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Mean-field theory and critical exponents for a random resistor network

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We consider the conductivities of a random resistor network on a regular lattice near the percolation point. It is shown that there is a close analogy between this problem and that of phase transitions. It is possible to construct a Hamiltonian and define an order parameter. The mean-field equation for the order parameter is found and the conductivities determined in the cases of a network of resistor and open circuits and a network of resistor and shorts. Exponent relations are discussed and the conductivity exponents given in $6-\epsilon$ dimensions. The analog of the resistor network arises in a number of other physical problems, e.g., spin waves in dilute ferromagnets, the elasticity of gels, hopping conductivity, and these results are also of interest in these systems.

I. INTRODUCTION

In this paper we consider a random resistor network on a regular lattice. Nearest-neighbor sites of the lattice are connected by conductors each of which is chosen randomly to have conductance σ_b (with probability p) or σ_a (with probability $1-p$) with $\sigma_b >> \sigma_a$. Of interest is the macroscopic conductivity Σ of this system in the vicinity of the percolation threshold. Early work on this problem has been reviewed by Kirkpatrick.[1] The analog of the random resistor network arises in a number of other physical problems and we mention a few: (i) spin waves in a random ferromagnet[2]; the spin-wave stiffness corresponds to the microscopic conductivity (see Sec. III); (ii) elasticity of gels[3]; (iii) hopping conductivity[4]; and (iv) phonons in disordered systems.

Recently, Straley[5] has suggested that the point $\sigma_a/\sigma_b=0$, $p=p_c$ is a critical point analogous to those studied in magnets. Using this analogy he has constructed a homogeneous function representation for the conductivity. Of particular interest are the two special cases: (i) $\sigma_a=0$, σ_b finite (a resistor lattice with a fraction of the resistors removed), and (ii) σ_a finite, $\sigma_b=\infty$ (a resistor lattice with a finite fraction of shorts). Close to p_c the conductivity has a power-law behavior; in (i) $\Sigma \sim (p-p_c)^t$ for $p>p_c$, and in (ii) $\Sigma \sim (p_c-p)^{-s}$ for $p<p_c$, where s and t are critical exponents. Scaling relations for resistor networks have been discussed by deGennes,[3] Skal and Shklovskii,[6] and Harris and Fisch.[7] These are discussed in Sec. VII.

The critical exponents t and s have been evaluated by Straley[8] for the Bethe lattice using some results of Stinchcombe.[9] Recently, Dasgupta *et al.*[10] have given a renormalization-group treatment of this problem in the special case $\sigma_a=0$, σ_b finite, and $p<p_c$, i.e., they have studied the average resistance of finite clusters and determined the exponent determining this resistance in $6-\epsilon$ dimensions. The conductivity exponent t has been inferred from scaling arguments. This treatment is based on a modification of the Potts model and it does not appear possible to extend it to calculate the macroscopic conductivity directly or to the case $\sigma_a \neq 0$.

In this paper we give a general formulation of this problem which allows us to consider general networks with both σ_a and σ_b finite or the case where the resistors are replaced by more complicated circuit elements with complex impedances. We show that there is a very close analogy between this problem and that of phase transitions in equilibrium thermodynamics. Using the replica method it is possible to construct a Hamiltonian for the network and from this Hamiltonian a natural definition of the order parameter emerges. The basic critical exponents are best defined in terms of the scaling properties of the order parameter. It is found that the critical behavior of the resistor network is described by the percolation exponents and two extra exponents which, in analogy with random magnets near the percolation point,[11] we call the crossover exponents. The macroscopic and microscopic conductivities of the network can be obtained from the response of the order parameter to a slowly varying (in space and time) applied potential and then are analogous to susceptibilities in magnetic phase transitions.

The paper is organized as follows: in Sec. II we begin with Kirchhoff's equations and define the macroscopic, frequency, and wave-vector-dependent conductivity. These results are well known but are included for completeness. In Sec. HI, using the replica method, the Hamiltonian is derived, the order parameter is defined and the relation of the order parameter to the macroscopic conductivity, the microscopic conductivity and the average resistance of finite clusters is given. In Secs. IV–VI the mean-field equations for the order parameter are obtained and the conductivities determined in three cases: (i) $\sigma_a = 0$, $\sigma_b \neq 0$; (ii) $\sigma_a \neq 0$, $\sigma_b = \infty$; and (iii) $\sigma_a \neq 0$, $\sigma_b \neq 0$, $p = p_c$. The mean-field theory presented here should not be confused with the effective-medium theory. In Sec. VII we discuss the relation of the conductivity exponents to the basic exponents appearing in the order parameter. Using the results of a renormalization-group calculation the conductivity exponents are given in $6-\epsilon$ dimensions.

II. KIRCHHOFF'S EQUATIONS

We consider a random resistor network on a regular lattice of N sites of coordination number z. Part of the network is shown in Fig. 1. At each site a time-dependent external potential $U_i e^{i\omega t}$ is applied through a capacitor. This network was introduced by Kirkpatrick.[1] If $Q_i e^{i\omega t}$ and $V_i e^{i\omega t}$ are the charge and potential at site i, the Kirchhoff equations are

$$i\omega Q_i = i\omega(V_i - U_i) = -\sum_j \sigma_{ij}(V_i - V_j) \quad . \tag{2.1}$$

For convenience we have taken all the capacitors in Fig. 1 to have the value unity. The conductors in Fig. 1 may be replaced by more general circuit elements with inductance and capacitance in which case σ_{ij} in (2.1) is replaced by the complex inverse inpedance $Z_{ij}^{-1}(\omega)$. For simplicity we assume that all the σ_{ij} are real. The σ_{ij} are all independent and each σ_{ij} may take on the two values σ_b and σ_a with probabilities p and $1-p$, respectively ($\sigma_b >> \sigma_a$).

