

Two possible types of superfluidity in crystals

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Using a modified procedure to derive hydrodynamics, it is shown from symmetry considerations that a superfluid velocity \vec{v}^s which is invariant under a Galilean transformation will be driven by the temperature: $\vec{v}^s + (k_B/m) \vec{\nabla} T = 0$. A crystal with such a velocity is able to sustain a persistent entropy flux rather than a mass current and has a propagating mode connected to temperature fluctuations. On the other hand, only a crystal with a \vec{v}^s which behaves like a true velocity under a Galilean transformation, will be able to sustain a persistent mass current and have vacancy propagation as its Goldstone mode. This spectrum differs, however, from that given by previous authors.

I. INTRODUCTION

Numerous theoretical investigations¹⁻⁶ of recent years have advanced the opinion that some form of superfluidity in crystals is a distinct possibility. Aside from microscopic approaches such as explicitly constructing a Bose condensed wave function which complies with a crystalline ordering,⁴⁻⁶ there were also attempts to study the problem from macroscopic symmetry considerations. Envisioning a quantum crystal capable of sustaining a persistent (liquidlike) flow of defects, Andreev and Lifshitz¹ derived its equations of motion in close analogy to the two-fluid hydrodynamics,^{7,8} and predicted a Goldstone mode, the vacancy propagation, which has a spectrum similar to fourth sound in He II. In addition, the velocities of elastic waves are found to be modified by a common factor proportional to the superfluid density, making it possible to detect superfluidity in a crystal by measuring the spectrum of elastic waves.

Recently, Saslow² suggested that, due to the presence of the lattice as a preferred inertial frame the superfluid velocity \vec{v}^s should be invariant under a Galilean transformation, i.e., \vec{v}^s should transform as a velocity difference. This alters one of the basic assumptions of the hydrodynamics as derived by Andreev and Lifshitz and makes a re-derivation necessary. Saslow² found that the structure of the hydrodynamic equations remains the same, and the only modification occurs in the thermodynamic relation between the superfluid velocity and the momentum density. But that was enough to cancel the changes in elastic waves: they now remain essentially unaltered going through the superfluid transition, making the measurement of elastic waves unsuitable for the detection of superfluidity.

The purpose of the present paper is to show that (i) only a "Galilean" \vec{v}^s (i.e., a \vec{v}^s which behaves as a true

velocity under a Galilean transformation: $\vec{v}^s \rightarrow \vec{v}^s + \vec{\omega}$) will lead to superfluidity as characterized by a persistent mass flow; while (ii) an "invariant" \vec{v}^s (namely, $\vec{v}^s \rightarrow \vec{v}^s$, as employed by Saslow), will give rise to a different kind of superfluidity—a persistent (or super) entropy flow. We will see that the modification made by Saslow suffers from internal inconsistency, and that the Andreev and Lifshitz equations are the only possible ones to describe superfluidity of mass flow in a crystal. (When rectified, Saslow's hydrodynamic equations are consistent with those of Andreev and Lifshitz.) We shall also see that (iii) the collective modes as calculated by Andreev and Lifshitz are not generally correct. This is easily recognized by the following argument. Since their equations of motion are given by generalizing two-fluid hydrodynamics, we must be able to obtain the spectrum of first (and second) sound in a superfluid liquid by setting to zero the appropriate elastic coefficients. Now, because the elastic waves as calculated by them are changed by a *common* factor, independent of the thermal expansion coefficient, one has to conclude that first sound, too, is changed by the same factor. This, of course, is in contradiction to the well-known fact that first sound is only altered to the extent that the thermal expansion coefficient is not neglected.⁷ By the same token, vacancy propagation cannot be given by a fourth-sound-like spectrum, but must be given by a generalization of second-sound spectrum. (The error can be traced back to the fact that Andreev and Lifshitz employed a perturbation calculation, in which the perturbation can be comparable to the zeroth-order term.) Our results will show that the change in the elastic waves is still proportional to ρ^s , although markedly different for transverse and longitudinal excitations. They remain a valid indicator for any superfluidity in crystals.

We then go on to explore (iv) the consequences

of a persistent entropy current as a result of introducing an invariant superfluid velocity to the crystal. We find that here the Goldstone mode is a temperature propagation. This conclusion leads naturally to two questions: first, why in a superfluid liquid, do a "Galilean" \vec{v}^s and a persistent mass flow, defying the above categorization, give rise to temperature propagation? An second, what are the connections between a super entropy flow and the familiar phenomenon of second sound in ordinary crystals, such as described by Enz's⁹ two-fluid hydrodynamics? Both questions are answered and after a detailed comparison with Enz's equations, a number of corrections and generalizations are suggested.

The usual way of deriving the hydrodynamic equations consists mainly in the manipulation of the redundant flux of the energy density. This procedure has been explained carefully in the two books by Khalatnikov⁷ and Putterman.⁸ However, one can construct examples to show that this method is not free of ambiguities, and especially in complicated systems, there is a tendency for one to mistake assumptions for results. We shall therefore (v) introduce a supplementary procedure to derive the linear-reactive terms which, when used in conjunction with the standard method, eliminates the ambiguities and yields a better understanding of the structure of the hydrodynamic equations.

This paper is organized as follows: In Sec. II, we define the superfluid crystal thermodynamically and discuss a number of thermodynamic derivatives which have given rise to some confusion in the literature. In Sec. III, the hydrodynamic equations are derived and investigated and their propagating modes calculated. It represents the main part of the present paper. In Sec. IV, we examine how to describe second sound in ordinary crystals employing the equations of Sec. III. Section V consists of summary and concluding remarks.

II. THERMODYNAMIC RELATIONS

The basic identity governing the thermodynamics of a superfluid crystal is given by^{1,2}

$$d\epsilon = \mu d\rho + T ds + v_i^n dg_i + (\hbar/m) j_i^s d\nabla_i \phi + \lambda_{ij} d\nabla_j u_i. \quad (2.1)$$

As in Ref. 2, ϵ , ρ , s , and g_i are the densities (per unit volume) of energy, mass, entropy, and momentum, respectively, u_i is the displacement vector, and ϕ is a scalar potential whose derivative $\vec{v}^s \equiv (\hbar/m)\vec{\nabla}\phi$ is required as an additional velocity to determine the equilibrium state (m is the mass of the atom). Note that ϕ is introduced

here as an unspecified velocity potential rather than a macroscopic phase variable because the latter interpretation is too restrictive, and it is never needed for the hydrodynamic derivation. The quantities μ , T , v_i^n , j_i^s , and λ_{ij} are the respective conjugate variables and defined by Eq. (2.1).

The change in the chemical potential $\delta\mu$, the temperature δT and $\delta\lambda_{ij}$ —linearized with respect to the normal and superfluid velocities, \vec{v}^n and \vec{v}^s —are given by the change in the thermodynamic variables $\delta\rho$, δs , $\delta\nabla_i u_j$, which are even under time inversion; \vec{g} and \vec{j}^s , on the other hand, are functions only of the odd variables, \vec{v}^s and \vec{v}^n , i.e.,

$$g_i = \frac{\partial g_i}{\partial v_j^n} v_j^n + \frac{\partial g_i}{\partial v_j^s} v_j^s, \quad j_i^s = \frac{\partial j_i^s}{\partial v_j^n} v_j^n + \frac{\partial j_i^s}{\partial v_j^s} v_j^s.$$

The Maxwell relation

$$\frac{\partial g_i}{\partial v_j^s} = - \frac{\partial j_i^s}{\partial v_j^n}, \quad (2.2)$$

reduces the four elastic coefficients to three independent ones. Here, one can employ the Galilean transformation properties to further reduce the number by one.¹⁰ The momentum and energy density will change under an infinitesimal Galilean transformation $d\vec{\omega}$ according to the well-known formulas⁷:

$$d\vec{g} = \rho d\vec{\omega} \quad \text{and} \quad d\epsilon = \vec{g} \cdot d\vec{\omega}. \quad (2.3)$$

We may now distinguish two cases:

(i) If \vec{v}^s is taken to be *invariant* under a Galilean transformation, as are all the other variables of Eq. (2.1), i.e., $d\rho = ds = d\nabla_j u_i = dv_i^s = 0$, then inserting Eq. (2.3) into Eq. (2.1) yields

$$\vec{g} = \rho \vec{v}^n, \quad (2.4)$$

and due to the Maxwell relation, Eq. (2.2), \vec{j}^s cannot be a function of \vec{v}^n . Linearized with respect to \vec{v}^s , we have

$$j_i^s = \rho_{ij}^s v_j^s. \quad (2.5)$$

(ii) On the other hand, if \vec{v}^s is a *Galilean* velocity, $d\vec{v}^s = d\vec{\omega}$, we will instead arrive at

$$\vec{g} = \rho \vec{v}^n + \vec{j}^s \quad (2.6)$$

or $\partial j_i^s / \partial v_j^s = \partial g_i / \partial v_j^s = -\partial j_j^s / \partial v_i^n$. Linearized, it gives

$$j_i^s = \rho_{ij}^s (v_j^s - v_j^n). \quad (2.7)$$

In arriving at Eq. (2.7), we have also used the Maxwell relation $\rho_{ij}^s = \rho_{ji}^s$.

