PHYSICAL REVIEW B VOLUME 17, NUMBER 11 1 JUNE 1978

Crossover near fluctuation-induced first-order phase transitions in superconductors

Jing-Huei Chen and T. C. Lubensky
Department of Physics and Laboratory for Research in the Structure of Matter, University of Pennsylvania, Philadelphia, Pennsylvania 19104

David R. Nelson
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 9 January 1978)

Crossover near the fluctuation-induced first-order transition in superconductors is studied to first order in $\epsilon = 4-d$. We find that the effective exponent γ_{eff} for the susceptibility of type-II systems falls from its value for the chargeless system to the mean-field value unity well before the first-order transition occurs. Although such effects are probably unobservable in real superconductors, these calculations may have some relevance to the nematic to smectic-A transition in liquid crystals.

I. INTRODUCTION

Several years ago, a combination of a fluctuation-corrected mean-field theory and an expansion in $\epsilon = 4 - d$ were used to show[1] that phase transitions in superconductors and liquid crystals[2] may actually be the first order in character. Independent field-theoretic investigations of massless scalar electrodynamics in four dimensions lead to the equivalent conclusion that both the scalar meson and the photon in this theory acquire a mass.[3] The crucial feature which led to these predictions was the coupling of the order parameter to a gauge field such as the vector potential in a superconductor, or the director in a liquid crystal.

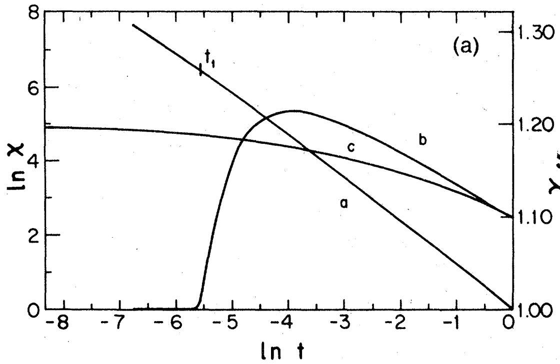
In Ref. 1, the ratio of the latent heat to the mean-field specific-heat jump was used to characterize the first-order nature of this transition. Here, we use an alternate, but equivalent characterization.

Let T_c be the fluctuation-induced first-order transition temperature of such a system, and let T^* be the temperature at the limit of metastability of the disordered phase. Then, the extent to which the transition differs from the continuous transition can be measured by the quantity $\Delta T = T_c - T^* > 0$. For type-I systems (i.e., $\kappa = \lambda/\xi \lesssim 1/\sqrt{2}$, where λ is the London penetration depth and ξ is the coherence length), ΔT was found to lie outside the critical region associated with T^*, so that mean-field critical behavior was predicted as T approached T_c from above. For type-II systems ($\kappa \gtrsim 1/\sqrt{2}$), however, ΔT was found to lie inside the critical region associated with T^*, so that nonclassical effective critical exponents were to be expected as the first-order transition was approached. For strongly-type-II systems, it was assumed that experimentally observed effective critical exponents would be approximately the same as for the gaugeless system (i.e., the same as for the λ transition in ^{4}He for both the normal-to-superconducing and nematic-smectic-A transitions).

It is possible, at least in the vicinity of four dimensions, to make quantitative computations which illuminate and check the qualitative picture described above. In this paper, we present detailed calculations of cross-over functions and ΔT for the normal to superconducting transition, correct to first order in $\epsilon = 4 - d$. The calculations were performed using techniques developed for computing the crossover scaling functions which arise in multicritical phenomena,[4] and make use of recursion relations derived from the ϵ expansion.[5] We find that the effective exponents are nonuniversal, and that the effective susceptibility exponent falls from a value near that associated with the λ transition to the mean-field result of unity as much as decade before the first-order transition occurs.

This changeover is also reflected in the renormalization-group flows, which (Ref. 1) gradually map type-II into type-I superconductors. Such behavior has recently been observed experimentally in the nematic to smectic-A transition.[6] A graphical summary of our results for effective critical exponents is presented in Fig. 1.

There exist several demonstrations that runaways at some second-order fixed points lead to first-order transitions.[7] As a result of these calculations, it is common practice to associate the absence of fixed points within the ϵ expansion with first-order transitions.[8] Such associations are, however, on shaky ground in the absence of detailed calculations of thermodynamic functions. Indeed, there are examples of runaways in random systems,[9] and in anisotropic systems in $2+\epsilon$ dimensions[10] that do not correspond to first-order transitions. The most-detailed calculations of thermodynamic functions indicating the existence of a first-order transition in systems where naive



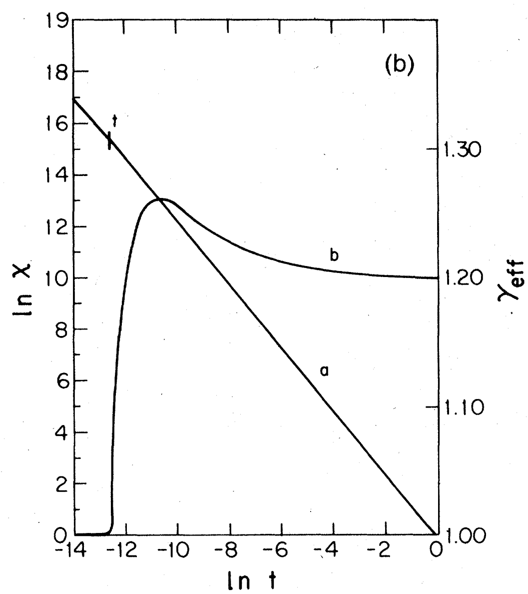


FIG. 1. (a) Curve a is the log-log plot of the order parameter susceptibility vs temperature for a two-component system with $u(0)=0.5u_H^*$, where u_H^* is the value of u at the Heisenberg fixed point, and $f(0)=10^{-2}f^*$. Curves b and c are effective exponent curves for two component systems with $u(0)=0.5_H^*$ and $f(0)=10^{-2}f^*$ and $f(0)=0$, respectively. (b) a susceptibility and b effective exponent curves for an extreme type-II system. The curves terminate at $t_c(0)$ calculated from Eq. (55).

Landau theory would predict a second-order transition are those of Rudnick[11] on xy systems with large cubic anisotropy. Preliminary versions of this work, in fact, inspired and set the stage for the present work. Bergman and Halperin[7d] also present careful calculations of thermodynamic functions for first-order transitions in compressible Ising ferromagnets.