Equation (2.1) can be written in the matrix form (we use a convention in which repeated indices are to be summed over)

$$B_{ij} V_j = i\omega U_i \quad . \tag{2.2}$$

The conductance matrix B has elements

$$B_{ii} = i\omega + \sum_j \sigma_{ij}, \quad B_{ij} = -\sigma_{ij} \quad . \tag{2.3}$$

We denote the elements of the inverse matrix B^{-1} by B_{ij}^{-1}, and from (2.2),

$$Q_i = V_i - U_i = i\omega B_{ij}^{-1} U_j - U_i \quad . \tag{2.4}$$

We use a notation $\langle Q \rangle_c$ to denote an average of Q over all configurations of conductors on the lattice and from (2.4) the average charge at site i is

$$\langle Q_i \rangle_c = i\omega \langle B_{ij}^{-1} \rangle_c U_j - U_i \quad . \tag{2.5}$$

Introducing Fourier transforms

$$Q_i = \frac{1}{\sqrt{N}} \sum_K e^{i\vec{k}\cdot\vec{r}_i} Q_k, \quad U_i = \frac{1}{\sqrt{N}} U e^{i\vec{k}\cdot\vec{r}_i} \tag{2.6}$$

into (2.5), we have

$$\langle Q_k \rangle_c = i\omega \langle k | \langle B^{-1} \rangle_c | k \rangle U - U \quad , \tag{2.7}$$

where

$$\langle k | \langle B^{-1} \rangle_c | k \rangle = \frac{1}{N} \sum_{ij} \langle B_{ij}^{-1} \rangle_c \times \exp[-i\vec{k}\cdot(\vec{r}_i - \vec{r}_j)] \quad . \tag{2.8}$$

We compare (2.7) with the equation of continuity for the charge Q_m and current $\vec{J}_m$ in a medium of macroscopic conductivity $\Sigma(k,\omega)$. Using $\vec{J}_{mk} = \Sigma(-i\vec{k}U)$, where $-i\vec{k}U$ is the electric field, we have

$$i\omega Q_{mk} + k^2 \Sigma U = 0 \quad . \tag{2.9}$$

We identify Q_{mk} with $\langle Q_k \rangle_c$ and from (2.7) and (2.9) the real part of the conductivity $\Sigma_R(k,\omega)$ is given by

$$\Sigma_R(k,\omega) = (\omega^2/k^2)\, \mathrm{Re} \langle k | \langle B^{-1} \rangle_c | k \rangle \quad . \tag{2.10}$$

The dc conductivity is given by

$$\Sigma_{\mathrm{dc}} = \lim_{\omega\to 0} \lim_{k\to 0} \Sigma_R(k,\omega) \quad , \tag{2.11}$$

where the order of limits is important.

Physically this macroscopic conductivity could be measured by placing the network between the plates of a capacitor and applying a slowly varying ac field. The losses in the cavity are determined by Σ_R.

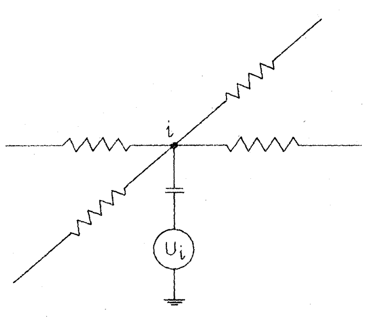

FIG. 1. Random-resistor network.

III. GAUSSIAN INTEGRAL FORMULATION AND THE ORDER PARAMETER

We introduce a "partition function" in the form of a Gaussian integral

$$Z = \int_{-\infty}^{\infty} \prod_i dV_i \exp(-\tfrac{1}{2} V_i B_{ij} V_j + i\omega V_i U_i) \quad , \tag{3.1}$$

where again repeated indices are to be summed over. The integration variable V_i may be thought of as the potential at site i and the expression under the integral sign as the probability distribution of these potentials. For the conductivity we require the configuration average of B^{-1}, and in order to obtain this average we introduce n replicas of the network. This is conveniently done by replacing V_i and U_i in (3.1) by n-component vectors $\vec{V}_i$ and $\vec{U}_i$ with components $V_{i\alpha}$ and $U_{i\alpha}=U_i$ $(\alpha=1,\ldots,n)$. The configuration average of Z^n is denoted by $Z(n)$:

$$Z(n)=\left\langle \int (dV)\exp(-\tfrac{1}{2}\vec{V}_i B_{ij}\vec{V}_j+i\omega\vec{V}_i\cdot\vec{U}_i)\right\rangle_c \quad , \tag{3.2}$$

where $(dV)=\prod_i d\vec{V}_i$. From (2.3), we have

$$\vec{V}_i B_{ij}\vec{V}_j=i\omega\sum_i \vec{V}_i^2+\sum_{nn}\sigma_{ij}(\vec{V}_i-\vec{V}_j)^2 \quad , \tag{3.3}$$

where the second sum is over all nearest-neighbor pairs. The configuration average in (3.2) is easily carried out and the result can be written [omitting a factor $(1-p)^{Nz/2}$]

$$Z(n)=\int (dV)e^{-H_0(V)-H_1(V)} \quad , \tag{3.4}$$

where

$$H_0(V)=\frac{i\omega}{2}\sum_i \vec{V}_i^2-i\omega\vec{V}_i\cdot\vec{U}_i+\frac{\sigma_a}{2}\sum_{nn}(\vec{V}_i-\vec{V}_j)^2 \quad , \tag{3.5}$$

$$H_1(V)=-\sum_{nn}\ln(1+\nu e^{-(\sigma_b/2)(\vec{V}_i-\vec{V}_j)^2}) \quad , \tag{3.6}$$

where $\nu=p/1-p$ and in the last term we have replaced $\sigma_b-\sigma_a$ by σ_b as $\sigma_b>>\sigma_a$.

The partition function is expressed in terms of the order parameter by expanding H_1 in a Fourier integral

$$H_1(V)=\frac{-1}{z}\sum_{nn}\sum_p B_p\psi_p(V_i)\psi_{-p}(V_j) \quad , \tag{3.7}$$

where $\psi_p(V)=e^{i\vec{p}\cdot\vec{V}}$, and for convenience, we write $\sum_p$ instead of $\int d\vec{p}$. By expanding the natural log in (3.6) we find (for $n=0$)

$$B_p=z\sum_{l=1}^{\infty}\frac{(-)^{l+1}}{l}\nu^l e^{-p^2/2\sigma_b l} \quad . \tag{3.8}$$

A factor z has been introduced into the definition of B_p. We thus have

$$e^{-H_1(V)}=\exp\left[\frac{1}{2}\sum_p\sum_{ij}B_p\psi_p(V_i)A_{ij}^{-1}\psi_{-p}(V_j)\right] \quad , \tag{3.9}$$

where $A_{ij}^{-1}=1/z$ if i,j are nearest neighbors and is zero otherwise. The exponent in (3.9) is a quadratic form in the $\psi_p(V_i)$ and can be written

$$e^{-H_1(V)}=C\int (ds)\exp\left[-\frac{1}{2}\sum_p\sum_{ij}B_p s_i(p)A_{ij}s_j(-p)+\sum_p\sum_i B_p s_i(p)\psi_p(V_i)\right] \quad , \tag{3.10}$$

where C is a constant and $(ds)=\prod_p\prod_i ds_i(p)$. Substituting this result in (3.4), we get

$$Z(n)=C\int (ds)(dV)e^{-H_0(s)-H_1(s,V)} \quad , \tag{3.11}$$

where

$$H_0(s)=\frac{1}{2}\sum_p\sum_{ij}B_p s_i(p)A_{ij}s_j(-p) \quad , \tag{3.12}$$

$$H_1(s,V)=\frac{i\omega}{2}\sum_i\vec{V}_i^2-i\omega\vec{V}_i\cdot\vec{U}_i+\frac{\sigma_a}{2}\sum_{nn}(\vec{V}_i-\vec{V}_j)^2-\sum_p\sum_i B_p s_i(p)\psi_p(V_i) \quad . \tag{3.13}$$

Equation (3.12) and (3.13) define the Hamiltonian.