Saslow² combined Eqs. (2.5) and (2.6), therefore his assumption of an invariant \vec{v}^s was not correctly executed. More technically, the Maxwell relation, Eq. (2.2), was violated. The correction of this one oversight, when made in compliance with a

number of interrelating symmetry requirements, leads amazingly enough to a quite different set of equations. In particular, the time rate of change of the superfluid velocity will no longer be given by the gradient of the chemical potential, but rather by that of the temperature. This is an excellent example of the cogency and rigidity of the structure of the hydrodynamic equations.

In deriving the equations of motion and calculating the collective modes, we shall need (or find convenient) alternative thermodynamic identities. First, we shall consider the energy of an arbitrary volume V inside the crystal. As long as the linear dimension of V (in the direction of the wave vector \vec{q}) is much smaller than the wavelength, the extensive variables are given by

$$E = \epsilon V, \quad M = \rho V, \quad S = sV, \quad G_i = g_i V. \quad (2.8)$$

With the help of the Duhem-Gibbs relation

$$P = -\epsilon + \mu\rho + Ts + v_i^n g_i, \quad (2.9)$$

Equation (2.1) can be rewritten

$$dE = \mu dM + T dS + v_i^n dG_i + V \left[\frac{\hbar}{m} j_i^s d\nabla_i \phi + \lambda_{ij} d\nabla_j u_i \right] - P dV.$$

This is valid for an arbitrary volume. We now define the change of this volume to be given by the motion of the lattice points, i.e.,

$$dV = V d\nabla_i u_i, \quad (2.10)$$

which gives

$$dE = \mu dM + T dS + v_i^n dG_i + V \left[\frac{\hbar}{m} j_i^s d\nabla_i \phi + \sigma_{ij} d\nabla_j u_i \right], \quad (2.11)$$

$$\begin{aligned} \left(\frac{\partial \sigma_{ij}}{\partial \nabla_k u_i} \right)_{M,S} &= \frac{\partial \sigma_{ij}}{\partial \nabla_k u_i} - \left(s \frac{\partial}{\partial s} + \rho \frac{\partial}{\partial \rho} \right) \sigma_{ij} \delta_{ki} \\ &= \frac{\partial \lambda_{ij}}{\partial \nabla_k u_i} - \left(s \frac{\partial}{\partial s} + \rho \frac{\partial}{\partial \rho} \right) (\lambda_{ij} \delta_{ki} + \lambda_{ki} \delta_{ij}) + \left(s^2 \frac{\partial T}{\partial s} + \rho^2 \frac{\partial \mu}{\partial \rho} + 2\rho s \frac{\partial T}{\partial \rho} \right) \delta_{ij} \delta_{ki}. \end{aligned} \quad (2.13)$$

(In the two expressions of the right-hand side $\nabla_k u_i$, s and ρ are the variables, and two of them are always held constant.)

We will next consider the fluctuation in the mass M (or particle number M/m), which is physically connected to the change in the vacancy concentration c : The difference D between the number of vacancies and interstitials in V is given by $D = N - M/m$, where N is the number of lattice sites. We have $dN = 0$, because the volume, as defined in Eq. (2.10)—though it may be changing with time—will keep its number of lattice sites constant. Therefore,

with

$$\sigma_{ij} = \lambda_{ij} - P \delta_{ij}. \quad (2.12)$$

We shall use Eq. (2.11) extensively. Its importance stems from the fact that the definition, Eq. (2.10), is a straightforward generalization of the case where V is the total volume of the crystal in global equilibrium¹¹ to the case where V is an infinitesimal volume inside the crystal in local equilibrium. The thermodynamic derivatives in these variables have simple and immediate experimental significance. For example, $(\partial \sigma_{ij} / \partial \nabla_k u_i)_{M,S}$ is the elastic tensor connected to changes in volume and shape of an isolated crystal, i.e., of one, whose total mass M and entropy S are kept constant. On the other hand, $(\partial \lambda_{ij} / \partial \nabla_k u_i)_{\rho,s}$ corresponds to the same changes of a crystal whose total particle number and entropy have to be varied to ensure that the respective densities are unaltered. This is obviously a much more intricate experimental prescription. Therefore, it is not surprising that the hydrodynamic equations have a much simpler structure in the extensive variables, and accordingly, the propagating modes are most conveniently expressed in the thermodynamic derivatives of Eq. (2.11). The elastic waves of an ordinary crystal, e.g., are given by

$$\omega^2 = (\partial \sigma_{ij} / \partial \nabla_k u_i)_{M,S} q_j q_k \hat{u}_i \hat{u}_i / \rho$$

[see Eq. (3.25) below]. Expressed in terms of the various elastic coefficients of Eq. (2.1), the spectrum is more difficult to interpret and looks correspondingly awkward:

$$dc \equiv d(D/N) = -dM/Nm. \quad (2.14)$$

(Concentration of long-lived defects is a more adequate name for c , since a negative c would mean presence of interstitials, and a vanishing c does not necessarily imply that the crystal is perfect.) Another useful thermodynamic identity is¹²

$$\begin{aligned} de &= T dS + v_i^n dv_i - \frac{P}{\rho(1-c)} dc \\ &+ \frac{\sigma_{ij}}{\rho} d\nabla_j u_i + \frac{j_i^s}{\rho} dv_i^s, \end{aligned} \quad (2.15)$$

where $e = \epsilon/\rho$, $S = s/\rho$, $v_i = g_i/\rho$ are the energy, entropy, and momentum per unit mass, respectively. The tensor σ_{ij} is the same as defined in Eq. (2.12). Note that in a liquid, the energy density ϵ as given by Eq. (2.1) does not depend on the strain tensor and λ_{ij} vanishes identically. Therefore σ_{ij} is given by

$$\sigma_{ij} = -P\delta_{ij} = (\epsilon - \mu\rho - Ts - \vec{\nabla}^n \cdot \vec{g})\delta_{ij}. \quad (2.16)$$

This equation not only states the inability of a liquid to sustain shear stresses (i.e., $\lambda_{ij} \sim \delta_{ij}$), it also implies the zero energy required in a liquid for displacing a particle from the bulk to the surface (i.e., $\lambda_{ii} = 0$, since $d\epsilon = \frac{1}{3}\lambda_{ii}d\nabla_k u_k$ gives the compressional energy at constant density, or the energy connected with the creation of vacancies and interstitials in a crystal).

It is instructive to examine more closely some of the thermodynamic derivatives which have given rise to confusion in the literature. First we observe that, setting $dc = 0$, or equivalently $dM = d(\rho V) = 0$, yields

$$d\rho = -\rho \frac{dV}{V} = -\rho d\nabla_i u_i. \quad (2.17)$$

Equation (2.17) is familiar as the condition of an ideal crystal. Defining

$$\rho_{ij} \equiv \left(\frac{\partial \rho}{\partial \nabla_i u_j} \right)_\mu + \rho \delta_{ij}, \quad (2.18)$$

Andreev and Lifshitz¹ argued that because $\partial\rho/\partial\nabla_i u_j = -\rho\delta_{ij}$ in an ideal crystal, $\rho_{ij} \ll \rho$ for not too high a vacancy concentration, and proceeded to treat ρ_{ij} as a small perturbation. To see why this argument is misleading we have to sharpen our notion of a perfect crystal, of which there are two kinds. We may call a crystal "globally" perfect if its total number of lattice sites equals its total particle number, but the fluctuation of its vacancy concentration does not identically vanish and varies, e.g., sinusoidally as $c \sim \sin\vec{q} \cdot \vec{r}$ with $\int c dV = 0$. Only if we also restrict the fluctuations, setting $c \equiv 0$, will the crystal become "locally" perfect. Equation (2.17) was arrived at by restricting the fluctuations of the vacancy concentration, setting $c = \text{const.}$ in a crystal with arbitrary defect number $\int c dV$. The equation is therefore valid only in a locally perfect crystal, but not in a globally perfect one. Hence, it cannot be approximately correct in a crystal which is nearly globally perfect. With a more precise statement of Eq. (2.17)

$$\left(\frac{\partial \rho}{\partial \nabla_i u_j} \right)_M = -\rho \delta_{ij},$$

we can in fact plausibly argue that ρ_{ij}/ρ may be a quantity of the order of unity. Inserting

$$\left(\frac{\partial \rho}{\partial \nabla_i u_j} \right)_\mu = \left(\frac{\partial \rho}{\partial \nabla_i u_j} \right)_M - \left(\frac{\partial \rho}{\partial \mu} \right)_{\nabla u} \left(\frac{\partial \mu}{\partial \nabla_i u_j} \right)_M$$

into Eq. (2.18) yields, e.g.,

$$\frac{\rho_{33}}{\rho} = - \left(\frac{1}{\rho} \frac{\partial \rho}{\partial \mu} \right)_{\nabla_3 u_3} \left(\frac{\partial \mu}{\partial \nabla_3 u_3} \right)_M = \left(\frac{\partial \mu}{\partial \rho} \right)_M / \left(\frac{\partial \mu}{\partial \rho} \right)_{\nabla_3 u_3}.$$