The estimated values for ΔT in superconductors are exceedingly small: of order 10^{-6} °K in aluminum, the "best" type-I case and even smaller for type-II systems. Furthermore, the critical region is never reached in real experiments on bulk superconductors. Thus, mean-field theory provides an adequate description for experiments that can be performed even on extreme type-II systems, and effects associated with the fluctuation induced first-order transition discussed in this paper have very little direct experimental relevance. Nevertheless, we feel that the calculations presented here are of some interest. First, it is important to understand phase transitions that cannot be described even qualitatively by standard Landau theory in as much detail and in as wide a variety of systems as possible. Second, the normal-to-superconducting (N-S) transition described here is similar in many respects to the nematic-to-smectic-A transition in liquid crystals.[12,13] Unlike the N-S transition, nematic-to-smectic-A the transition seems to have an experimentally accessible critical region.[6,14,15] Although such transitions are very complicated and as yet incompletely understood, calculations based on the ϵ expansion[2,16] suggest that they should be in the same universality class as the N-S transition very close to T_c. One might hope that the unusual crossover from nonclassical to mean-field exponents as $T \to T_c$ (rather than the other way around) found in this paper for superconductors would have some relevance to liquid crystals. At the very least, the calculations presented here are a necessary prelude to the computation of the more complicated and experimentally more interesting crossover functions for the nematic-to-smectic-A transition. Finally, some of the technical details in this paper are of interest in their own right.

The techniques employed in this paper have been applied with some success to tricritical[3] and bicritical points,[17] and are, in principle, quite straightforward. Near a critical point, standard perturbation theory cannot be applied, because various terms in the usual diagrammatic expansion are nonanalytic and divergent. Far from a critical point, however, calculations can be carried out using standard techniques (e.g., high-temperature series expansions). The renormalization group maps Hamiltonians in the critical region into a more tractable region far from T_c. Thus, one can calculate quantities near a critical point by mapping the Hamiltonians along a trajectory determined by the recursion relations into a noncritical region where calculations can be performed using standard techniques.

For the superconducting problem considered here, the Hamiltonian space can be parametrized to order ϵ by the temperature $t \equiv (T-T^*)/T^*$, the quartic coupling u, and a parameter f, which measures the coupling of the order parameter to the gauge field. The quantity f is proportional to $4\pi\mu e^2$, where e is the charge and μ is the magnetic permeability. Our principal interest will be in calculating the critical value $t_c = \Delta T/T^* > 0$ of t at which the first-order transition occurs and the order-parameter susceptibility

$$\chi(t, u, f) = \lim_{q\to 0} \langle |\psi(q)|^2\rangle ,$$

where $\psi(q)$ is Fourier component of the order parameter. The vector-potential correlation function

$$D_{ij}(q, t, u, f) = \langle A_i(q) A_j(-q)\rangle ,$$

and the specific heat C_v are also of some interest but are more difficult to calculate, and will be given little attention in this paper. Indeed, it appears that a calculation to at least second order in ϵ is necessary before meaningful statements about the crossover behavior of D_{ij} can be made. We will also calculate $|\psi_c|$, the value of the order parameter at $t = t_c^-$, and the ratio $\chi(t_c^-)/\chi(t_c^+)$.

This paper is divided into five sections. In Sec. II, Hamiltonian flows are calculated from differential recursion relations valid to first order in ϵ. In Sec. III, the crossover behavior of the susceptibility χ is calculated. In Sec. IV, the first-order transition temperature t_c, the value of the order parameter just below t_c, and the ratio of the susceptibility just above and just below t_c are determined. Finally, in Sec. V, there is a brief summary of results. In the appendices, recursion relations are explicitly integrated and the crossover function for the specific heat is evaluated.

II. RECURSION RELATIONS

Our starting point is the Landau-Ginzburg free-energy functional $F(\psi, \vec{A})$ for a generalized superconductor with order parameter ψ consisting of $\frac{1}{2}n$ complex components and vector potential $\vec{A}$.[1] Let T_0^* be the mean-field transition temperature and $t_0 = (T - T_0^*)/T_0^*$ be the corresponding reduced temperature. Then the reduced free-energy functional $\mathcal{H} \equiv H/k_B T$, where T is the temperature is given by

$$\mathcal{H} = \int d^d x \Big(r_0 |\psi|^2 + |(\vec{\nabla} - i q_0 \vec{A})\psi|^2 + \tfrac{1}{2} u_0 |\psi|^4 + \frac{1}{8\pi\mu_0} (\vec{\nabla}\times\vec{A})^2 \Big), \qquad (1)$$

where $r_0 = a t_0$ (a is a constant), $q_0 = 2e/\hbar c$, and μ_0 is the magnetic permeability of a normal metal which is close to unity. The Coulomb gauge with $\vec{\nabla}\cdot\vec{A} = 0$ will be used throughout this paper.

In this paper, we employ the differential[18] form of the original finite cutoff formulation of the renormalization group.[5] This formulation presents problems in that a finite cutoff Λ in wave numbers cannot be introduced arbitrarily in Eq. (1) without destroying gauge invariance. To first order in ϵ, however, gauge invariance may be successfully restored[1] merely by ignoring any finite mass for the $\vec{A}$ field that is generated by the removal of degrees of freedom with the magnitude of momentum between Λ and Λ/b. To obtain consistent equations to second order in ϵ, it is probably necessary to employ the Callan-Symanzik formulation[19] of the renormalization group which is manifestly gauge invariant. (We have verified that the Callan-Symanzik approach and that presented here give identical results to first order in ϵ for the quantities we calculate here.) Thus to obtain recursion relations, we simply remove degrees of freedom with wave number $\vec{q}$ satisfying $b^{-1} < |\vec{q}| < 1$ (setting $\Lambda = 1$) and rescale wave number and fields via $\vec{q} \to b\vec{q}$, $\psi \to b^{(d+2-\eta)/2}\psi$, and $\vec{A} \to b^{(d+2-\eta_A)/2} \vec{A}$. To obtain differential recursion relations, let $b = e^l$ and eliminate an infinitesimal shell at each iteration. The resulting recursion relations are

$$\frac{dr(l)}{dl} = [2 - \eta(l)] r(l) + \tfrac{1}{2}(n+2) u(l) C_d \frac{1}{1 + r(l)} + 3q^2(l) 4\pi\mu(l) C_d, \qquad (2a)$$

$$\frac{d\mu^{-1}(l)}{dl} = -\eta_A(l)\,\mu^{-1}(l) + \tfrac{1}{6} n 4\pi q^2(l) C_d, \qquad (2b)$$

$$\frac{d\mu(l)}{dl} = [\epsilon - 2\eta(l)] u(l) - \tfrac{1}{2}(n+8) \frac{u^2(l) C_d}{[1 + r(l)]^2} - 6[4\pi\mu(l) q^2(l)]^2 C_d, \qquad (2c)$$

$$\frac{dq(l)}{dl} = \tfrac{1}{2}[\epsilon - \eta_A(l)] q(l) , \qquad (2d)$$

$$\eta(l) = -3[4\pi\mu(l) q^2(l)] C_d \{1/[1 + r(l)]\} , \qquad (2e)$$

where $C_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(\frac{1}{2}d)$. Since $q(l)$ represents an odd vertex (it connects two χ's and one D_{ij} propagator) it always appears to an even power. We, therefore, introduce the quantity $f(l) = 4\pi\mu(l) q^2(l)$ which satisfies the recursion relation

$$\frac{df(l)}{dl} = \epsilon f(l) - \tfrac{1}{6} n f^2(l) C_d . \qquad (3)$$

The 0 subscript has been removed in all quantities to indicate renormalization. Note that η_A does not appear in this equation or in the equation for u. Thus the fixed-point structure is completely independent of η_A, and η_A can be chosen arbitrarily. We will choose $\eta_A = \epsilon$ so that the charge remains unchanged under renormalization. Another choice that is often convenient is $\eta_A = 0$.