The order parameter is $\langle s_i(p)\rangle$, where the angular brackets denote a canonical average

$$\langle s_i(p)\rangle=\frac{1}{Z(n)}\int (ds)(dV)s_i(p)e^{-H} \quad . \tag{3.14}$$

To obtain an expression for the order parameter, we add a term

$$H_\eta=-\sum_p\sum_i B_p\eta_i(p)A_{ij}s_j(-p) \tag{3.15}$$

to the Hamiltonian in (3.11). Then

$$Z_\eta=\int (ds)(dV)e^{-H_0(s)-H_1(s,V)-H_\eta} \quad , \tag{3.16}$$

and, by differentiation,

$$\left(\frac{\partial\ln Z_\eta}{\partial\eta_i(p)}\right)_{\eta=0}=B_p\sum_j A_{ij}\langle s_j(-p)\rangle \quad . \tag{3.17}$$

By making the transformation $s_i(p)=s_i'(p)+\eta_i(p)$ in (3.16), we find

$$Z_\eta=\int (ds')(dV)\exp\left[-H_0(s')-H_1(s',V)+\sum_{pi}B_p\eta_i(p)\psi_p(V_i)\right] \quad , \tag{3.18}$$

where terms of order η^2 have been omitted. Then

$$\left(\frac{\partial\ln Z_\eta}{\partial\eta_i(p)}\right)_{\eta=0}=B_p\langle\psi_p(V_i)\rangle \quad , \tag{3.19}$$

and from (3.17),

$$\sum_j A_{ij}\langle s_j(-p)\rangle=\langle\psi_p(V_i)\rangle \quad . \tag{3.20}$$

$\langle\psi_p(V_i)\rangle$ is the generating function for voltage fluctuations at site i. The Fourier transform of A_{ij} is $A(k)=1+O(k^2)$ and we may replace $A(k)$ by unity or A_{ij} by δ_{ij} in (3.20) so that

$$\langle s_i(p)\rangle = \langle \psi_{-p}(V_i)\rangle \quad . \tag{3.21}$$

The Fourier transform with respect to p of $\langle s_i(p)\rangle$ is

$$\langle s_i(W)\rangle = \frac{1}{(2\pi)^n}\sum_p e^{i\vec{p}\cdot\vec{W}}\langle s_i(p)\rangle \tag{3.22}$$

Equations (3.21) and (3.22) are easily evaluated from (3.2) and for $n=0$, we find

$$\langle s_i(p)\rangle = \langle e^{(-p^2B_{ii}^{-1})/2+\omega\vec{p}\cdot B_{ij}^{-1}\vec{U}_j}\rangle_c \quad , \tag{3.23}$$

$$\langle s_i(W)\rangle = \langle e^{-(W^2-i\omega\vec{W}\cdot B_{ij}^{-1}\vec{U}_j)/2B_{ii}^{-1}}\rangle_c \tag{3.24}$$

The order parameter has the following convenient properties:

A. Macroscopic conductivity

The response of the network to the external potential can be obtained from the response of the order parameter to the potential. Thus form (3.23),

$$\frac{1}{\omega}\left[\frac{\partial}{\partial p_\alpha}\langle s_i(p)\rangle\right]_{p=0} = \langle B_{ij}^{-1}\rangle_c U_j \tag{3.25}$$

The Fourier transform of the order parameter is

$$\langle s_i(p)\rangle = \frac{1}{\sqrt{N}}\sum_k e^{i\vec{k}\cdot\vec{r}_i}\langle s_k(p)\rangle \quad . \tag{3.26}$$

Taking the potential in the form (2.6) and using (2.8) and (2.10), we have

$$\Sigma_R(k,\omega) = \frac{\omega}{k^2U}\mathrm{Re}\left[\frac{\partial}{\partial p_\alpha}\langle s_k(p)\rangle\right]_{p=0} \quad , \tag{3.27}$$

or in terms of the Fourier transform (3.22),

$$\Sigma_R(k,\omega) = \frac{\omega}{k^2U}\lim_{n\to 0}\mathrm{Im}\int W_\alpha d\vec{W}\langle s_k(W)\rangle \quad . \tag{3.28}$$

B. Average resistance of finite clusters

Consider the network for $p<p_c$ in the case $\sigma_a=0$, σ_b finite, and $\omega \ll \sigma_b$. The network only consists of finite clusters and in the absence of an external potential

$$\langle s(W)\rangle = \langle e^{-W^2/2B_{ii}^{-1}}\rangle_c \tag{3.29}$$

and is independent of site i. It is shown in Appendix A that

$$\langle 1/B_{ii}^{-1}\rangle_c = i\omega[\overline{m} - i\omega\overline{R} + O(\omega/\sigma_b)^2] \quad , \tag{3.30}$$

where $\overline{m}$ is the average number of sites in a cluster per site and $\overline{R}\sim 1/\sigma_b$ is the average resistance of a cluster per site. These are defined as follows: the average resistance of a cluster is $R=\sum_{i>j}R_{ij}$, where R_{ij} is the resistance between sites i and j in the cluster and the sum extends over all pairs of sites in the cluster. If $P_m(R)$ is the probability that a site lies in a cluster of m sites and average resistance R then $\overline{m}=\sum_{m,R} mP_m(R)$ and $\overline{R}=\sum_{m,R} RP_m(R)$. The definition of $\overline{m}$ is the conventional one and $\overline{R}$ same way (a different definition of the average resistance of a cluster has been used in Ref. 10). From (3.29) and (3.30) it follows that by expansion of the order parameter in powers of W^2 and ω/σ_b, we can obtain $\overline{R}$.

C. Voltage correlations

We define the voltage correlations in the following way: suppose we apply a potential $U_i=U$ at site i of the network and set all the other $U_j=0$. Then from (2.4),

$$V_i = i\omega B_{ii}^{-1}U, \quad V_j = i\omega B_{ji}^{-1}U \quad , \tag{3.31}$$

and we define the voltage correlation function

$$F_{ij}(\omega) = \langle V_j/V_i\rangle_c = \langle B_{ji}^{-1}/B_{ii}^{-1}\rangle_c \quad , \tag{3.32}$$

and its Fourier transform

$$F(k,\omega) = \frac{1}{N}\sum_{ij}\langle\frac{B_{ji}^{-1}}{B_{ii}^{-1}}\rangle_c e^{i\vec{k}\cdot(\vec{r}_i-\vec{r}_j)} \quad . \tag{3.33}$$

This correlation function can be obtained from (3.24) by choosing the potential $U_i=(1/\sqrt{N})Ue^{i\vec{k}\cdot\vec{r}_i}$. Then

$$F(k,\omega) = -\frac{2i}{\omega U}\left[\frac{\partial}{\partial W_\alpha}\langle s_k(W)\rangle\right]_{W=0} \quad . \tag{3.34}$$

For $p<p_c$, $\sigma_a=0$, σ_b finite, and $\omega<\sigma_b$, F_{ij} reduces to the probability that i and j are in the same cluster. In the conducting state, $F(k,\omega)$ will have a pole at $\omega=i\sigma_{mi}k^2$, where σ_{mi} is the microscopic conductivity. Thus (3.34) allows us to determine the microscopic conductivity. The microscopic conductivity corresponds to the spin wave stiffness in a ferromagnet.