[The strain tensor $\nabla_i u_j$ aside from the (3,3) component is held constant in all the derivatives of the last equation.] When the density is varied, while either the vacancy concentration is kept constant ($dM = 0$) or the lattice rigid ($d\nabla_3 u_3 = 0$), the respective changes in the chemical potential are unlikely to be orders of magnitude apart. However, microscopic calculations of ρ_{ij} seem lacking.

For later convenience, we define

$$\begin{aligned} \beta_{ij} &\equiv \left(\frac{\partial \mu}{\partial \nabla_i u_j} \right)_{M, S} = \left(\frac{\partial \mu}{\partial \nabla_i u_j} \right)_{c, S} \\ &= V \left(\frac{\partial \sigma_{ji}}{\partial M} \right)_{\nabla u, S} = \left(\frac{\partial \sigma_{ji}}{\partial \rho} \right)_{\nabla u, S}. \end{aligned} \quad (2.19)$$

[The third equal sign here, and also in Eq. (2.20) below, is a Maxwell relation.] As discussed above, β_{ij} is a quantity of the same order of magnitude as $-\rho(\partial\mu/\partial\rho)_{\nabla u, S}$. (In liquid at zero temperature, $d\sigma_{ij} = -\rho d\mu\delta_{ij}$, and the two quantities are in fact equal.)

Another thermodynamic cross derivative,

$$\begin{aligned} \alpha_{ij} &\equiv \left(\frac{\partial T}{\partial \nabla_i u_j} \right)_{M, S} = \left(\frac{\partial T}{\partial \nabla_i u_j} \right)_{c, S} \\ &= V \left(\frac{\partial \sigma_{ji}}{\partial S} \right)_{M, \nabla u} = \left(\frac{\partial \sigma_{ji}}{\partial S} \right)_{\rho, \nabla u} \end{aligned} \quad (2.20)$$

is connected to the thermal expansion coefficient

$$\left(\frac{\partial \nabla_i u_j}{\partial T} \right)_{c, M} = - \left(\frac{\partial \nabla_i u_j}{\partial \sigma_{kl}} \right)_{M, T} \left(\frac{\partial S}{\partial T} \right)_{\nabla u, M} \left(\frac{\partial \sigma_{kl}}{\partial S} \right)_{M, \nabla u}$$

and can be treated as a small quantity. Note that

$$\begin{aligned} \left(\frac{\partial S}{\partial \nabla_i u_j} \right)_{T, c} &= - \left(\frac{\partial S}{\partial T} \right)_{\nabla u, c} \left(\frac{\partial T}{\partial \nabla_i u_j} \right)_{s, c} \\ &= - \left(\frac{\partial S}{\partial T} \right)_{\nabla u, c} \alpha_{ij} - s \delta_{ij} \end{aligned} \quad (2.21)$$

is not proportional to α_{ij} or the thermal expansion coefficient.⁹ In a cubic crystal we have¹¹

$$\alpha_{ij} = \alpha \delta_{ij}, \quad \beta_{ij} = \beta \delta_{ij}, \quad \rho_{ij}^s = \rho^s \delta_{ij}. \quad (2.22)$$

III. EQUATIONS OF MOTION

A. Linear reactive terms

The derivation of the hydrodynamic equations is usually accomplished by first writing down the conservation laws, the law of increase of entropy and the equations of motion for the order parameter (in our case, ϕ and u_i). One then determines the

various unknown fluxes by observing that the energy current is redundant and the energy conservation must be consistent with, and implied by, the form of all the other fluxes. The classical introduction to this procedure is given in Ref. 7 by Khalatnikov, a more detailed and careful explanation can be found in Ref. 8 by Putterman. The famous two-fluid hydrodynamics of He II,⁷ nematodynamics,¹³ hydrodynamics of superfluid ³He,¹⁴ and of superfluid^{1,2} and normal crystals¹² are all derived by employing this standard procedure. Also, the method is widely used to derive various nonlinear terms.^{2,7,15}

Partly due to the success of Landau's two-fluid hydrodynamics, complete rigor of the derived equations is claimed on occasions on the sole ground that the fluxes satisfy the differential equation of the standard procedure. However, one can construct examples to show that the standard method is, when treating the reactive part especially in complicated systems, not free of ambiguities. As has been first observed by Halperin and Hohenberg¹⁶ and discussed in more detail by Putterman,⁸ the differential equation of the

standard procedure is compatible with the equation of motion $\dot{\phi} + \mu + A = 0$, where A is a function of temperature alone. The entropy current would be simultaneously modified to include the additional term $j_i^s \partial A / \partial T$. There are other ambiguities. Restricting ourselves for the present to the reactive terms only, we can write

$$\dot{\epsilon} + \nabla_k Q_k = 0, \quad (3.1)$$

$$\dot{\rho} + \nabla_k j_k = 0, \quad (3.2)$$

$$\dot{s} + \nabla_k f_k = 0, \quad (3.3)$$

$$\dot{g}_i + \nabla_i P + \nabla_j \pi_{ij} = 0, \quad (3.4)$$

$$(\hbar/m)\dot{\phi} + Z = 0, \quad (3.5)$$

$$\dot{u}_i + Y_i = 0, \quad (3.6)$$

where Z and Y_i , according to the assumption of local equilibrium, are arbitrary functions of μ , T , \vec{v}^n , \vec{v}^s , and $\nabla_i u_j$. The standard procedure then yields the two equations

$$Q_k = \mu j_k + T f_k + v_i^n \pi_{ik}, \quad (3.7)$$

and

$$\begin{aligned} 0 = & \nabla_k \mu \left(-j_k + \rho v_k^n + j_k^s \frac{\partial Z}{\partial \mu} + \lambda_{jk} \frac{\partial Y_j}{\partial \mu} \right) + \nabla_k T \left(-f_k + s v_k^n + j_k^s \frac{\partial Z}{\partial T} + \lambda_{jk} \frac{\partial Y_j}{\partial T} \right) \\ & + \nabla_k v_i^n \left(-\pi_{ik} + v_k^n g_i + j_k^s \frac{\partial Z}{\partial v_i^n} + \lambda_{jk} \frac{\partial Y_j}{\partial v_i^n} \right) + \nabla_k v_i^s \left(-j_k^s v_i^n + j_k^s \frac{\partial Z}{\partial v_i^s} + \lambda_{jk} \frac{\partial Y_j}{\partial v_i^s} \right) \\ & + \nabla_k \nabla_i u_i \left(-\lambda_{ik} v_i^n + j_k^s \frac{\partial Z}{\partial \nabla_i u_i} + \lambda_{jk} \frac{\partial Y_j}{\partial \nabla_i u_i} \right). \end{aligned} \quad (3.8)$$

Due to the independence of the quantities \vec{v}^n , \vec{v}^s , \dots , the terms in each parentheses have to vanish separately, determining the most general form of the fluxes which is compatible with the standard procedure. The Andreev and Lifshitz¹ equations (supplemented by the nonlinear stress tensor of Saslow²) are obtained by setting

$$Y_i = v_k^n \nabla_k u_i - v_i^n, \quad Z = v_k^n v_k^s + \mu. \quad (3.9)$$

In the following, we shall introduce an alternative procedure in the extensive variables to show the uniqueness of this choice with respect to its linear terms if \vec{v}^s is Galilean, and also to work out the form of Z and Y_i , if \vec{v}^s is invariant. The reader who is more interested in the results can skip the rather formal derivation and turn directly to the hydrodynamic equations [Eqs. (3.19)–(3.24)] and the ensuing discussion of their physical significance. He only needs to know that the superscript T in $\vec{v}^T = (\hbar/m)\vec{v}^T$, \vec{j}^T and $\vec{\rho}^T$ labels the invariant counterparts of \vec{v}^s , \vec{j}^s , and ρ^s .