Equations (2) and (3) and the initial conditions $r(0) = r_0$, $u(0) = u_0$, $f(0) = 4\pi\mu_0 q_0^2$ completely determine the fixed-point structure and renormalization trajectories to first order in ϵ. To find the fixed points, we set the left-hand sides of Eqs.

(2) and (3) equal to zero and solve for u^*, f^*, and r^*. We then linearize the equations about the fixed points to obtain the eigenfunctions g_i and associated eigenvalues λ_{g_i}. We will denote the eigenvalue associated with r by $\lambda_t \equiv 1/\nu$, where ν is the correlation-length exponent. Crossover exponents[20] for the fields g_i are by the definition $\varphi_{g_i} = \nu\lambda_{g_i}$ and Eqs. (2) and (3) have the following fixed points:

(i) Gaussian: $u^* = f^* = 0$; $\lambda_u = \lambda_f = \epsilon$, $\lambda_t = 2$; this fixed point describes mean-field behavior with $\nu = \frac{1}{2}$.

(ii) Heisenberg: $f^* = 0$, $u^* = [2/(n+8)C_d]\epsilon$; $\lambda_g = -\epsilon$; $\lambda_f = \epsilon$; $\lambda_t = 2 - [(n+2)/(n+8)]\epsilon$, where $g = u - 2\epsilon/(n+8)C_d - [6/(n+8)]f$. When $n = 2$, this fixed point describes the λ transition in helium.

(iii) Superconducting: In general there are two new fixed points (denoted by subscripts "+" and "−") associated with a finite value of the charge. They are characterized by

$$f^* = 6\epsilon/nC_d\,, \tag{4a}$$

$$u^*_\pm = [\epsilon/(n+8)C_d][(1+36/n) \pm (1/n)\Delta] \tag{4b}$$

and

$$\lambda_{t_\pm} = 2 + (18/n)\epsilon - \tfrac{1}{2}(n+2)u^*_\pm C_d\,, \tag{5a}$$

$$\lambda_{g_\pm} = \mp\epsilon\Delta/n\,, \tag{5b}$$

$$\lambda_{f'} = -\epsilon\,, \tag{5c}$$

where $\Delta = (n^2 - 360n - 2160)^{1/2}$, $f' = f - f^*$, and $g_\pm = (1 \mp \Delta/n)u/8 - [(33n + 180 \mp 3\Delta)/4n(n+8)]f$. Δ is imaginary for $n < n_c = 365.9$. Thus for $n < n_c$, u is complex for both fixed points and physically inaccessible. The only accessible fixed points for $n < n_c$ are the Gaussian and Heisenberg fixed points, both of which are unstable with respect to f (i.e., with respect to turning on charge). There is therefore, a "runaway"[21] which was interpreted in Ref. 2 to correspond to a first-order transition. We will pursue this question in greater detail in Sec. IV. For $n > n_c$, these fixed points have real values for u^* and are physically accessible. Both are stable with respect to f' ($\lambda_{f'} < 0$). The "plus" fixed point (i.e., the one with the larger value for u^*, $u^* = u^*_+$) is stable with respect to g_+ whereas the "minus" fixed point is unstable with respect to g_-. At $n = n_c$, $\Delta = 0$, and the "plus" and "minus" fixed points merge to a single marginally stable fixed point with

$$u^* = \frac{\epsilon}{(n_c+8)C_d}\left(1 + \frac{36}{n_c}\right), \quad f^* = \frac{6\epsilon}{n_c C_d}\,, \quad \lambda_g = 0, \quad \lambda_{f'} = -\epsilon \tag{6a}$$

and

$$\lambda_t = 2 + \frac{18}{n_c}\epsilon - \frac{1}{2}\frac{n_c+2}{n_c+8}\left(1 + \frac{36}{n_c}\right)\epsilon\,. \tag{6b}$$

Fixed points and renormalization trajectories are depicted in Figs. 2(a), 2(b), 2(c), and 2(d). Note that for $n > n_c$, the line joining the Gaussian and Superconducting "minus" fixed point divides the f-u plane into two parts: points to the right of this line flow to the stable "plus" fixed point and points to the left flow towards negative u and a first-order transition.

The Hamiltonian flows, shown in Fig. 2 were obtained by solving Eqs. (2) and (3) analytically to first order in ϵ. The results (see Appendix A) can be expressed in terms of $f(l)$, $u(l)$ and a temperature variable

$$t(l) = r(l) + \tfrac{3}{2}f(l)C_d + \tfrac{1}{4}(n+2)u(l)C_d + r(l)\ln[1 + r(l)][\tfrac{3}{2}f(l)C_d - \tfrac{1}{4}(n+2)u(l)C_d]\,. \tag{7}$$

The solution for $f(l)$ is independent of the other variables

$$f(l) = e^{\epsilon l}\tilde{f}(l) = \frac{e^{\epsilon l}f(0)}{1 + (n/6\epsilon)f(0)C_d(e^{\epsilon l} - 1)}\,, \tag{8}$$

while the solution for $u(l)$ is best expressed as

$$u(l) = R(l)f(l)\,, \tag{9}$$

where $R(l)$ is given below. From Eq. (1) it is easy to see that the renormalized correlation lengths and London penetration depths are, respectively, $\xi_0^2 = r_0^{-1}$ and $\lambda^{-2} = 2(f_0/u_0)r_0$. We therefore have

$$R(0) = 2\kappa^2\,, \tag{10}$$

where κ is the Ginzburg parameter λ/ξ. Finally, we write

$$t(l) = P(l)t(0)\,. \tag{11}$$

The precise forms of $R(l)$ and $P(l)$ depend on whether $n < n_c$, $n > n_c$, or $n = n_c$, and we consider the three cases separately:

(i) $n < n_c$:

$$R(l) = (1/A)\{B + |\Delta|\tan[\theta_0 - \theta(l)]\}\,, \tag{12}$$

$$P(l) = e^{2l}\left(\frac{f(0)}{\tilde{f}(l)}\right)^{E_1}\left|\frac{\cos[\theta_0 - \theta(l)]}{\cos\theta_0}\right|^{-E_2} \tag{13}$$

where

$$\theta(l) = (|\Delta|/2n)\ln[f(0)/\tilde{f}(l)]\,, \tag{14}$$

$$\theta_0 = \tan^{-1}\{[AR(0) - B]/|\Delta|\}\,, \tag{15}$$

and

$$A = 6(n+8), \quad B = n + 36\,, \quad E_1 = \frac{-n^2 - 2n + 216}{2n(n+8)}\,, \quad E_2 = \frac{n+2}{n+8}\,; \tag{16}$$

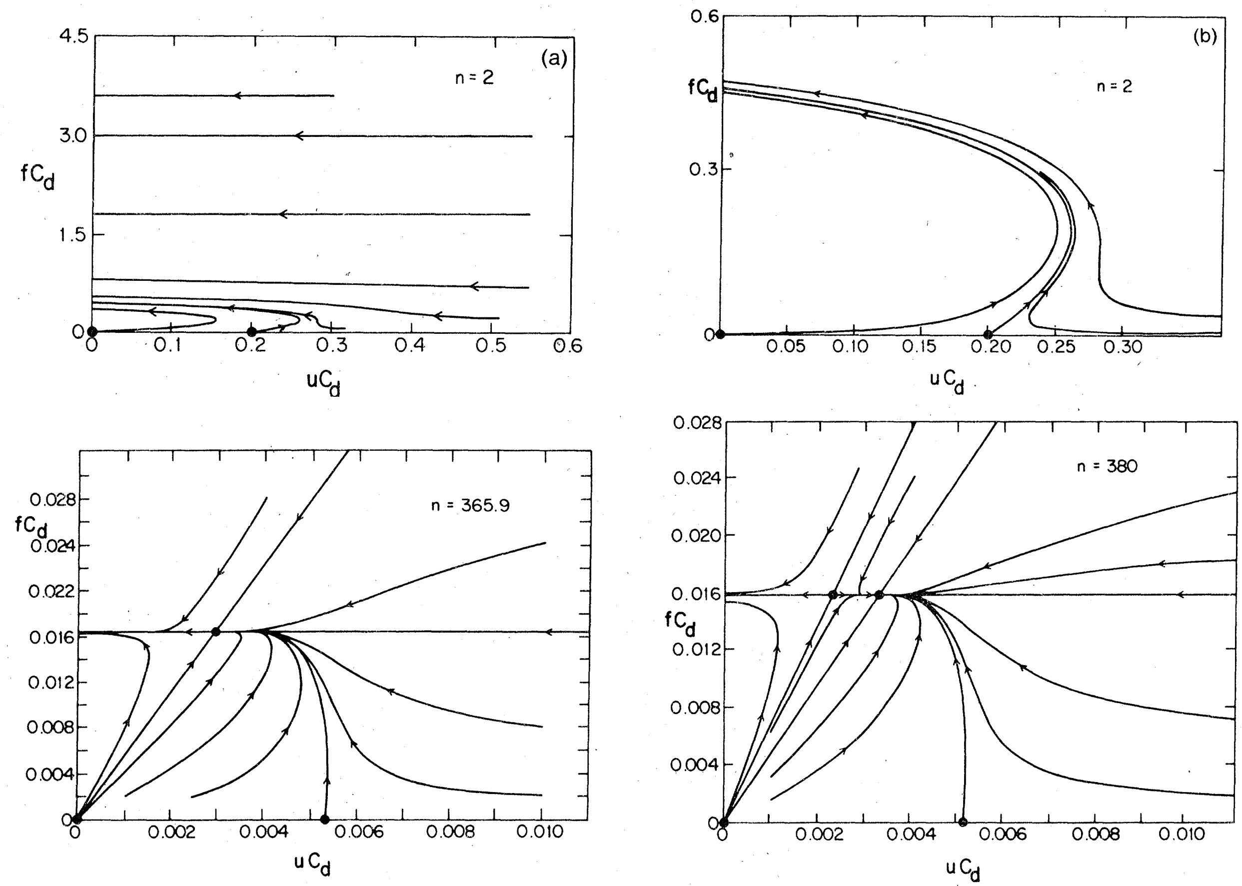


FIG. 2. Hamiltonian flows induced by the renormalization-group transformation. The arrows indicate the direction of flow under iteration. The trajectory emerging from the Heisenberg fixed point is the critical trajectory along which the irrelevant eigenvector g_2 is zero. (a) Shows flows for $n=2$. (b) Expanded version of (a) for small f. (c) Shows flows at $\kappa=\kappa_c=365.9$ where the superconducting fixed points first become real. (b) Shows flows for $\kappa=380>\kappa_c$ where there are two real superconducting fixed points.

(ii) $n>n_c$:

$$R(l)=(1/A)\{B+|\Delta|\tanh[\theta_0'+\theta(l)]\}\,, \tag{17}$$

$$P(l)=e^{2l}\left(\frac{f(0)}{\tilde f(l)}\right)^{E_1}\left|\frac{\cos[\theta_0'+\theta(l)]}{\cosh\theta_0'}\right|^{-E_2}, \tag{18}$$

where

$$\theta_0'=\tanh\{[AR(0)-B]/|\Delta|\}\,; \tag{19}$$

(iii) $n=n_c$:

$$R(l)=\frac{1}{A}\left(B+\frac{AR(0)-B}{1+[AR(0)-B](1/2n)\ln[f(0)/\tilde f(l)]}\right) \tag{20}$$

and

$$P(l)=e^{2l}\frac{f(0)}{\tilde f(l)}^{E_1}\left|1+[AR(0)-B]\frac{1}{2n}\ln\frac{f(0)}{\tilde f(l)}\right|^{-E_2}. \tag{21}$$

Equation (7)–(16) can be used to obtain nonlinear scaling fields[22] for $n<n_c$. These are

$$g_t=t\left(\frac{n}{6\epsilon}C_d f\right)^{-E_2}\left(1-\frac{n}{6\epsilon}C_d f\right)^{E_1+E_2}\times\left|\frac{2n}{|\Delta|}\cos\theta_0\right|^{E_2}, \tag{22a}$$

$$g_f=(n/6\epsilon)C_d f\,[1-(n/6\epsilon)C_d f]^{-1}, \tag{22b}$$

$$g_2=\left[1-e^{(2n/|\Delta|)(\pi/2)-\theta_0)}\left(1-\frac{n}{6\epsilon}C_d f\right)\right]\times\frac{1-(n/6\epsilon)C_d f}{(n/6\epsilon)C_d f}, \tag{22c}$$

which satisfy

$$g_t(l)=g_t(0)e^{\lambda_t l},\quad \lambda_t=2-[(n+2)/(n+8)]\epsilon, \tag{23a}$$

$$g_f(l)=g_f(0)e^{\epsilon l}, \tag{23b}$$

$$g_2(l)=g_2(0)e^{-\epsilon l}. \tag{23c}$$

Note that as $f \to 0$, g_t and g_2 reduce to the Heisenberg scaling fields [Eqs. (2)–(36) of Ref. 4]

$$g_t \to t(u/u_H^*)^{-E_2}, \tag{24a}$$

$$g_2 \to 1 - u_H^*/u, \tag{24b}$$

where u_H^* is the value of u at the Heisenberg fixed point.