We thus see that the order parameter (3.23) or (3.24) contains most of the interesting information about the conducting properties of the lattice. In the remainder of this paper we establish the mean-field equation for this order parameter and determine the conducting properties of the lattice near p_c.

The mean-field equation for the order parameter is obtained by evaluating the s integral in (3.11) by steepest descents. Differentiating the exponent with respect to $s_i(p)$ gives the mean-field equation

$$A_{ij}s_j(-p) = \frac{1}{Z_1}\int(dV)e^{i\vec{p}\cdot\vec{V}_i}e^{-H_1(s,V)} \quad , \tag{3.35}$$

where

$$Z_1 = \int(dV)e^{-H_1(s,V)} \quad . \tag{3.36}$$

Taking the Fourier transform with respect to p of (3.35) gives

$$A_{ij}s_i(W)=\frac{1}{Z_1}\int(dV)\delta(\vec{W}-\vec{V}_i)e^{-H_1(s,V)} \quad . \tag{3.37}$$

We now omit the angular brackets from s so that s is the mean-field value of $\langle s\rangle$. The factor Z_1 is to be determined from the condition $s(p=0)=1$. We now consider the solution of (3.37) in some special cases.

IV. NETWORK OF OPEN CIRCUITS AND CONDUCTORS ($\sigma_a=0$, $\sigma_b\neq 0$)

When $\sigma_a=0$, $H_1(s,V)$ separates into a sum of terms for each site i and the integral in (3.37) is a product of N integrals. $N-1$ of these integrals (for sites $j\neq i$) cancel with identical integrals in Z_1 and we obtain

$$A_{ij}s_j(W)=\frac{1}{Z_{i1}}\exp\left[-\frac{i\omega W^2}{2}-i\omega\vec{W}\cdot\vec{U}_i+\sum_p B_p s_i(p)e^{i\vec{p}\cdot\vec{W}}\right] \quad , \tag{4.1}$$

where Z_{i1} is determined from the condition

$$\int d\vec{W}\, s_i(W)=1 \quad . \tag{4.2}$$

In the absence of a spatially dependent potential, $s_i(W)$ is independent of site i and (4.1) reduces to

$$s(W)=\frac{1}{Z_1}\exp\left[-\frac{i\omega W^2}{2}+\sum_p B_p s(p)e^{i\vec{p}\cdot\vec{W}}\right] \tag{4.3}$$

We consider some special cases.

A. $\sigma_b=\infty$

In this case from (3.8), $B_p=z\ln(1+v)=x$ say and is independent of p. It is shown below that $Z_1=e^x$ and (4.3) becomes

$$s(W)=e^{-i\omega W^2/2+x[s(W)-1]} \quad . \tag{4.4}$$

The percolation point is determined by $x=1$ and we let $r_0=1-x$. Solutions to Eq. (4.4) can be obtained as follows:

(i) For $p<p_c$, Eq. (4.4) is readily solved by Lagranges method

$$s(W)=\sum_{m=1}^{\infty}P_m e^{-im\omega W^2/2} \quad , \tag{4.5}$$

where P_m is the probability that a site lies in a cluster of m sites and is given by

$$P_m=\frac{(mx)^{m-1}}{m!}e^{-mx}\simeq\frac{1}{(2\pi m^3)^{1/2}}e^{-mr_0^2/2} \quad . \tag{4.6}$$

The second form is appropriate near p_c for large m. The average size of clusters diverges at $r_0=0$ which gives $x=1$ for the percolation point. This result has been given previously.[12] We note that $s(W)$ is normalized as in (4.2) (for $n=0$) so that our choice of $Z_1=e^x$ is correct. It is interesting to note that equations of the type (4.1) and (4.4) arise in percolation problems where there are contributions from clusters of all sizes.

(ii) Equation (4.4) is readily solved by expansion in W^2 and in this way we can also obtain the solution for $p>p_c$:

$$s(W)=\begin{cases}1-i\omega W^2/2(1-x), & p<p_c \ ,\\ 1-P-i\omega W^2/2(x-1), & p>p_c \ ,\end{cases} \tag{4.7}$$

where $P=2(x-1)=2|r_0|$ is the percolation probability. From (3.29) and (3.30) the coefficient of $-\frac{1}{2}i\omega W^2$ in (4.7) is the average number of sites in a cluster $\bar{m}=1/|1-x|$.

(iii) close to p_c we expect that $s(W)$ will scale in the variables W^2 and r_0 and we look for a solution of the form

$$s(W)=1-r_0^{\beta}f(\omega W^2/r_0^{\Delta}) \quad , \tag{4.8}$$

where the exponents β and Δ must be determined and we have used the conventional symbols for these percolation exponents.[12] Equation (4.8) is substituted in (4.4) and the natural log of both sides is taken. Assuming $\beta>0$ we expand in powers of $r_0^{\beta}f$. It is readily shown that $\beta=1$, $\Delta=2$, and f satisfies

$$\tfrac{1}{2}f^2+f+iW^2/2r_0^2=0 \quad , \tag{4.9}$$

$$f=-1\pm(1+i\omega W^2r_0^{-2})^{1/2}\left.\begin{matrix}p<p_c\\ p>p_c\end{matrix}\right\} \quad . \tag{4.10}$$

This solution for f is in agreement with (4.7), and this method for the solution of the mean-field equations will be used below. We note that the scaling solution breaks down if W^2 is large. Thus from (4.4) or (4.5), $s(W\to\infty)=0$, a property which does not hold for (4.8) and (4.10). This is not surprising as we do not expect scaling to hold for the whole range of the variables. After these preliminary considerations we turn to the case σ_b finite.