We will begin with Eq. (2.11), or rather its partially integrated form:

$$\begin{aligned} dE = & \mu dM + T dS + v_i^n dG_i \\ & - V [\nabla_i j_i^s d(\hbar/m)\phi + \nabla_j \sigma_{ij} du_j]. \end{aligned} \quad (3.10)$$

Note that Eq. (3.10) is obtained by partially integrating Eq. (2.1), changing terms such as $\lambda_{ij} d\nabla_j u_i$ to $-\nabla_j \lambda_{ij} du_i$ and then transforming it to the extensive variables as has been done to obtain Eq. (2.11). Equation (3.10) differs from Eq. (2.11) only by surface terms; it does not break the invariance of the system, e.g., under a uniform translation, as might be inferred at first glance. In fact, it states explicitly that $\partial E / \partial u_j$ approaches zero¹⁷ when the wave vector q vanishes.

The advantage of Eq. (3.10) is that we can immediately write down the time rate of change of the variables which, for small deviations from equilibrium, are generally given by linear combinations of the conjugate variables:

$$\frac{d}{dt} \begin{bmatrix} M \\ S \\ G_i \\ (\bar{n}/m)\phi \\ u_i \end{bmatrix} = \begin{bmatrix} & & b & f_j \\ & \text{I} & a & h_j \\ & & d_i & c_{ij} \\ -b & -a & -d_j & \\ -f_i & -h_i & -c_{ji} & \text{II} \end{bmatrix} \begin{bmatrix} \mu \\ T \\ v_i^n \\ -V\nabla_i j_i^s \\ -V\nabla_k \sigma_{jk} \end{bmatrix}. \quad (3.11)$$

Obviously Eq. (3.11) has enough structure to contain the Andreev and Lifshitz equations, whose linear reactive part is given by setting $b=1$, $c_{ij}=-\delta_{ij}$, and the rest of the coefficients to zero [Eqs. (3.13)–(3.16) below].

In a uniform system, the forces μ , T , and v_i^n are not necessarily zero, but the time rate of change of the conserved quantities N , S , and G_i must still vanish, therefore the part of the matrix numbered as I can contain only terms at least proportional to q . They are of higher orders in q and will be discussed in Sec. III C. The same applies to the II part of the matrix. Although it can contain q -independent elements, these give rise to fluxes of the same order in q as the elements in I. So for our present purpose, we can set I and II to zero. This leaves us with the two off-diagonal blocks, which by virtue of the Onsager reciprocal relation display the antisymmetry indicated in Eq. (3.11). (The three vectors \vec{d} , \vec{f} , and \vec{h} have to be odd under time inversion to give reactive couplings.)

It is important to realize that the time rate of change of the energy E is determined by Eq. (3.11) and always given as a total divergence. Inserting Eq. (3.11) into Eq. (2.11) we have

$$(V^{-1}d/dt)E + \nabla_k [j_k^s (b\mu + aT + d_i v_i^n) + \sigma_{jk} (\mu f_j + Th_j + v_i^n c_{ij})] = 0. \quad (3.12)$$

This shows that each of the coefficients of the off-diagonal blocks will lead to terms which are not eliminated by the standard procedure and have to be examined more closely.

First, we note that the lattice points of a crystal in equilibrium and in the frame where \vec{v}^n vanishes, will not change their position, i.e., $d\vec{u}/dt=0$, and therefore $\vec{f}=\vec{h}=0$. A Galilean transformation of $\vec{\omega}$ from this frame gives $\vec{v}^n=\vec{\omega}$ and $d\vec{u}/dt=\vec{\omega}$, so we may set $c_{ij}=-\delta_{ij}$ or

$$\frac{du_i}{dt} - v_i^n = 0.$$

This equation in conjunction with Eq. (2.10) connects the extensive with the intensive variables. Since in Eq. (3.11) time derivatives were taken of variables of a volume moving with \vec{v}^n , we have $d/dt = (\partial/\partial t) + \vec{v}^n \cdot \nabla$, and hence

$$\begin{aligned} \frac{1}{V} \frac{dM}{dt} &= \frac{d\rho}{dt} + \rho \frac{d\nabla_i u}{dt} \\ &= \dot{\rho} + \vec{\nabla} \cdot (\rho \vec{v}^n). \end{aligned} \quad (3.13)$$

(The dot denotes the partial time derivative. This type of nonlinear term enters the discussion quite naturally, and we shall keep track of these for the benefit of later discussions.) Quite analogously, we also have

$$(V^{-1}d/dt)S = \dot{s} + \vec{\nabla} \cdot (s\vec{v}^n), \quad (3.14)$$

$$(V^{-1}d/dt)E = \dot{\epsilon} + \vec{\nabla} \cdot (\epsilon\vec{v}^n), \quad (3.15)$$

$$(V^{-1}d/dt)G_i = \dot{g}_i + \vec{\nabla} \cdot (g_i \vec{v}^n). \quad (3.16)$$

The following reasoning will show that the three remaining coefficients a , b , and d_i are universally constant quantities, and due to the lack of such a vector, d_i must vanish. This can be expected because, e.g., $(V^{-1}d/dt)M = -b\vec{\nabla} \cdot \vec{j}^s = -\vec{\nabla} \cdot (b\vec{j}^s)$ states that ρ is a conserved quantity. We can prove it by inserting $Z=A(T)+B(\mu)+D(v_i^n)$ and Y_i as given by Eq. (3.9) in Eq. (3.8) and rewriting, e.g., Eqs. (3.3) and (3.5) according to Eq. (3.11) as

$$(V^{-1}d/dt)S + (\partial A/\partial T)(\nabla_i j_i^s) = 0, \quad (3.17)$$

$$(d/dt)\phi + (A/T)T + \dots = 0. \quad (3.18)$$

The Onsager reciprocal relation enforces $\partial A/\partial T=A/T$, i.e., $A=aT$, and similarly, $B=b\mu$, $D=d_i v_i^n$ with universally constant a , b , and d_i . This argument is of course not valid for nonlinear terms such as $D=v_i^n^2$.

Next we shall examine the continuity equation $\dot{\rho} + \vec{\nabla} \cdot (\rho\vec{v}^n + b\vec{j}^s) = 0$. It is a usual assumption in deriving the hydrodynamic equations that the mass current \vec{j} is equal to the momentum density \vec{g} . However, Putterman⁸ pointed out that there are cases where this equality is not valid. We shall defer the proof of $\vec{g}=\vec{j}$ for any isolated systems to the appendix and proceed by taking them as equal. This is important because the momentum density is already known to be completely determined by the Galilean transformation property of the additional velocity \vec{v}^s [Eqs. (2.4) and (2.6)] and we may conclude that $b=1$, if \vec{v}^s is a Galilean velocity and $b=0$, if \vec{v}^s is invariant.

In order to distinguish these two case, we shall

relabel the invariant velocity, its conjugate variables, the elastic tensor, and the transport coefficient a with T (which stands for superthermal): $\vec{v}^T \equiv (\hbar/m)\vec{\nabla}\phi^T, \vec{j}^T, \vec{\rho}^T, a^T$, while keeping the superscript s for their Galilean counterparts: $\vec{v}^s \equiv (\hbar/m)\vec{\nabla}\phi^s, \vec{j}^s, \vec{\rho}^s, a^s$. (As we have seen above, $b^s = 1$ and $b^T = 0$.) By setting \vec{j}^s or \vec{j}^T to zero, we can arrive at the two cases conveniently. The equation for the entropy is hence given by

$$\dot{s} + \vec{\nabla} \cdot (s\vec{v}^n + a^s \vec{j}^s + a^T \vec{j}^T) = 0.$$

Subject to the assumption that ρ^T vanishes for zero temperature and ρ^s does not, we can set $a^s = 0$ and $a^T = a(k_B/m)$. The first equation complies with the requirement that the total entropy current must vanish for zero temperature.¹⁸ The second equation is obtained by dimensional considerations, where a is dimensionless and easily eliminated by redefining $\vec{\phi}^T = \phi^T/a$ and $\vec{j}_i^T = a\vec{j}_i^T$. This leaves the energy $\vec{j}^T \cdot d\vec{v}^T$ unchanged, but gives $\vec{a}^T = k_B/m$.