III. CROSSOVER FUNCTIONS

The ψ and $\vec{A}$ susceptibilities satisfy the following homogeneity relations:

$$\chi(r_0, u_0, f_0) = \exp\left(2l - \int_0^l \eta(l')\, dl'\right) \times \chi(r(l), u(l), f(l)), \tag{25a}$$

$$D_{ij}(q, r_0, u_0, f_0) = e^{(2-\epsilon)l} D_{ij}(e^l q, r(l), u(l), f(l)), \tag{25b}$$

where $D_{ij}(q) = (\delta_{ij} - q_i q_j/q^2)D(q)$, and where we have set $\eta_A = \epsilon$. By gauge invariance $D(q) = 1/Kq^2$ for small q, where $K = 1/4\pi\mu_0$ far from $t=0$. Eq. (25b) can then be expressed in terms of K,

$$K(r_0, u_0, f_0) = e^{\epsilon l} K(r(l), u(l), f(l)). \tag{26}$$

This equation implies that K satisfies the Josephson[23] relation $K \sim t^{-\epsilon\nu}$ near any fixed point. Since the exponent is explicitly of order ϵ, no meaningful crossover behavior can be calculated for K to first order in ϵ. We will, therefore, not give further consideration to this function. To determine crossover functions and effective critical exponents, we use the solutions to the recursion relations of Sec. II to map the Hamiltonian out of the critical regime. Once out of the critical regime, χ can be calculated using standard perturbation theory to first order in u and f. The result of this calculation is

$$\chi^{-1}(l) = t(l) - \tfrac{3}{2} f(l) C_d t(l) \ln[1 + t(l)] + \tfrac{1}{4}(n+2) u(l) C_d t(l) \ln t(l). \tag{27}$$

Combining Eqs. (25), (26), and (27), we obtain

$$\chi(r_0, u_0, f_0) = \frac{e^{2l^*}}{t(l^*)}\left(1 + \frac{n}{6\epsilon} f(0) C_d (e^{\epsilon l^*} - 1)\right)^{18/n} \times [1 - \tfrac{1}{4}(n+2) u(l^*) C_d \ln t(l^*)], \tag{28}$$

where l^* is some suitably chosen value of l to be discussed below. Explicit differentiation of Eq. (28) with respect to l shows that χ is independent of the precise choice of l^* up to order ϵ^2 as required.[3] Equation (28) reduces to Eq. (2.26) of Ref. 4 when $f=0$.

We now consider the choice of l^*. Calculations of χ are particularly simple when $u(l^*) \ln t(l^*) = 0$. We therefore introduce l_1 and l_2 via the conditions

$$u(l_1) = 0, \quad t(l_2) = 1, \tag{29}$$

and define $l^* = \min(l_1, l_2)$. By stopping the integration of the renormalization group equations at this value of l, we ensure that $u(l^*) \ln t(l^*) = 0$, and that $u(l^*)$ never becomes negative. This definition of l^* leads to a "matching" temperature t_1: if $t > t_1$, $l^* = l_2$, if $t < t_1$, $l^* = l_1$. Since $t(l) = P(l) t(0)$ and $P(l)$ behaves roughly like e^{2l}, it is easy to see that t_1 must exist. What is not so obvious is that t_1 is greater than t_c, the value of the reduced temperature at the first-order transition. In the next section, we will verify explicitly that this is so for all trajectories when ϵ is small and for most trajectories of interest when ϵ is set equal to unity in the first-order solutions of this paper. Using Eqs. (9) and (12), we can solve for l_1 in closed form

$$e^{\epsilon l_1} = 1 + \frac{6\epsilon}{n C_d f(0)} \left\{ \exp\left[\frac{2n}{|\Delta|}\left(\theta_0 + \tan^{-1} \frac{B}{|\Delta|}\right) - 1\right]\right\}. \tag{30}$$

This formula will be of some use in Sec. IV. A closed form analytic solution for l_2 is not possible; in practice, it is evaluated numerically for given initial conditions.

Equations (28), (29), and (30) fully determine $\chi(t)$ to first order in ϵ. The critical exponent γ can be defined in the vicinity of any fixed point

$$\chi \sim t^{-\gamma}. \tag{31}$$

More generally an effective exponent[24] can be defined via

$$\gamma_{\text{eff}} = -\frac{\partial \ln\chi}{\partial \ln t}. \tag{32}$$

Explicit differentiation of Eq. (28) with respect to $\ln t$ yields

$$\gamma_{\text{eff}} = \begin{cases} 1 + \frac{1}{4}(n+2) C_d u(l^*), & t > t_1, \\ 1, & t_c < t < t_1. \end{cases} \tag{33}$$

The discontinuity in slope of γ_{eff} at t_1 presumably would disappear if a calculation to all orders in ϵ could be one. This artifact of the two distinct matching conditions [Eq. (29)] is easily removed by numerical differentiation of the function $\chi(t)$. χ and γ_{eff} are plotted in Fig. 1 for two initial values of u and f. These curves are terminated at the first-order transition temperatures appropriate to the initial values of u and f. This temperature is calculated in Sec. IV. Note that γ_{eff} assumes the mean-field value of unity for $t_c < t < t_1$. Furthermore, t_1 is a factor of 2 or more greater than t_c for starting points in the vicinity of the Heisenberg fixed point, so that this mean-field behavior should in principle be observable prior to the actual transition.

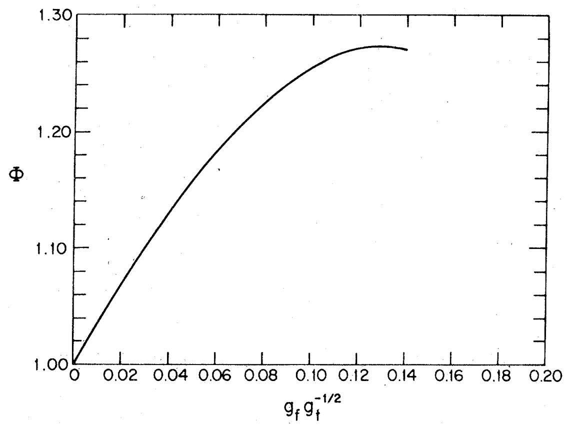


FIG. 3. Critical susceptibility scaling functions $\Phi(x, y=0)$ (cf. end of Sec. III) in the disordered phase. This function crosses over from constant behavior at large t to critical behavior governed by the first-order transition.