B. σ_b finite

We expect the solution near p_c for $s(p)$ in (4.3) to be sharply peaked around $p=0$. This is confirmed below and allows us to expand B_p in powers of p^2,

$$B_p=x-p^2/b+O(p^4) \quad , \tag{4.11}$$

$$\frac{1}{b}=\frac{z}{2\sigma_b}\sum_{l=1}^{\infty}\frac{(-)^{l+1}}{l^2}v^l \quad . \tag{4.12}$$

In what follows, b will be used in place of σ_b. Substituting (4.11) in (4.3) (with $Z_1=e^x$), we obtain

$$s(W)=\exp\left[\frac{-i\omega W^2}{2}+x[s(W)-1]+\frac{1}{b}\sum_\alpha\frac{\partial^2 s}{\partial W_\alpha{}^2}\right] . \tag{4.13}$$

Again this equation may be solved in several ways.

(i) $p<p_c$: To obtain the average resistance of finite clusters we solve (4.13) as a power series in W^2 and b^{-1}. Let

$$s(W)=1-\tfrac{1}{2}s_1 i\omega W^2+(1/4!)s_2(i\omega W^2)^2\cdots . \tag{4.14}$$

Substituting in (4.13) it is found (for $n=0$)

$$s_1(1-x)=1-(2i\omega/3b)s_2 ,$$

$$s_2(1-x)=3(1+xs_1)^2 ,$$

which give

$$s_1=1/(1-x)-2i\omega/b(1-x)^4,\quad s_2=3/(1-x)^3 . \tag{4.15}$$

From (3.30) the average resistance of finite clusters is

$$\bar{R}=2/br_0^4 , \tag{4.16}$$

and diverges with an exponent 4 as $p\to p_c$.

(ii) close to p_c we expect the solution of (4.13) to scale in its variables r_0, W^2, and b, and we choose a solution of the form

$$s(W)-1-r_0^\beta F_0(\tfrac{1}{4}bW^2r_0^\phi,\omega/br_0^{\Delta+\phi}) . \tag{4.17}$$

It is readily shown using the method described above that $\beta=\phi=1$ and $\Delta+\phi=3$ and F_0 satisfies

$$\tfrac{1}{2}F_0^2+F_0-\frac{i\omega W^2}{2r_0^2}-\frac{1}{br_0}\sum_\alpha\frac{\partial^2F_0}{\partial W_\alpha{}^2}=0 . \tag{4.18}$$

It may be shown that the higher order derivatives omitted in (4.13), arising from the expansion of B_p in powers of p^2, do not contribute in the scaling region to (4.18). Introducing the scaling variables $u=\frac{1}{4}bW^2|r_0|$ and $\omega_b=\omega/b|r_0|^3$, Eq. (4.18) takes the form ($n=0$)

$$\left.\tfrac{1}{2}F_0^2+F_0-2iu\omega_b\mp u\frac{\partial^2F_0}{\partial u^2}=0\right\}\begin{matrix}p<p_c ,\\ p>p_c .\end{matrix} \tag{4.19}$$

We note that $\bar{R}$ in (4.16) is in agreement with (4.17) and that $\bar{R}\sim r_0^{-2\Delta-\phi+\beta}$.

For the dc conductivity we will require $s(W)$ in the case for $\omega=0$ and for $p>p_c$. In this case we put

$$F_0(u,0)=-2+2g(u),\quad s(W)=1-P[1-g(u)] , \tag{4.20}$$

where g satisfies

$$g^2-g+u\frac{\partial^2 g}{\partial u^2}=0 . \tag{4.21}$$

the boundary conditions for this equation are $g(0)=1$, $g(\infty)=0$ and follow from $s(W=0)=1$ and $s(W=\infty)=1-P$. This latter result comes from the fact that when $\omega=0$, the contribution of finite clusters to $s(W)$ is a constant independent of W, and for $p>p_c$, this constant is $1-P$ [see Eq. (4.7)]. It is interesting to note that an identical equation to (4.21) was obtained by Stinchcombe[9] in his determination of the conductivity of a Bethe lattice.

The scaling properties of the order parameter are given in Eq. (4.17) and there are three independent exponents. We now determine the conductivity.

C. Response of the order parameter to an applied potential

In the presence of an applied potential $U_i=(1/\sqrt{N})Ue^{i\vec{k}\cdot\vec{r}_i}$, the order parameter is spatially dependent. For small k the Fourier transform of A_{ij}, $A(k)=1+\gamma^2k^2\cdots$, where γ is proportional to the nearest-neighbor lattice spacing. Thus for long wavelengths in Eq. (4.1) we may replace $A_{ij}\to 1-\gamma^2\nabla^2$. We take the order parameter in the form $s(W)=1-r_0^\beta F(\vec{r})$ and then following the steps that lead to (4.18), we obtain the equation for F (with $\beta=1$):

$$\tfrac{1}{2}F^2+F-\frac{i\omega W^2}{2r_0^2}-\frac{1}{br_0}\sum_\alpha\frac{\partial^2F}{\partial W_\alpha{}^2}-\frac{\gamma^2}{r_0}\nabla^2F=\frac{i\omega}{r_0^2}\vec{W}\cdot\vec{U}(\vec{r}) . \tag{4.22}$$

We note that if the potential $\vec{U}$ is spatially uniform that the solution of (4.22) is $F=F_0(\vec{W}-\vec{U})$. Physically this corresponds to changing the potential at each site by a constant amount U which does not alter the Kirchhoff equations. This suggests that we take the solution of (4.22) in the form

$$F=F_0+\frac{\alpha}{\sqrt{N}}e^{i\vec{k}\cdot\vec{r}}\sum_\alpha\frac{\partial F_0}{\partial W_\alpha} , \tag{4.23}$$

when the potential is small and slowly varying. Substituting in (4.22) and neglecting the derivatives of α with respect to W (we show below that this is a good approximation for small k), we obtain

$$\alpha\left[F_0+1-\frac{1}{br_0}\sum_\beta\frac{\partial^2}{\partial W_\beta{}^2}\pm\xi^2k^2\right]\sum_\alpha\frac{\partial F_0}{\partial W_\alpha}=\frac{i\omega}{r_0^2}\vec{W}\cdot\vec{U} , \tag{4.24}$$

where $\xi^2=\gamma^2/|r_0|$ is the correlation length and the upper and lower signs apply for $p<p_c$ and $p>p_c$. Differentiating (4.18) with respect to W_α we obtain

$$\left[F_0+1-\frac{1}{br_0}\sum_\beta\frac{\partial^2}{\partial W_\beta^2}\right]\frac{\partial F_0}{\partial W_\alpha}=\frac{i\omega W_\alpha}{r_0^2} . \tag{4.25}$$

Substituting in (4.24) and using the fact that F_0 depends on W^2 we obtain

$$\alpha = -i\omega U \Bigg/ \left(i\omega \pm 2\xi^2 k^2 r_0^2 \frac{\partial F_0}{\partial W^2} \right) . \tag{4.26}$$