After this lengthy discussion, we finally arrive at

$$(V^{-1}d/dt)M + \vec{\nabla} \cdot \vec{j}^s = 0, \quad (3.19)$$

$$(\hbar/m)(d/dt)\phi^s + \mu = 0, \quad (3.20)$$

$$(V^{-1}d/dt)S + (k_B/m)\vec{\nabla} \cdot \vec{j}^T = 0, \quad (3.21)$$

$$(\hbar/k_B)(d/dt)\phi^T + T = 0, \quad (3.22)$$

$$(V^{-1}d/dt)G_i - \nabla_j \sigma_{ij} = 0, \quad (3.23)$$

$$(d/dt)u_i - v_i^n = 0. \quad (3.24)$$

Equations (3.19)–(3.24) represent the linear reactive part of a “three-fluid hydrodynamics” with \vec{v}^n , \vec{v}^s , and \vec{v}^T as three independent velocities. They contain all the information needed for calculating the velocities of the propagating modes, whose general pattern can be easily recognized here as coupling of a conserved quantity to an order parameter, leading to propagation of this conserved quantity. Hence, in an ordinary crystal, where $\vec{j}^s = \vec{j}^T = 0$, Eqs. (3.23) and (3.24) yield three pairs of elastic waves, which are the propagation of the momentum. Introducing a Galilean velocity potential ϕ^s provides the system with an independent and persistent mass current \vec{j}^s and gives rise to propagation of the mass M [or because of Eq. (2.14), of the vacancy concentration c .] An invariant velocity potential ϕ^T , on the other hand, provides an independent and persistent entropy current \vec{j}^T (or equivalently, energy current), and sustains entropy propagation. The phenomena connected to ϕ^s are known as superfluidity, those connected to ϕ^T we may call “superthermality.” Note that because of the correspondence between the conserved quantities and the propagating modes, there can be no more propagating modes than are found here—aside, of course, from spin

waves,¹⁶ which may propagate in a system when the spin angular momentum is independently conserved.

The distinct difference made above between superfluidity and superthermality lends itself naturally to the question as to why in the two-fluid hydrodynamics of He II a Galilean velocity can after all lead to temperature (or equivalently, entropy) propagation? The answer lies in the fact that in a liquid there is no preferred inertial frame: In Eqs. (3.19)–(3.24) we have three velocities \vec{v}^n , \vec{v}^s , and \vec{v}^T ; or to put it more physically, the velocity of the mass flow $\vec{v}^n + \vec{j}^s/\rho$, the velocity of the lattice points \vec{v}^n , and the velocity of the entropy current $\vec{v}^n + \vec{j}^T/s$. By setting either the first two equal (i.e., $\vec{j}^s = 0$), or the last two (i.e., $\vec{j}^T = 0$), we can reduce the three-fluid hydrodynamics to two completely different two-fluid hydrodynamics: the superfluid and the superthermal one. In liquids, on the other hand, due to the lack of lattice points as the reference flow, we have only two fluids to start with, the entropy and the particle current. They can either be equal (ordinary fluid) or different (superfluid). In fact, a persistent mass flow in an unbounded liquid in the frame, where $\vec{v}^n = 0$ can be equivalently viewed as a persistent entropy flow in the frame where $\vec{g} = 0$, and we can indeed define an invariant velocity $\vec{v}^T \equiv \vec{v}^s - \vec{v}^n$, and a barycentric velocity $\vec{v} = \vec{g}/\rho$ to rewrite the kinetic energy

$$d\epsilon = \vec{v}^n \cdot d\vec{g} + \vec{j}^s \cdot d\vec{v}^s = \vec{v} \cdot d\vec{g} + \vec{j}^T \cdot d\vec{v}^T,$$

where $\vec{j}^T = \rho^s \rho^n (\vec{v}^s - \vec{v}^n)/\rho$. In these variables, the usual two-fluid hydrodynamics is given as

$$\dot{\rho} + \vec{\nabla} \cdot (\rho\vec{v}) = 0, \quad \rho\vec{\nabla} + \vec{\nabla}\rho = 0,$$

$$\dot{s} + \vec{\nabla} \cdot \left(s\vec{v} - \frac{s}{\rho^n} \vec{j}^T \right) = 0, \quad \dot{T} - \frac{s}{\rho^n} \vec{\nabla} T = 0.$$

These equations are obviously in complete agreement with the superthermal part of Eqs. (3.19)–(3.24). Of course He II is after all superfluid rather than superthermal. This is, e.g., revealed by the fourth sound experiment, where an artificial reference flow \vec{v}_{pd} , that of the powder, is introduced into the system. By setting $\vec{v}^n = 0$ to calculate the fourth sound velocity, we supplement the information not contained in the linear reactive part of the two-fluid hydrodynamics, which is $\vec{v}^n = \vec{v}_{pd}$, rather than $\vec{v} = \vec{v}_{pd}$ (as would have been the case, if the system were superthermal).

B. Propagation of sound

1. Ordinary crystals

We shall first reproduce the results for elastic waves of an ordinary crystal by setting $\vec{j}^T = \vec{j}^s = 0$

in Eqs. (3.19)–(3.24), yielding

$$\dot{S}=0, \quad \dot{M}=0.$$

It is therefore advantageous to take S , M , and $\nabla_i u_j$ as the variables, where the derivatives $\partial\sigma_{ij}/\partial S$ and $\partial\sigma_{ij}/\partial M$ do not have to be considered. This gives immediately the spectrum of the elastic waves

$$c_{10}^2 = \frac{\omega^2}{q^2} = \left(\frac{\partial\sigma_{ij}}{\partial\nabla_k u_l} \right)_{M,S} \frac{\hat{q}_j \hat{q}_k \hat{u}_i \hat{u}_l}{\rho}, \quad (3.25)$$

(\hat{u} is the unit polarization vector, $\hat{q} = \vec{q}/q$ the unit wave vector.) Since $(\partial\sigma_{ij}/\partial\nabla_k u_l)_{M,S} = (\partial\sigma_{ij}/\partial\nabla_k u_l)_C$, [Eq. (2.15)], the result agrees with that of Refs. 11 and 12.

2. Superfluid crystals

Next we shall examine the superfluid crystal, where only \vec{j}^T vanishes, i.e., $\dot{S}=0$. Note first that keeping \vec{v}^n zero, only Eqs. (3.19) and (3.20) contribute to a propagating mode:

$$\begin{aligned} & \left(-\omega^2 + V \frac{\partial\mu}{\partial M} \rho_{ij}^s q_i q_j \right) \delta\phi^s + \omega \left(V \frac{\partial\mu}{\partial M} \rho_{ij}^s + \frac{\partial\mu}{\partial\nabla_i u_j} \right) q_i \delta u_j = 0, \\ & \left(\omega^2 q_j \rho_{ij}^s + q_j q_k q_l \rho_{kl}^s V \frac{\partial\sigma_{ij}}{\partial M} \right) \delta\phi^s + \omega \left(-\omega^2 (\rho \delta_{il} - \rho_{il}^s) + q_j q_k \frac{\partial\sigma_{ij}}{\partial\nabla_k u_l} + q_j q_k \rho_{kl}^s V \frac{\partial\sigma_{ij}}{\partial M} \right) \delta u_l = 0. \end{aligned} \quad (3.27)$$

(Note that these two equations are generally correct, not only for zero temperature.) To linear order in the superfluid density, Eqs. (3.27) can be solved to yield

$$c_2^2 = (\rho_{ij}^s \hat{q}_i \hat{q}_j) \left[\left(\frac{\partial\mu}{\partial\rho} \right)_{s,\nabla u} - \frac{1}{c_{10}^2 \rho} (\beta_{ij} \hat{q}_i \hat{u}_j)^2 \right], \quad (3.28)$$

$$\begin{aligned} c_1^2 &= c_{10}^2 \left(1 + \frac{\rho_{ij}^s \hat{u}_i \hat{u}_j}{\rho} \right) + \frac{1}{\rho} (\beta_{ij} \hat{q}_i \hat{u}_j) \\ &\times \left[2(\rho_{ij}^s \hat{q}_i \hat{u}_j) + \frac{\rho_{ij}^s \hat{q}_i \hat{q}_j}{c_{10}^2} (\beta_{ij} \hat{q}_i \hat{u}_j) \right]. \end{aligned} \quad (3.29)$$

[β_{ij} is defined in Eq. (2.19).] For a cubic crystal, these two equations reduce to [Eq. (2.22)]

$$c_2^2 = \rho^s \left(\frac{\partial\mu}{\partial\rho} - \frac{(\hat{q}_i \hat{u}_i)^2 \beta^2}{\rho c_{10}^2} \right), \quad (3.30)$$

$$c_1^2 = c_{10}^2 (1 + \rho^s/\rho) + (\hat{q}_i \hat{u}_i)^2 (\rho^s/\rho) (\beta/c_{10}^2 + 2)\beta. \quad (3.31)$$

To arrive at simple expressions, we have pretended that the orientation of the polarization with respect to the wave vector was a free parameter, while it is in fact determined by Eq. (3.27). Although the general results will be too complicated to be illuminating, it is easy to see that for special

$$\begin{aligned} \omega^2 &= V \left(\frac{\partial\mu}{\partial M} \right)_{s,\nabla u} \rho_{ij}^s q_i q_j \\ &= \left(\frac{\partial\mu}{\partial\rho} \right)_{s,\nabla u} \rho_{ij}^s q_i q_j. \end{aligned} \quad (3.26)$$

In analogy to two-fluid hydrodynamics, we may call it fourth sound. Equation (3.26) is in agreement with the microscopic result of Saslow,² where \vec{v}^n is also set to zero. Because this mode is a propagation of the vacancy concentration, it can only exist in a globally perfect crystal [as defined in the discussion following Eq. (2.18)], and not in a locally perfect one.