A scaling prediction for χ near the Heisenberg fixed point expressed as a function of the nonlinear scaling fields g_t, g_f, and g_2 follows from (25a), namely

$$\chi(g_t, g_f, g_2) \approx g_t^{-\gamma}\Phi(g_f/g_t^{\epsilon\nu}, g_2 g_t^{\epsilon\nu})$$

for g_f small. A plot of $\Phi(x, y=0)$ is shown in Fig. 3.

IV. CALCULATION OF t_c

Halperin, Lubensky, and Ma[1] (HLM) showed that t_c could be calculated in type-I systems using a modified mean-field theory. We begin this section with a review of this calculation valid for all dimensions between three and four, indicating explicitly the region in the f-u plane for which the cutoff independent solution is applicable. We will then use the recursion relations of the previous sections to derive t_c for an arbitrary point in the f-u plain (subject to limitations to be discussed).

A. Mean-field theory in the type-I region ($\eta \ll 1$)

An effective Hamiltonian involving ψ only can be obtained by removing the vector potential

$$e^{-\mathcal{H}_{\text{eff}}(\psi)} = \int \mathfrak{D}\vec{A}\, e^{-\mathcal{H}(\psi, \vec{A})}. \tag{34}$$

In general the evaluation of this expression is very difficult. When spatial variations in ψ can be ignored, as is the case in type-I super-conductors it can, however, be evaluated exactly. Spatial variations in ψ are unimportant for reduced temperatures t greater than the Ginzburg reduced temperature t_G. Ginzburg's three-dimensional calculation[25] of t_G can easily be generalized to arbitrary dimension less than four to obtain

$$t_G = \tfrac{1}{2}\{[\tfrac{1}{2}\pi/\sin(\tfrac{1}{2}\kappa\epsilon)]C_d u\}^{2/\epsilon}. \tag{35}$$

Thus, if t_c calculated by ignoring spatial variation in ψ is greater than t_G, it is correct and self-consistent. If, on the other hand, $t_c < t_G$, other techniques must be employed.

Ignoring the spatial variations in ψ, we obtain

$$\frac{1}{2\Omega}\frac{dH_{\text{eff}}}{d|\psi|} = r_0|\psi| + u_0|\psi|^3 + q_0^2|\psi|\langle A^2\rangle_\psi, \tag{36}$$

where Ω is the volume of the system and

$$\langle A^2\rangle_\psi = \frac{d-1}{K}\int_0^\Lambda \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+k_s^2}. \tag{37}$$

The quantity k_s is the inverse of the London penetration depth λ :

$$k_s^2 = \lambda^{-2} = (2q_0^2/K)|\psi|^2 = 2f|\psi|^2. \tag{38}$$

$\langle A^2\rangle_\psi$ can be evaluated by first calculating $d\langle A^2\rangle_\psi/dk_s^2$:

$$\begin{aligned} d\langle A^2\rangle_\psi &= \frac{d-1}{K}\left(-\int_0^\Lambda \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+k_s^2)^2}\right) \\ &= \frac{d-1}{K}\left(-\int_0^\infty \frac{d^dk}{(2\pi)}\frac{1}{(k^2+k_s^2)^2} + \int_\Lambda^\infty \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+k_s^2)^2}\right). \end{aligned} \tag{39}$$

Note that the first term in Eq. (39) is completely independent of the cutoff Λ, whereas the second term depends on Λ and can be expanded in analytic power series in k_s^2 if $k_s < \Lambda$. In four dimensions, we find

$$\begin{aligned} \frac{1}{\Omega}\mathcal{H}_{\text{eff}} &= \tilde{r}|\psi|^2 + \tfrac{1}{2}(u_0 - \tfrac{3}{2}C_d f^2)|\psi|^4 + \tfrac{3}{2}C_d f^2|\psi|^4 \ln\left(\frac{2f|\psi|^2}{\Lambda^2}\right) \\ &\quad - \tfrac{3}{8}\Lambda^4 C_d\left[\left(\frac{(2f|\psi|^2)^2}{\Lambda^2} - 1\right)\ln\left(1 + \frac{2f|\psi|^2}{\Lambda^2}\right) + \frac{2f|\psi|^2}{\Lambda^2} - \frac{1}{2}\left(\frac{2f|\psi|^2}{\Lambda^2}\right)^2\right] \end{aligned} \tag{40}$$

for $0 < \epsilon < 1$, Eq. (36) can be used to give

$$(1/\Omega)H_{\text{eff}} = \tilde{r}|\psi|^2 - v|\psi|^{4-\epsilon} + \tfrac{1}{2}\tilde{u}|\psi|^4 + \cdots, \tag{41}$$

where

$$\tilde{r} = r + [(3-\epsilon)/(2-\epsilon)]fC_d\Lambda^{2-\epsilon}, \tag{42a}$$

$$v = [(3-\epsilon)/(4-\epsilon)][\pi/2\sin(\tfrac{1}{2}\pi\epsilon)]C_d(2f)^{2-\epsilon/2}, \tag{42b}$$

$$\tilde{u} = u + [2(3-\epsilon)/\epsilon]C_d f^2\Lambda^\epsilon. \tag{42c}$$

The neglected terms in Eq. (41) are small as $k_s^2 \ll \Lambda^2$ just below the transition. The $\tilde{r}$ appearing in Eq. (40) is the $\epsilon = 0$ version of Eq. (42a). It differs from t [Eq. (7)] by a term of order u which is small in type-I systems and terms of order f^2 which are small in an ϵ expansion near four dimensions.

Neglecting higher-order terms in Eq. (41), we find the first-order transition temperature

$$\tilde{r}_c = [\epsilon/(2-\epsilon)](\tilde{u}/2)[(2-\epsilon)v/\tilde{u}]^{2/\epsilon}, \tag{43}$$

and the value of ψ just below the transition

$$|\psi_c| = [(2-\epsilon)v/\tilde{u}]^{1/\epsilon}. \tag{44}$$

When $\epsilon u/f^2 \ll \Lambda^\epsilon$, $\tilde{u}$ can be replaced by u, and $\tilde{r}_c$ is asymptotically cutoff independent

$$\tilde{r}_c = \frac{\epsilon}{2-\epsilon}\left(\frac{(2-\epsilon)(3-\epsilon)}{(4-\epsilon)}\right)^{2/\epsilon} t_G \kappa^{-2(4-\epsilon)/\epsilon}. \tag{45}$$