In obtaining this result we have neglected derivatives with respect to W of α. It may easily be shown that the corrections to (4.26) are of order $k^2\xi^2/\omega_b^2$ and are negligible at long wavelengths. Introducing the scaling variables u and ω_b in (4.26) we get

$$\alpha = -i\omega_b U \Bigg/ \left(i\omega_b \pm \tfrac{1}{2}\xi^2 k^2 \frac{\partial F_0}{\partial u} \right) . \tag{4.27}$$

We first use this result to determine the conductivity at long wavelengths for $p > p_c$. When $\xi^2 k^2/\omega_b < 1$, we can neglect the frequency dependence of F_0 and put $F_0(u,0) = -2 + 2g(u)$ as in (4.20) and then

$$\alpha = -i\omega_b U/(i\omega_b - \xi^2 k^2 g') . \tag{4.28}$$

The Fourier transform of the order parameter is then obtained from (4.23) [with $\partial F_0/\partial W_\alpha$ replaced by $4(\partial g/\partial W^2) W_\alpha$]:

$$s_k(W) = -4 r_0^\beta \alpha \frac{\partial g}{\partial W^2} \sum_\alpha W_\alpha . \tag{4.29}$$

When this is substituted in (3.28) we require an integral of the form (where A is a function of W^2)

$$\lim_{n\to 0} \int d\vec{W}\, W_\alpha \sum_\beta W_\beta A(W^2) = \lim_{n\to 0} \frac{1}{n} \int d\vec{W}\, W^2 A(W^2) = \int_0^\infty W\, dW\, A(W^2) . \tag{4.30}$$

The last result follows because the volume element in n-dimensional space is $d\vec{W} = \Omega_n W^{n-1} dW$, where Ω_n is the area of a unit hypersphere in n dimensions and $\lim_{n\to 0} n^{-1}\Omega_n = 1$.

Substituting (4.28) and (4.29) in (3.28), and using (4.30), we find

$$\Sigma_R(k,\omega) = \Sigma_{\rm dc} R(\omega_b^2/k^2\xi^2) , \tag{4.31}$$

where

$$\Sigma_{\rm dc} = 2\gamma^2 b |r_0|^3 \int_0^\infty du\,(g')^2 = \tfrac{2}{3}\gamma^2 b |r_0|^3 \tag{4.32}$$

and

$$R(y^2) = 3y^2 \int_0^\infty du \frac{g'^2}{y^2 + g'^2} . \tag{4.33}$$

The dc conductivity thus vanishes at p_c with a mean-field exponent of 3. This is in agreement with the results of Straley[8] on the Bethe lattice. The integral in (4.32) has also been évaluated by Straley. The frequency dependence of the conductivity is determined by the scaling variable $\omega_b = \omega/b|r_0|^3$. Owing to the approximations made in obtaining (4.26), Eq. (4.31) is only correct for $k\xi < 1$.

The voltage correlation function for $p > p_c$ from (3.34) and (4.29) is given by

$$F(k,\omega) = \frac{2g'(0)}{|r_0|} \frac{1}{\xi^2 k^2 g'(0) - i\omega_b} , \tag{4.34}$$

where $g'(0) = -0.76$ and has been determined by Stinchcombe.[9] The pole in (4.34), $\omega = i\sigma_{mi} k^2$, determines the microscopic conductivity

$$\sigma_{\rm mi} = -b\gamma^2 |r_0|^2 g'(0) , \tag{4.35}$$

and has an exponent of 2. This is in agreement with the Stinchcombe[9] result for the Bethe lattice. The fact that the macroscopic conductivity (4.32) and the microscopic conductivity have different exponents has been discussed by deGennes.[3]

V. NETWORK OF TWO CONDUCTORS ($\sigma_a << \sigma_b$, $p < p_c$)

The order parameter in this case is determined by (3.37) and we first consider the case where $U = 0$ so that $s_i(W) = s(W)$, independent of the site index i. We approximate the last term in (3.13) by expanding B_p as in (4.11). We look for a scaling solution of (3.37) of the form $s(W) = 1 - r_0^\beta M$, where M is a function of the variables σ_a, $b \sim \sigma_b$, ω, W^2, and r_0. We can write this scaling solution in the form

$$M = M\left(\frac{\sigma_a W^2}{r_0^{\phi_1}}, \frac{\omega}{\sigma_a r_0^{\Delta - \phi_1}}, \frac{\sigma_a}{b r_0^{\phi + \phi_1}} \right) , \tag{5.1}$$

where, in analogy with (4.17), we have introduced a crossover exponent ϕ_1 associated with the σ_a variable. The other exponents, Δ and ϕ, are the same as in (4.17).

The above form for s is substituted in (3.37) and after taking the natural log of both sides we get

$$\ln(1 - r_0^\beta M) = -x r_0^\beta M - b^{-1} r_0^\beta \times \sum_\alpha \frac{\partial^2 M}{\partial W_\alpha^2} + \ln R(W) , \tag{5.2}$$

where $R(W)$ is all the remaining terms in the integral on the right-hand side of (3.37). R is written in Appendix B, but for the present purposes its exact form is not important. From the condition $s(W=0) = 1$ it follows that $M(W=0) = 0$ and $R(0) = 1$. For small W we may take

$$R(W) = 1 - \tfrac{1}{2} a' W^2 + \cdots , \tag{5.3}$$

where a' is to be determined. We show in Appendix B that it is given by

$$a' = \tfrac{1}{2}\{i\omega + \sigma_a z + [(i\omega + \sigma_a z)^2 - 4\sigma_a^2 z]^{1/2}\} \quad . \tag{5.4}$$

Thus for $\sigma_a = 0$, $a' = i\omega$ and (5.2) reduces to (4.18) after expansion of the logs. For $\omega = 0$, $a' = a$ where

$$a = \tfrac{1}{2}\sigma_a z[1 + (1 - 4/z)^{1/2}] \quad . \tag{5.5}$$

In what follows we will use the variable a instead of σ_a. Equation (5.4) also shows that σ_a and ω scale in the same way so that we conclude that $\phi_1 = \Delta$ (in mean-field theory).

The equation determining M is obtained from (5.2) by expanding the logs and we find

$$\tfrac{1}{2}M^2 + M - \frac{a' W^2}{2r_0^2} - \frac{1}{br_0}\sum_\alpha \frac{\partial^2 M}{\partial W_\alpha^{\,2}} = 0 \quad , \tag{5.6}$$

where we have set $\beta = 1$. This equation shows that M is of the form (5.1) with $\phi_1 = \Delta = 2$, $\phi + \phi_1 = 3$.