The general spectrum of the four propagating modes is given by solving Eqs. (3.19), (3.20), (3.23), and (3.24) simultaneously. They are the three elastic waves and the Goldstone mode, which we, again by analogy, shall call second sound. Taking M , S , $\nabla_i u_j$, ϕ^s , and v_i^n as the variables and eliminating M and v_i^n gives after some straightforward algebra:

directions of propagation (such as [100], [110], and [111] in a cubic crystal), whenever a pure longitudinal wave is an eigenmode, we have $\hat{q}_i \hat{u}_i = 1$ in the expression for second sound; and setting $\hat{q}_i \hat{u}_i$ to one or zero in Eq. (3.31) we arrive at the longitudinal and transverse sound velocities, respectively. We see that the modification is different for longitudinal or transverse sound. For propagations off these special directions, $\hat{q}_i \hat{u}_i$ will no longer remain one in Eq. (3.30), and because (as discussed in Sec. II) β is of the same order of magnitude as $\rho(\partial\mu/\partial\rho)$ or c_{10}^2 , the increase in the second sound velocity will be significant.

It is instructive to see how longitudinal elastic waves and longitudinal second (vacancy) sound reduce to the familiar spectrum of first and second (temperature) sound in He II. In a liquid $d\sigma_{ij} = -dP\delta_{ij}$ [Eq. (2.16)] therefore c_{10}^2 and β become, respectively,

$$\begin{aligned} c_{10}^2 &= -\frac{V}{\rho} \left(\frac{\partial P}{\partial V} \right)_{M,S} = \left(\frac{\partial P}{\partial\rho} \right)_s, \\ \beta &= -\left(\frac{\partial P}{\partial\rho} \right)_s. \end{aligned}$$

Neglecting the thermal expansion coefficient, $\alpha = -(\partial P/\partial s)_\rho = 0$ [Eq. (2.20)], we have

$$\beta = -c_{10}^2. \quad (3.32)$$

Inserting this equality into Eq. (3.31) gives $c_1 = c_{10}$, which states the well-known fact⁷ that neglecting the thermal expansion coefficient, first sound remains unchanged going through the λ point. Again neglecting the thermal expansion coefficient, we have, due to a Maxwell relation

$$0 = \left(\frac{\partial P}{\partial s}\right)_\rho = \rho \left(\frac{\partial \mu}{\partial s}\right)_\rho + s \left(\frac{\partial T}{\partial s}\right)_\rho \\ = \rho \left(\frac{\partial T}{\partial \rho}\right)_s + s \left(\frac{\partial T}{\partial s}\right)_\rho,$$

or

$$-\beta = \rho \left(\frac{\partial \mu}{\partial \rho}\right)_s + s \left(\frac{\partial T}{\partial \rho}\right)_s \\ = \rho \left(\frac{\partial \mu}{\partial \rho}\right)_s - \frac{s^2}{\rho} \left(\frac{\partial T}{\partial s}\right)_\rho. \quad (3.33)$$

Inserting Eqs. (3.32) and (3.33) into Eq. (3.30) yields

$$c_2^2 = \rho^2 s^2 (\partial T / \partial s)_\rho. \quad (3.34)$$

It agrees with the second-sound velocity of He II (see Ref. 7) to first order in ρ^s . Treating ρ_{ij} [defined in Eq. (2.18)] as a small perturbation, Andreev and Lifshitz had to identify $(\partial \lambda_{ij} / \partial \nabla_k u_l) (\hat{q}_j \hat{q}_k \hat{u}_i \hat{u}_l)$ as the velocity of the unperturbed elastic waves [Eq. (2.13)]. Therefore, their results neither reduce to first and second sound for the superfluid liquid limit, $\lambda_{ij} = 0$, nor to elastic waves in the normal crystal limit, $\vec{j}^s = 0$.

3. Superthermal crystals

Here \vec{j}^s vanishes and \vec{j}^T does not, yielding $\dot{M} = 0$. The two variables \vec{g} and \vec{j}^T are diagonal [Eqs. (2.4) and (2.5)] and, neglecting the thermal expansion coefficient, so are σ_{ij} and T , therefore, elastic waves and temperature propagation are decoupled for vanishing α_{ij} . To the order of the thermal expansion coefficient squared, we have

$$c_1^2 = c_{10}^2 + a, \quad (3.35)$$

$$c_2^2 = c_{20}^2 - a, \quad (3.36)$$

with

$$c_{20}^2 = (k_B/m)^2 (\rho_{ij}^T \hat{q}_i \hat{q}_j) V (\partial T / \partial s)_{M, \nabla u} \\ = (k_B/m)^2 (\rho_{ij}^T \hat{q}_i \hat{q}_j) (\partial T / \partial s)_{\rho, \nabla u}$$

and

$$a = \frac{(\rho_{ij}^T \hat{q}_i \hat{q}_j) (\alpha_{ij} \hat{u}_i \hat{q}_j)^2}{\rho} \left(\frac{k_B}{m}\right)^2,$$

where c_{10}^2 and α_{ij} are given by Eq. (3.25) and Eq. (2.20), respectively. For a cubic crystal,

$$a = (k_B/m)^2 (\rho^T / \rho) (\hat{u}_i \hat{q}_i)^2 \alpha^2 / (c_{10}^2 - c_{20}^2).$$

As in the case of superfluid crystal, $\hat{u} \cdot \hat{q}$ is given by the eigenvectors. And again for special directions of propagation, whenever a pure longitudinal wave is an eigenmode, we have $\hat{u} \cdot \hat{q} = 1$ in Eq. (3.36). Note that in the superthermal case we have $\vec{v} = \vec{v}^n$, and the "fourth sound" velocity, defined as given by setting $\vec{v}^n = 0$, is equal to c_{20} .

C. Nonlinear and dissipative terms

The nonlinear terms are somewhat difficult to determine. Assuming Eqs. (3.20), (3.22), and (3.24) to be the correct nonlinear equations brings us back to Eq. (3.9) and $Z^T = v_k^n v_k^T + T$. It is easy to convince oneself that Eqs. (3.19)–(3.24) satisfy the suitably extended differential equation Eq. (3.8), if Eq. (3.23) is changed to

$$(V^{-1} d/dt) G_i - \nabla_j (\sigma_{ij} - \pi_{ij}^0) = 0,$$

where²

$$\pi_{ij}^0 = j_j^s v_i^s + j_j^T v_i^T + \lambda_{kj} \nabla_i u_k. \quad (3.37)$$

π_{ij}^0 is formally the same as the Erikson stress tensor¹⁹ of nematodynamics. Andreev and Lifshitz were only interested in a linear theory with respect to the normal crystal motion and had $\pi_{ij}^0 = j_j^s v_i^s$, of which $j_j^T v_i^T$ is a corollary extension. $\lambda_{kj} \nabla_i u_k$ was first proposed by Saslow. In a system with no long-range force, σ_{ij} is symmetric.¹¹ Therefore, to insure the conservation of angular momentum, $\pi_{ij} = \pi_{ij}^0 + v_j^n g_i$ has to be symmetric, too. Employing the invariance of the energy under a rotation,²⁰ the antisymmetric part of π_{ij} is seen to be of the order of the displacement vector squared (which is bound to be small in a three-dimensional crystal). So although Eq. (3.37) exactly satisfies the differential equation of the standard procedure, it seems reliable only to linear order in u_i . Besides, there is the problem of uniqueness.²¹