This result, when evaluated at $\epsilon = 1$, is identical to that obtained by HLM. It is easy to verify that $k_s^2 = 2f|\psi_c|^2 \ll \Lambda^2$ as long as $\epsilon u/f^2 \ll \Lambda^\epsilon$. In order for $\tilde{r}_c$ to be greater than t_G and have mean-field theory be valid, $\kappa^{-2} = f/2u$ must be greater than unity. Thus, the simple cutoff-independent mean field applies only in the small shaded region of Fig. 4 where $f^2\Lambda^\epsilon/\epsilon \ll u \ll f$. We stress that this theory is valid in all dimensions between two and four. Such a cutoff-independent mean-field theory exists to our knowledge only in systems with gaugelike couplings to massless fields. It does not exist, for example, for the xy model with strong cubic anisotropy discussed by Rudnick.[11]

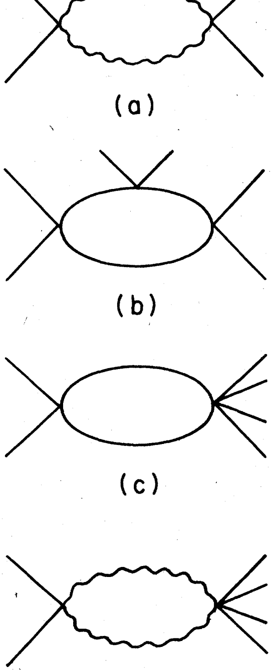


FIG. 4. Diagrams contributing to recursion relation Eq. (49) for the six point function $|\psi|^6$. Solid lines indicate order parameter propagators; wavy lines indicate gauge propagators.

B. Calculations for type-II superconductors ($\kappa > 1$)

When $\kappa > 1$, $\tilde{r}_c$ is less than t_G, and we cannot use the treatment just described. In Ref. 1, it was predicted that the first-order transition temperature satisfied

$$t_c = \tfrac{4}{9} Z t_G \kappa^{-2/\nu_H}, \tag{46}$$

in the extreme type-II case, where ν_H is the correlation-length exponent of the chargeless n-component transition. The constant of proportionality Z in the above transition (that was estimated to be of order unity in Ref. 1) can be defined in a manner that permits convenient extrapolation from four to three dimensions by exploiting the existence of a cutoff-independent mean-field theory

$$Z = \frac{t_C^{\mathrm{II}*}(\kappa^{-2})^{1/\epsilon\nu_H}}{t_c^{\mathrm{I}}/(\kappa^{-2})^{(4-\epsilon)/\epsilon}}, \tag{47}$$

where the superscripts I and II refer to type-I and type-II systems, respectively. In this section, we will give a concrete estimate of Z to lowest order in ϵ.

We begin with the recursion relations of Sec. III which map the Hamiltonian out of the critical regime. If we follow a trajectory until $l = l^*$ such that the Hamiltonian lies in a region where we can calculate $t_c(l^*)$ by the HLM technique, we find the original transition temperature by mapping backwards:

$$t_c(0) = P^{-1}(l^*)t_c(l^*), \tag{48}$$

where $P(l^*)$ is given by Eq. (13). The most convenient choice of l^* is $l^* = l_1$ such that $u(l_1) = 0$, $t_c(l_1)$; however, it cannot be calculated directly in the manner just described because the effective free energy in Eq. (39) tends to negative infinity for large $|\psi|$ below some critical temperature when $u = 0$. This unphysical feature is due to the neglect of higher-order potentials that are generated by recursion relations even if they are initially zero. Consider for example the recursion

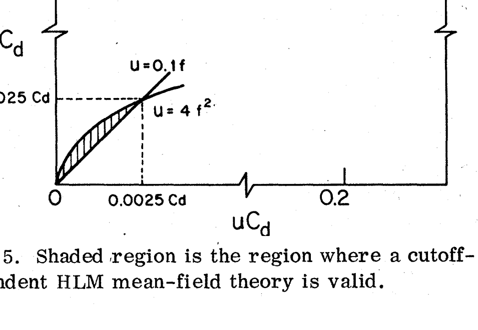


FIG. 5. Shaded region is the region where a cutoff-independent HLM mean-field theory is valid.

relations for the coefficient of u_6 of $|\psi|^6$ to lowest order (one loop order) in u and f resulting from the diagrams shown in Fig. 5:

$$\frac{du_6(l)}{dl} = -[2-2\epsilon-3\eta(l)]u_6(l) + [4f^3(l)C_d + C_1u^3(l) + C_2u(l)u_6(l) + C_3f(l)v_6(l)]\Lambda^{d-6}, \quad (49)$$

where C_1, C_2, and C_3 are constants which we do not need in the following calculations. v_6 is the coefficient of $\vec{A}\cdot\vec{A}|\psi|^4$. Equation (49) can be integrated to yield

$$u_6(l) = \tfrac{1}{2}[4f^3(l)C_d + C_1u^3(l)]\Lambda^d + e^{-2l}[u_6(0) - 2f^3(0)C_d - \tfrac{1}{2}C_1u^3(0)]\Lambda^{d-6}. \quad (50)$$

Thus if we choose l^* such that $u(l^*)=0$, and if e^{-2l^*} or its coefficient in the above is much less than one, $u_6(l^*)$ depends only on $f(l^*)$. Similar considerations apply to $u_{2m}(l^*)$, we find

$$u_{2m}(l^*) = \frac{3\times 2^m(-1)^{m+1}}{4m(m-2)} f^m(l^*)C_d\Lambda^{d-2m}. \quad (51)$$

The series $\sum_{m=3}^{\infty} u_{2m}(l^*)|\psi|^{2m}$ can be summed explicitly, giving

$$\sum_{m=3}^{\infty} u_{2m}|\psi|^{2m} = \tfrac{3}{8}\Lambda^d C_d\left[\left(\frac{2f|\psi|^2}{\Lambda^2}-1\right)\ln\left(1+\frac{2f|\psi|^2}{\Lambda^2}\right) + \frac{2f|\psi|^2}{\Lambda^2} - \frac{1}{2}\left(\frac{2f|\psi|^2}{\Lambda^2}\right)^2\right]. \quad (52)$$

