of such a range is by itself partly fortuitous. After having located approximately the range of validity of Eq. (25), we may perform a more detailed comparison with the exact results by going to lower fields such as those considered in Table I. We focus attention on the values of the critical exponent $\delta_P \equiv \delta_i$, $z=i$, and on the ratio q_3/q_4 of the magnetizations for the two lattices, $z=3$ and $z=4$. According to Eq. (25) the latter ratio should take the field-independent value $q_3/q_4 = \frac{3}{4}\sqrt{3} \simeq 1.299$. Values of δ_i may be extracted from our exact results by assuming Eq. (1) to be valid at low fields and adjusting the latter equation to the numerical results for $\langle m \rangle \equiv q_i$ at two neighboring values of $2\beta h$. Values obtained in this way for δ_3 and δ_4 are listed in Table I, along with the ration q_3/q_4. It is seen that δ_3 and δ_4 gradually decrease with decreasing fields to reach the values $\delta_3 \sim 2.03$ and $\delta_4 \sim 2.02$ at our lowest-field values, $h \sim 10^{-4}(2\beta)^{-1}$. This lends further support to the approximate validity of Eq. (25) for very low fields. However, in view of the fact that the value $\delta_P = 2$ given by Eq. (25) cannot be exact, the results of Table I suggest that the actual exponent δ_P for $h \to 0$ will depend, albeit quite weakly, on the lattice coordination number z. From From (27) it follows that for $h \to 0$ the Eq. (26) [or (23)] may be written approximately as the sum of a term proportional to $\sqrt{h}$ and a remainder involving the high-order moments F_r which must be treated exactly as discussed above. This shows that $\langle m \rangle$ is a nonanalytic function of $\sqrt{h}$ for finite h [see, e.g., Eqs. (24a) and (24b)] and so the value $\delta_P = 2$ cannot be regarded as the exact critical exponent, in spite of the fact that the term proportional to $\sqrt{h}$ in $\langle m \rangle$ does give the dominant contribution for $h \to 0$ (see Table I). Finally, we note that the ratio q_3/q_4 increases gradually towards a value of 1.295 for $2\beta h = 10^{-4}$, which is again consistent with Eq. (25). The weak field dependence of the ratio q_3/q_4 in Table I is, of course, nothing but another manifestation of the z dependence of δ_P in Eq. (1).

III. CONCLUDING REMARKS

From an exact moment expansion for the magnetization of a Bethe lattice with a concentration p of Ising spins we have obtained detailed numerical results both for the magnetization and the critical exponent δ_P for $p=p_c$, in the range of low fields ($2\beta h \ll 1$) down to values $2\beta h \sim 10^{-4}$. For the lowest-field values these exact results compare favorably with the approximate analytic expression for the magnetization of Essam *et al.*,[2] although some deviations with respect to the value $\delta_P = 2$ of the latter equation are found. Finally the present exact study of the magnetization near $p=p_c$ in the percolation limit should be useful in view of the increasing interest in analyzing the more difficult problem of the critical behavior of dilute ferromagnets at finite temperature (see Ref. 3 and references quoted therein) and its relation to the above limit.

The deviation of the exponent δ_P from the value of 2 suggested by the discussion of Sec. II has some interesting general consequences. First, it implies a weak violation of the general scaling relation for percolation processes, namely,[2]

$$1/\delta_P = 1 - \gamma_P/\Delta_P, \tag{28}$$

in the case of the Bethe lattice, where $\gamma_P = 1$ and the gap exponent Δ_P takes the value $\Delta_P = 2$.[8]

A second consequence has to do with the known relation between percolation processes, such as the one considered here, and the ferromagnetic transition in a pure system described by the Potts model.[9] In a previous discussion[10] it has been implied that this relation is supported by the comparison of the critical exponents for the mean field treatment of the s-state Potts model for $s<2$ and the percolation exponents for the dilute Ising model on a Bethe lattice. Indeed the mean-field treatment for the Potts model yields $\beta=1$ and $\gamma=1$ in agreement with the percolation exponents discussed in Sec. II. However, Harris *et al.*[10] did not consider the exponent δ, whose analysis reveals that the mean-field treatment of the Potts model confirms, in fact, only partially the relationship between this model and the percolation processes. The exponent δ may be obtained from the mean-field free energy per site F for the Potts model, as given by Eq. (6) of Harris *et al.*,[10] by solving the equation $h = (\partial F/\partial m)_T$ giving the magnetic field h in terms of the magnetization per site m for $T=T_c$. This yields the value $\delta = 2$ for $s<2$, which disagrees slightly with the exact value of δ_p suggested by the analysis of Sec. II. Note that this disagreement is not in contradiction with the general relationship between the Potts model and percolation processes; it only shows that the mean-field treatment of the Potts model is not exact from the point of view of the comparison of δ with the exact value of the percolation exponent δ_p for a Bethe lattice. In fact, this is not surprising since the mean-field treatment of the Potts model is expected to provide an accurate description of phase transitions in two or higher dimensions only when the number of states s is large.[11]

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[1]For a review see V. H. S. Shante and S. Kirkpatrick, Adv. Phys. 20, 325 (1971).

[2]J. W. Essam, K. M. Gwilym, and J. M. Loveluck, J. Phys. C 9, 365 (1976).

[3]A. P. Young, J. Phys. C 9, 2103 (1976).

[4]M. E. Fisher and J. W. Essam, J. Math. Phys. 2, 609 (1961).

[5]A. B. Harris, J. Phys. C 7, 1671 (1974).

[6]G. M. Bell, J. Phys. C 8, 669 (1975).

[7]The mean magnetization per site $\langle M\rangle$ is simply $\langle M\rangle = p\langle m\rangle$.

[8]J. W. Essam and K. M. Gwilym, J. Phys. C 4, L228 (1971).

[9]C. M. Fortuin and P. W. Kasteleyn, Physica (Utr.) 57, 536 (1972).

[10]A. B. Harris, T. C. Lubensky, W. M. Holcomb, and C. Dasgupta, Phys. Rev. Lett. 35, 327 (1975).

[11]L. Mittag and M. G. Stephen, J. Phys. A 7, L109 (1974).