Having determined the linear-reactive terms, it is straightforward to derive the linear dissipative part using the standard procedure.^{7,8} We have

$$\dot{\rho} + \vec{\nabla} \cdot (\rho \vec{v}^n + \vec{j}^s) = 0, \quad (3.38)$$

$$\dot{s} + \vec{\nabla} \cdot [s \vec{v}^n + (k_B/m) \vec{j}^T + \vec{f}^D] = R/T, \quad (3.39)$$

$$\dot{g}_i + \nabla_j (g_i v_j^n - \sigma_{ij} + \pi_{ij}^0 + \pi_{ij}^D) = 0, \quad (3.40)$$

$$\dot{\vec{v}}^s + \vec{\nabla} (\mu + \vec{v}^n \cdot \vec{v}^s + Z_s^D) = 0, \quad (3.41)$$

$$\dot{\vec{j}}^T + \vec{\nabla} [(k_B/m) T + \vec{v}^n \cdot \vec{v}^T + Z_T^D] = 0, \quad (3.42)$$

$$\dot{\vec{u}} + (\vec{v}^n \cdot \vec{\nabla}) \vec{u} - \vec{v}^n + \vec{Y}^D = 0, \quad (3.43)$$

where

$$-R = f_i^D \nabla_i T + \pi_{ij}^D \nabla_j v_i^n + Y_i^D \nabla_j \lambda_{ij} \\ + Z_s^D \nabla_i j_i^s + Z_T^D \nabla_i j_i^T$$

gives the dissipative terms, denoted by superscript D [and, if allowed by symmetry, the reactive ones contained in the I and II parts of the matrix in Eq. (3.11)]. In other words, the currents f_i^D, π_{ij}^D, \dots are expanded to linear order in the forces $\nabla_i T, \nabla_j v_i^n, \dots$, where the expansion coefficients are required to be compatible with the symmetry of the system. Equations (3.38)–(3.43) represent the complete three-fluid hydrodynamics. Omitting Eq. (3.42) and setting $\vec{j}^T = 0$ in the other equations reduces the hydrodynamics to its superfluid part, which is the same as given by Andreev and Lifshitz¹ and, with proper interpretation, by Saslow.²

IV. SECOND SOUND IN AN ORDINARY CRYSTAL

The superthermal crystal is a system in which one assumes that the energy depends on an additional velocity $\vec{v}^T = (\hbar/m)\vec{\nabla}\phi^T$. Minimizing the energy with respect to ϕ^T , one arrives at $\vec{\nabla}\cdot\vec{j}^T = 0$, and concludes that a constant but nonvanishing \vec{j}^T would be a true equilibrium state. Since \vec{j}^T is essentially an additional entropy current, it would also be a persistent one. A phonon gas in a crystal does not consist of such a system because although a uniform drift velocity of the phonon gas represents an independent entropy flux, it dissipates through umklapp processes and defect scatterings and is not persistent. Still the decay of the drift velocity may become so slow in a perfect crystal at low temperature that its relaxation rate is orders of magnitude smaller than that of any other q -independent microscopic relaxation, then a hydrodynamic description becomes again possible. The idea pursued here is very similar to that of the generalized hydrodynamics near phase transitions.²²

Because the hydrodynamic derivation in the last section is based on very general arguments, any crystal that can sustain temperature propagation must display the same structure in its equations of motion as the Eqs. (3.38)–(3.43). However, to account for the facts that (i) \vec{v}^T relaxes even for $\vec{\nabla}\cdot\vec{j}^T = 0$, and (ii) \vec{v}^T is not necessarily a potential flow, we have to substitute

$$\frac{dv_i^T}{dt} + \nabla_k \left(\frac{k_B}{m} T \delta_{ik} + Z_{ik}^D \right) + \tau_i^D = 0 \quad (4.1)$$

for Eq. (3.42). At the same time the dissipation function R has to be changed to

$$-R = \int \nabla_i^D T + \pi_{ij}^D \nabla_j v_i^n + Y_i^D \nabla_j \lambda_{ij} + Z_s^D \nabla_i j_i^s + Z_{ij}^D \nabla_j j_i^T - \tau_{ij}^D j_i^T. \quad (4.2)$$

Also, the term $j_j^T v_i^T$ in Eq. (3.37) must be omitted for the hydrodynamic equations to be self-consistent.

With these changes, Eqs. (3.38)–(3.43) describe the temperature propagation as a quasihydrodynamic mode in a superfluid crystal, and, setting $\vec{j}^s = 0$, in an ordinary crystal.

Second sound in ordinary crystals has been extensively studied^{9, 23} but to the knowledge of the author, no previous derivation of the hydrodynamics based solely on symmetry considerations and conservation laws has been given. No where in our discussions have we needed to specify how the independent entropy flux is realized, or to rely on any other aspect of a microscopic model. Making full use of the phonon-gas model, Enz⁹ arrived at equations similar to ours, and it is useful to compare them.

Enz has defined two velocities: v_L and v_p , the lattice and phonon drift velocities, respectively. Due to the structure of his continuity equation (E3.8) (the original number of Enz's equations are preceded by E)

$$\dot{\rho} + \vec{\nabla} \cdot (\rho \vec{v}_L) = 0, \quad (4.3)$$

and the equation of motion for v_p (E3.28)

$$\dot{\vec{v}}_p + (s_p/\rho_p) \vec{\nabla} T = -\vec{v}_p/\tau_J, \quad (4.4)$$

we can identify them as

$$\vec{v}_L = \vec{v}^n \quad \text{and} \quad \vec{v}_p \sim \vec{v}^T, \quad (4.5)$$

since \vec{v}^T is an invariant velocity, we have to interpret \vec{v}_p as the phonon drift velocity *with respect to the lattice*. But this is in contradiction with the equation (E3.52)

$$\dot{s} + \vec{\nabla} \cdot (s_p \vec{v}_p - \kappa \vec{\nabla} T) = \sigma, \quad (4.6)$$

because the entropy current given here is unchanged under a Galilean transformation, while we expect any entropy flow \vec{f} to transform as $\vec{f} \rightarrow \vec{f} + \vec{\omega}s$. Therefore, the term $s\vec{v}_L$ must be added to Eq. (4.6). The lattice momentum density (E3.9)

$$\vec{j}_L = \rho \vec{v}_L = \rho \vec{v}^n, \quad (4.7)$$

is obviously our total momentum density \vec{g} . Comparing the equation of motion for j_L , (E3.10) with the momentum conservation Eq. (3.40), we find that the two quantities $-\pi_L$ (denoted in this paper as σ_{ij}) and Σ (denoted as λ_{ij}) have been set equal, while we believe they are connected by Eq. (2.12). Besides, being a conserved quantity the time rate of change of \vec{j}_L has to be given by a total divergence; this is violated by the term $v_L \nabla j_L$. However, together with the oversight discussed in Eq. (2.21), these four terms cancel each other when calculating the sound velocities, and Eqs. (3.35) and (3.36) for the special case $\rho_{ij}^T = \rho^T \delta_{ij}$ agree with Enz's results, if one identifies $\rho^T (k_B/m)^2$ with s_p^2/ρ_p .

Comparing the dissipative terms of the equations

(E3.1), (E3.42), (E3.43), and (E3.56) with the dissipative function Eq. (4.2), we find that $\nabla_j \lambda_{ij}$ has been neglected by Enz as a thermodynamic force. For instance Y_i^D is generally given as¹²

$$Y_i^D = \xi_{ij} \nabla_k \lambda_{jk} + \xi_{ij} \nabla_j T. \quad (4.8)$$

V. CONCLUSIONS

A three-fluid hydrodynamics has been derived by employing only symmetry considerations and conservation laws, where a supplementary procedure to deal with the linear reactive terms in the extensive variables via a transparent Onsager matrix was introduced to achieve uniqueness in the fluxes. This hydrodynamic theory describes a system which can sustain the maximum number of propagating modes, each engaging one of the five conserved quantities. The three independent fluids of such a system are the mass current, the entropy flux and the flow of lattice points (or equivalently, \vec{v}^n , \vec{v}^s , and \vec{v}^T). In an ordinary crystal, all three currents are equal and given by \vec{v}^n . Introducing an additional Galilean velocity \vec{v}^s , one provides the system with an independent mass current which makes it superfluid. If the additional velocity \vec{v}^T is invariant, the entropy current becomes independent and the system may be called superthermal. This distinction is unique to the crystal. In a liquid, due to the lack of a reference flow, which is provided in a crystal by the lattice, the entropy flux and the mass current can be either equal (ordinary fluid) or different (superfluid) with no third choice.