Of particular interest is the case $b = \infty$, and in order to obtain the conductivity for $k\xi < 1$, we only require M in the case $\omega = 0$. Then setting $a' = a$ and $M(aW^2/r_0^2, 0) = m(\nu)$ with $\nu = aW^2/r_0^2$, we find

$$\tfrac{1}{2}m^2 + m - \tfrac{1}{2}\nu = 0 \quad , \tag{5.7}$$

with the solution for $p < p_c$,

$$m = -1 + (1 + \nu)^{1/2} \quad . \tag{5.8}$$

This scaling solution breaks down for $\nu > \nu_c$ and we can estimate ν_c by including the leading nonscaling term in (5.7). From (5.2) this leading correction comes from the expansion of the log on the left-hand side and is $\frac{1}{3}r_0 m^3$. This term is of the same order as the terms in (5.7) when

$$\nu = \nu_c \approx 9/4r_0^2 \quad . \tag{5.9}$$

The response of the order parameter to an external potential is taken in the form (4.23) with M replacing F_0. It is found for $k\xi < 1$ that α is given by (4.26) (with M replacing F_0). For $k\xi < 1$ and $b = \infty$, we replace M by m and

$$\alpha = -i\omega_a U/(i\omega_a + 2\xi^2 k^2 m') \quad , \tag{5.10}$$

where $\omega_a = \omega/a$. Substituting this result in Eq. (3.28) for the conductivity gives

$$\Sigma_R(k,\omega) = 2\gamma^2 a\,\omega_a^2 \int_0^{\nu_c} d\nu \frac{m'^2}{\omega_a^2 + (2k^2\xi^2 m')^2} \quad ,$$

where we have cut off the integral at ν_c. Using (5.8) and (5.9) the dc conductivity is given by

$$\Sigma_{\rm dc} = \tfrac{1}{2}\gamma^2 a \ln(1 + \nu_c) \tag{5.12}$$

and diverges like $-\ln(p_c - p)$ close to p_c. This is in agreement with a recent result of Straley[8] on the Bethe lattice. The ac conductivity from (5.11) is given by

$$\Sigma_R(k,\omega) = \Sigma_{\rm dc} - \tfrac{1}{2}\gamma^2 a \ln[1 + (k^2\xi^2/\omega_a)^2] \quad . \tag{5.13}$$

The microscopic conductivity is determined by the pole (5.10) and is given by

$$\sigma_{\rm mi} = \gamma^2 a/r_0 \quad , \tag{5.14}$$

and diverges linearly as $(p_c - p)^{-1}$.

VI. CONDUCTIVITY AT $p = p_c$ ($\sigma_a << \sigma_b$)

From the result of Sec. V we anticipate that the order parameter at $p = p_c$ may be taken in the form

$$s(W) = 1 - (a/b)^{\beta/(\Delta+\phi)} M_1 \times (a^{\phi/(\Delta+\phi)} b^{\Delta(\Delta+\phi)} W^2, \omega/a) \quad , \tag{6.1}$$

where we have set $\phi_1 = \Delta$. If $\phi_1 \neq \Delta$, a more general form, which follows from (5.1), must be used. Following the method of Sec. V, and making use of the fact that $a/b << 1$, it is easily shown that M_1 satisfies ($n = 0$)

$$M_1^2 - \frac{a'}{a}w - 8w\frac{\partial^2 M_1}{\partial w^2} = 0 \quad , \tag{6.2}$$

where $w = a^{1/3}b^{2/3}W^2$, we have set $\beta = \phi = 1$ and $\Delta = 2$, and a' and a are given by (5.4) and (5.5), respectively. In the case $\omega = 0$, we put $M_1(w, 0) = m_1(w)$ which satisfies

$$m_1^2 - w - 8w\frac{\partial^2 m_1}{\partial w^2} = 0 \quad . \tag{6.3}$$

This equation breaks down for large w and including the leading nonscaling correction, $\frac{2}{3}(a/b)^{1/3}m_1^3$, in (6.3) we find that the cutoff is given by

$$w_c = \tfrac{9}{4}(b/a)^{2/3} \quad . \tag{6.4}$$

The solution to (6.3) is conveniently taken in the form

$$m_1 = (1 + w)^{1/2} + m_2 \quad , \tag{6.5}$$

where m_2 vanishes at $w = \infty$. The most divergent part of the conductivity is determined by the first term in (6.5) and is

$$\Sigma_{\rm dc} = \tfrac{1}{2}\gamma^2 a \ln(1 + w_c) \quad , \tag{6.6}$$

and depends logarithmically on the ratio b/a. The microscopic conductivity is given by

$$\sigma_{\rm mi} = 2\gamma^2 a^{2/3} b^{1/3} m_1'(0) \quad , \tag{6.7}$$

and diverges as $(b/a)^{1/3}$ for large b/a.

VII. SCALING LAWS AND CRITICAL EXPONENTS

The scaling behavior of the order parameter, Eq. (5.1), shows that a description of the critical behavior of the resistor network involves the percolation exponents and two crossover exponents ϕ and ϕ_1. Whether these exponents are independent cannot be decided from the mean-field theory. The present theory can be extended to include the effects of fluctuations and these exponents have been calculated in $6-\epsilon$ dimensions[13] with the results

$$\phi=1, \quad \phi_1=\Delta \quad . \tag{7.1}$$

The exponent ϕ is the same as the crossover exponent in random magnets.[11] The same exponent $\phi=1$ has been determined by Dasgupta *et al.*[10] by a different method and shown to be unity to all orders in ϵ by Young and Wallace.[14] The result $\phi_1=\Delta$ follows because σ_a and ω scale in the same way (they both scale as "magnetic fields" as they couple linearly to the order parameter). The results (7.1), while correct to all order in ϵ, may not be true in low dimensions.

It is useful to express the conductivity exponents in terms of the exponents appearing in the order parameter. In this way calculation of the conductivity exponents is reduced to determining the scaling properties of the order parameter. We define the two conductivity exponents

$$\begin{aligned} \Sigma_{\mathrm{dc}}(a=0,b) &\sim |r_0|^t, \quad p>p_c \ , \\ \Sigma_{\mathrm{dc}}(a,b=\infty) &\sim r_0^{-s}, \quad p<p_c \ , \end{aligned} \tag{7.2}$$

where we have used Straley's[5] notation. We also define exponents for the microscopic conductivities

$$\begin{aligned} \sigma_{\mathrm{mi}}(a=0,b) &\sim |r_0|^{t'}, \quad p>p_c \ , \\ \sigma_{\mathrm{mi}}(a,b=\infty) &\sim r_0^{-s'}, \quad p<p_c \ . \end{aligned} \tag{7.3}$$