Superfluidity present in a crystal will modify the spectrum of the elastic waves enough to make them a useful indicator. We found that the changes in the velocity are different for shear and compressional waves. Accordingly, second sound as the new Goldstone mode, which is a vacancy propagation, has a velocity strongly dependent on the direction of propagation. It will be slowest along those axes of high symmetry, where a coupling between transverse and longitudinal sound is lacking. While the spectrum of this second sound is given by a slightly intricate generalization of second sound in He II, a true temperature propagation as the direct generalization of second sound exists only in a superthermal crystal. It is decoupled from the elastic waves to the order of the thermal expansion coefficient squared.

Because the derivation of the superthermal hydrodynamics is based on very general arguments, any crystal that can sustain a temperature propagation must display the same structure in its equations of motion. Therefore, with only a slight modification to account for the eventual decay of the independent entropy current, which consists of

drifting phonons, these equations describe second sound in an ordinary crystal. Comparing our equations to those of Enz,⁹ which were derived relying on various aspects of the phonon-gas model, some disagreements were found and studied.

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APPENDIX: EQUALITY OF THE MOMENTUM DENSITY AND MASS CURRENT IN AN ISOLATED SYSTEM

The following reasoning represents the completion of a proof in a footnote of the fluid dynamic textbook²⁴ by Landau and Lifshitz, Sec. 49. The footnote omits the surface terms, which we believe are important.

With \vec{g} and ρ defined in Eq. (2.1), and \vec{j} given by the continuity equation Eq. (3.2), we first observe that quite generally the total mass $M = \int \rho dV$ and momentum $\vec{G} = \int \vec{g} dV$ of the system must satisfy

$$\vec{G} = M \dot{\vec{R}}, \quad (A1)$$

with $\vec{R} = M^{-1} \int \rho \vec{r} dV$ as the center of mass coordinate. The time derivative of the mass,

$$\dot{M} = \int \dot{\rho} dV + \oint \rho \dot{n}_i da_i = \oint (\rho \dot{n}_i - j_i) da_i,$$

has two contributions, one from a change in the density, the other from that of the volume. da_i is a surface area element, \dot{n}_i is perpendicular to the surface with $\dot{n}_i da_i = dV$. \dot{n}_i is the velocity of the volume's boundary. Similarly, $M \dot{\vec{R}}$ also has two contributions,

$$\begin{aligned} \int \vec{g} dV &= M \dot{\vec{R}} = \int \dot{\rho} \vec{r} dV + \int \rho \vec{r} \dot{n}_i da_i - \vec{R} \dot{M} \\ &= \int \vec{j} dV + \oint (\vec{r} - \vec{R})(\rho \dot{n}_i - j_i) da_i. \end{aligned}$$

Since the boundary of an isolated system obviously follows the mass flow at the surface, we have

$$\rho \dot{n}_i = \vec{j} \quad (A2)$$

which gives $\vec{g} = \vec{j} + \vec{A}$ where $\int \vec{A} dV$ identically vanishes. However, the option from the hydrodynamics is $\vec{g} = \vec{j} + \alpha \vec{j}^s + \beta \vec{\nabla} \mu + \gamma \vec{\nabla} T \dots$, and none of the additional terms integrate identically to zero. Therefore we may conclude $\alpha = \beta = \gamma = \dots = 0$, and

$$\vec{g} = \vec{j}. \quad (A3)$$

We note that there is a problem of internal consistency if one wants to generalize the volume to

be an arbitrary one inside the system: Combining Eq. (A3) with a different time evolution of the volume $\dot{V} = \int \dot{n}_i da_i$, such as given by Eq. (2.10) or by setting $\dot{n} = 0$, Eq. (A1) is no longer valid. So in a local description, only two of the above three equations can be specified. If V is the total volume of an isolated system, however, because Eq. (A1)

and (A2) are trivially true, so is Eq. (A3).

Equation (A3) was also discussed by Putterman in Appendix 1 of Ref. 8. However, his argument does not rule out the possibility of an invariant \vec{v}^s (i.e., $\vec{g} = \rho \vec{v}^n$) in connection with superfluidity (i.e., $\vec{j} = \rho \vec{v}^n + \vec{j}^s$) or vice versa, a Galilean \vec{v}^s ($\vec{g} = \rho \vec{v}^n + \vec{j}^s$) in connection with no superfluidity ($\vec{j} = \rho \vec{v}^n$).

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¹A. F. Andreev and I. M. Lifshitz, Zh. Eksp. Teor. Fiz. **56**, 2057 (1969) [Sov. Phys.-JETP **29**, 1107 (1969)].

²W. M. Saslow, Phys. Rev. B **15**, 173 (1977).

³For a review, see C. M. Varma, in *The Physics of Liquid and Solid Helium*, edited by K. H. Bennemann and J. B. Ketterson (Wiley, New York, 1976).

⁴G. V. Chester, Phys. Rev. A **2**, 256 (1970).

⁵A. J. Leggett, Phys. Rev. Lett. **25**, 2543 (1970).

⁶Y. Imry and M. Schwartz, J. Low Temp. Phys. **21**, 543 (1975), and references cited therein.

⁷I. M. Khalatnikov, *Theory of Superfluidity* (Benjamin, New York, 1965), Part II.

⁸S. J. Putterman, *Superfluid Hydrodynamics* (North-Holland, Amsterdam, 1974), Sec. 3.

⁹C. P. Enz, Rev. Mod. Phys. **46**, 705 (1974).

¹⁰Part of the proof was provided by M. Cross; see also Ho, Ref. 15.

¹¹L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, Oxford, 1970), Chap. 1.

¹²P. D. Fleming and C. Cohen, Phys. Rev. B **13**, 500 (1976), the corresponding and instructive discussion of vacancy concentration and thermodynamic identities. Note, however, that their Eq. (2.3) should be modified to $\delta n_0 = -n_0 \vec{\nabla} \cdot \vec{R}$ (where $n_0 = N/V$ and $\vec{R} = \vec{u}$ in our notations) with some corollary changes in the ensuing equations.

¹³F. Jähnig and H. Schmidt, Ann. Phys. (N.Y.) **71**, 129 (1972); D. Forster, T. C. Lubensky, P. C. Martin, J. Swift, and P. C. Pershan, Phys. Rev. Lett. **26**, 1016 (1971).

¹⁴R. Graham, Phys. Rev. Lett. **33**, 1431 (1974); R. Graham and H. Pleiner, *ibid.* **34**, 792 (1975).

¹⁵C. R. Hu and W. M. Saslow, Phys. Rev. Lett. **38**, 605 (1977); J. M. Delrieu, J. Phys. (Paris) **L38**, 127 (1977); D. Lhuillier, *ibid.* **L38**, 121 (1977); and T. L.

Ho, *Sanibel Proceedings*, edited by S. B. Trickey, E. D. Adams, and J. W. Dufty (Plenum, New York, 1977).

¹⁶B. I. Halperin and P. C. Hohenberg, Phys. Rev. **188**, 898 (1969).

¹⁷P. C. Martin, P. Parodi, and P. S. Pershan, Phys. Rev. A **6**, 2401 (1972).

¹⁸See the corresponding discussion in Appendix III of Ref. 8, where the problem is clarified. However, interdependency of variables at zero temperature does not seem to provide conclusions for general temperature.

¹⁹P. G. deGennes, *Physics of Liquid Crystals* (Oxford U.P., Oxford, 1974), Chap. 5.

²⁰T. L. Ho, Ref. 15.

²¹Note that in the simpler case of the two-fluid hydrodynamics of an isotropic superfluid no such difficulties and ambiguities exist. Setting $\lambda_{ij} = 0$ in Eq. (3.8) the ∇v^s parentheses and isotropy of the system yield $Z = Z_1(\mu, T, v_n^2) + \vec{v}^n \cdot \vec{v}^s$. The Galilean transformation property of the stress tensor in conjunction with the ∇v^n parentheses then gives $Z_1 = Z_1(\mu, T)$, which is further reduced to $Z_1 = \mu + Z_2(T)$ by the $\nabla \mu$ parentheses and the equality $g_i = j_i$. We have already shown that $Z_2(T) = 0$ because $\vec{j}^s(T=0) \neq 0$ [Eqs. (3.17) and (3.18)]. Therefore, $Z = \mu + \vec{v}^n \cdot \vec{v}^s$ is the only solution of the differential equation which complies with all the symmetry requirements.

²²I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **57**, 489 (1969) [Sov. Phys.-JETP **30**, 268 (1970)]; M. Liu, Phys. Rev. Lett. **35**, 1577 (1975); *Physica (Utr.) B* **90**, 78 (1977).

²³W. Götze and K. H. Michel, Phys. Rev. **156**, 963 (1967).

²⁴L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, New York, 1959), Sec. 49, footnote.