The relation of these exponents to the order-parameter exponents may be inferred from (4.28), (4.29), (5.10), and formula (3.28) for the conductivity. It is assumed that correlation length $\xi \sim r_0^{-\nu}$ and enters in the form $k\xi$ as in (4.28) (mean-field theory gives $2\nu=\beta$). The following relations are then obtained:

$$t=\beta+\Delta+\phi-2\nu=\phi+(d-2)\nu \ , \tag{7.4}$$

$$s=2\nu+\phi_1-\beta-\Delta=\phi_1-(d-2)\nu \ , \tag{7.5}$$

$$t'=t-\beta, \quad s'=s+\beta \ . \tag{7.6}$$

A relation of the form (7.4) has been given by de Gennes[3] (with $\phi=1$), Skal and Shklovskii,[6] and Harris and Fisch.[7] Using (7.1) and the known percolation exponents,[12,15] we obtain the conductivity exponents in $6-\epsilon$ dimensions

$$\begin{aligned} t &= 3-\tfrac{11}{42}\epsilon, \quad s=\tfrac{11}{42}\epsilon \ , \\ t' &= 2-\tfrac{5}{42}\epsilon, \quad s'=1+\tfrac{5}{42}\epsilon \ . \end{aligned} \tag{7.7}$$

The result for t is in agreement with that of Dasgupta *et al.*[10]

The frequency dependence of the conductivity is determined by the scaled frequency variables

$$\omega_b=\omega/b|r_0|^{\Delta+\phi}\sim\omega/b|r_0|^3 \ ,$$

when $a=0$, and

$$\omega_a=\omega/ar_0^{\Delta-\phi_1}\sim\omega/a \ ,$$

when $b=\infty$. In the case where a and b are nonzero with $a \ll b$, the conductivity depends on the scaling variable $a/br_0^{\phi+\phi_1} \sim a/br_0^3$.

A further quantity of interest is the average resistance $\bar{R}$ of finite clusters when $a=0$, $p<p_c$. This diverges like

$$\bar{R}\sim r_0^{-\gamma_R}, \quad \gamma_R=2\Delta+\phi-\beta \ . \tag{7.8}$$

The exponent γ_R is related to the exponent γ_r of Harris and Fisch[7] for their average resistance by $\gamma_R=\gamma_r+\Delta$.

In conclusion the Gaussian integral method introduced in this paper enables one to derive a "Hamiltonian" for the resistor network and a natural order parameter for the problem is found. The conductivity is obtained from the response of the order parameter to an external potential and is thus similar to a susceptibility. Using scaling arguments familiar in critical phenomena we are then able to obtain considerable insight into the properties of the random network near the percolation threshold.

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APPENDIX A

The conductance matrix (2.3) has the form $B=i\omega+A$, where the matrix A has elements

$$A_{ii}=\sum_j \sigma_{ij}, \quad A_{ij}=-\sigma_{ij} \ . \tag{A1}$$

The cofactors of A are denoted by $\tilde{A}_{ii}$, $\tilde{A}_{ijij}$. Thus $\tilde{A}_{ii}$ is the determinant obtained by omitting row and column i of A and A_{ijij} is the determinant obtained by omitting row i and j and column i and j in A. It is easily shown that $A_{ii}=A_{jj}$, etc. The resistance between sites i and j in the same cluster is given by

$$R_{ij}=\tilde{A}_{ijij}/\tilde{A}_{ii} \ . \tag{A2}$$

Now consider B_{ii}^{-1} in the case $p<p_c$, $\sigma_a=0$, and $\sigma_b\neq 0$, and suppose that site i lies in a cluster of m sites. Then expanding $(B_{ii}^{-1})^{-1}$ in powers of $i\omega/\sigma_b$ we have

$$\frac{1}{B_{ii}^{-1}}=\frac{|B|}{\tilde{B}_{ii}}=\left[i\omega m\tilde{A}_{ii}+(i\omega)^2\sum_{i>j}\tilde{A}_{ijij}+\cdots\right]$$

$$\times\left[\tilde{A}_{ii}+i\omega\sum_{j}\tilde{A}_{ijij}+\cdots\right]^{-1} \quad \text{(A3)}$$

$$=i\omega\left[m+i\omega\sum_{i>j}R_{ij}-i\omega m\sum_{j}R_{ij}+\cdots\right], \quad \text{(A4)}$$

where the sums extend over all the sites in the cluster. When (A4) is averaged over all configurations of conductors on the lattice, we obtain Eq. (3.30).

APPENDIX B

Here we determine a' in (5.3) self-consistently. From (3.37) and (5.2) we have

$$R(W)=\frac{1}{Z}\int(dV)\delta(\vec{W}-\vec{V}_i)$$

$$\times\exp\left[-H_0(V)-r_0\sum_j{}' M(V_j)\right], \quad \text{(B1)}$$

where H_0 is given by (3.5) (with $U=0$) and the prime on the summation means that the term $j=i$ is omitted. We have set $b=\infty$ for simplicity and this can be shown not to change the final result. We can easily expand $R(W)$ in powers of W by setting $\vec{V}_i=\vec{W}$ in the right-hand side of (B1), omitting the $\vec{V}_i$ integral, and expanding the exponent in powers of W. For $n=0$, $Z=1$, and in this way we find

$$a'=i\omega+\sigma_a z-\sigma_a^2\int(dV)'\left[\sum_j{}'' V_j\alpha\right]^2$$

$$\times\exp\left[-H_0'(V)-r_0\sum_j{}' M(V_j)\right], \quad \text{(B2)}$$

where H_0' is obtained from H_0 by setting $\vec{V}_i=0$ $(dV)'=\prod_{j\neq i}d\vec{V}_j$, and the double prime on the summation means that j must be a nearest neighbor to i. The scaling properties of the order parameter M are given in (5.1) and we thus introduce new variables $\vec{V}_j'=(\sigma_a^{1/2}/r_0)\vec{V}_j$ into (B2). Then $H_0'(V)=(r_0^2/\sigma_a)H_0'(V')$ and can be neglected compared with the term $r_0\sum_j M(V_j')$. With this approximation all the integrals in (B2) factorize and

$$a'=i\omega+\sigma_a z-(I_0)^{N-2}I_2\sigma_a z r_0^2\ , \quad \text{(B3)}$$

where

$$I_0=\int d\vec{V}'e^{-r_0M(V')}\ , \quad \text{(B4)}$$

$$I_2=\int d\vec{V}'\vec{V}_\alpha'^2e^{-r_0M(V')}\ . \quad \text{(B5)}$$

From (5.6) (with $b=\infty$),

$$M(V')=-1+[1+(a'/\sigma_a)\vec{V}'^2]^{1/2}$$

$$\simeq[(a'/\sigma_a)\vec{V}'^2]^{1/2}\ , \quad \text{(B6)}$$

where the second form is valid for large V'. Using this large V' approximation in (B4) and (B5) it is easily shown (for $n=0$) that

$$I_0=1,\quad I_2=\sigma_a/a'r_0^2\ . \quad \text{(B7)}$$

When this is substituted in (B3) we find the equation

$$a'^2-a'(i\omega+\sigma_a z)+\sigma_a^2 z=0\ . \quad \text{(B8)}$$

The solution is given by (5.4) where we have chosen the solution which reduces to $i\omega$ when $\sigma_a=0$.

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