This exactly cancels the last term in Eq. (40) leaving

$$1/\Omega H_{\rm eff} = \tilde{r}|\psi|^2 - \tfrac{3}{4}C_d f^2|\psi|^4 + \tfrac{3}{2}C_d f^2|\psi|^4\ln(2f|\psi|^2/\Lambda^2), \quad (53)$$

which leads immediately to predictions for $t_c(l_1)$ and $\psi_c(l_1)$,

$$t_c(l_1) = \tfrac{3}{4}e^{-0.5}f(l_1)C_d, \quad (54a)$$

$$|\psi_c(l_1)|^2 = [\Lambda^2/2f(l_1)]e^{-0.5}. \quad (54b)$$

These are the $\epsilon=0$ limits of Eqs. (43) and (44) with $u=0$. We note that these results are similar to those obtained by Rudnick[11] for the XY model with a strong cubic anisotropy. He calculated the free energy directly by using trajectory integrals rather than summing an infinite sequence of one loop diagrams as we have done. We therefore have

$$t_c(0) = P^{-1}(l_1)\tfrac{3}{4}e^{-0.5}C_d f(l_1). \quad (55)$$

We first show that this formula reproduces the correct form for $t_c(0)$ in the shaded region of Fig. 4. In this region, $R \ll 1$ and $u \gg f^2/f^*$, where $f^* = 6\epsilon/nC_d$. From Eq. (30), we have

$$e^{\epsilon l_1} \simeq \tfrac{1}{36}nf^*R(0)/f(0), \quad (56a)$$

$$P(l_1) \simeq [\tfrac{1}{36}nf^*R(0)/f(0)]^{2/\epsilon}. \quad (56b)$$

Inserting this into Eq. (55), we obtain

$$t_c(0) = \epsilon t_G(\tfrac{1}{9}nC_d f^*/\epsilon)^{1-2/\epsilon}(\tfrac{3}{4}e^{-0.5})\kappa^{-2(4-\epsilon)/\epsilon}. \quad (57)$$

It is easy to see from Eq. (43) that order ϵ^2 contributions from $f(l_1)$ and $u(l_1)$ will change the overall factor $\frac{3}{4}e^{-0.5}$ in this result. Because of this fact, $t_c(0)$ in Eq. (57) differs from the correct value given by the small ϵ limit of Eq. (45) by a constant factor $e^{2/3}$. Nevertheless, it is encouraging that the dependence on κ is correctly given by Eq. (57). In the extreme type-II case, $R=2\kappa^2 \gg 1$, and we have

$$t_c(0) = \epsilon t_G[(Q-1)C_d f^*/\epsilon]^{1-2/\epsilon} \times [\tfrac{1}{12}(n+8)]^{E_2/2}\frac{2^{1-1/\epsilon\nu_H}}{Q^1+E_1}(\tfrac{3}{4}e^{-0.5})\kappa^{-2/\epsilon\nu_H}, \quad (58)$$

where E_1 and E_2 are given by Eq. (16), $1/\nu_H = 2 - \epsilon E_2$, and

$$Q = \exp\{(2n/|\Delta|)[\tfrac{1}{2}\pi + \tan^{-1}(B/|\Delta|)]\}. \quad (59)$$

Again, an order ϵ^2 calculation would be necessary to obtain the correct prefactor in Eq. (58). Because of its potential importance to experimentalists, we will nevertheless attempt to estimate Z by using Eqs. (57) and (58) in Eq. (46). We obtain

$$Z = 2^{1-1/\epsilon\nu_H}\left(\frac{9(Q-1)}{n}\right)^{1-2/\epsilon} \times\left(\frac{n+8}{12}\right)^{E_2/2} Q^{-(1+E_1)}. \quad (60)$$

Note that this expression contains no explicit dependence on f^*. For a two component system at $\epsilon=1$, we take the experimental value of $\frac{2}{3}$ for ν_H and obtain $Z=0.30$. Alternatively, we may use values for ν_H preceding Eq. (59) to first order in ϵ; from this, we obtain $\nu_H=\frac{3}{5}$ for $n=2$ and $\epsilon=1$, and $Z=0.25$. Z depends on n. At $n=n_c$, Q becomes infinite, and Z goes to zero indicating a continuous transition. It is expected that n_c is a decreasing function of ϵ.[26] It is, therefore likely that Q is larger in three dimensions than our estimate based on the first-order expansion in ϵ. If this is so, Z could well be smaller than 0.3.

We can also estimate the jump of the order parameter at the first-order transition ψ_c for both types of superconductors by the following scaling relation:

$$|\psi_c(0)|^2 = \exp\left(-(2-\epsilon)l_1 + \int_0^{l_1}\eta(l)\,dl\right)|\psi_c(l_1)|^2. \quad (61)$$

We obtain

$$|\psi_c(0)|^2_{\rm I} = \frac{\Lambda^2}{2f(0)} e^{-0.5}\left(\frac{nf^*}{36f(0)} R(0)\right)^{-2/\epsilon} \quad (62)$$

for systems in the shaded region, and

$$|\psi_c(0)|^2_{\rm II} = [\Lambda^2/2f(0)]e^{-0.5}Q^{1+18/n} \times (Q-1)^{-2/\epsilon}[f^*/f(0)]^{-2/\epsilon} \quad (63)$$

for extreme type-II systems. The ratio $|\psi_c(0)|^2_{\rm II}/|\psi_c(0)|^2_{\rm I}$ is related to the Ginzburg parameter κ,

$$|\psi_c(0)|^2_{\rm II}/|\psi_c(0)|^2_{\rm I} = Z_1\kappa^{4/\epsilon}, \quad (64)$$

with $Z_1 = Q^{1+18/n}(Q-1)^{-2/\epsilon}(\frac{1}{18}n)^{2/\epsilon}$. For $n=2$, $Z_1 = 2.01$ at $\epsilon = 1$. Another quantity which is of interest is the ratio between the order-parameter susceptibilities right above and below t_c,

$$\chi_-/\chi_+ = \tfrac{1}{2} + O(\epsilon^2). \quad (65)$$

This can be derived directly from Eqs. (27) and (53).

One can easily verify that $\ln t_c(0)$ and $\ln|\psi_c(0)|$ are independent of l up to order ϵ by including the u-dependent terms in Eqs. (54), (55), and (61). It remains to be shown that t_c is always less than t_1 for any values of $u(0)$ and $f(0)$. By definition of t_1, we have $t_1(l_1) = 1$. However, $t_c(l_1)$ is of order ϵ [Eq. (54)]. Near four dimensions, then $t_c(l_1) < t_1(l_1)$ and $t_c < t_1$:

$$\ln t_c - \ln t_1 = \ln[\tfrac{3}{4}e^{-0.5}f(l_1)C_d]. \quad (66)$$

At three dimensions $f(l_1)$ is order unity, thus there are trajectories for which $t_1 < t_c$. However, as can be seen by Fig. 2(b), all trajectories that pass anywhere near the Heisenberg fixed point have $t_c(l_1) < 1$ so that $t_1 > t_c$ for most trajectories of interest.

V. SUMMARY

In this paper, we have presented detailed calculations to first order in $\epsilon = 4-d$ of thermodynamic functions in the vicinity of the fluctuations induced first-order normal to superconducting transition. In particular, we calculated the order-parameter susceptibility χ and its associated effective critical exponent $\gamma_{\rm eff}$. A particularly interesting feature of this calculation (cf. Fig. 1) is that $\gamma_{\rm eff}$ decreases from a value of order that of the pure xy system to unity as much as a decade before the transition takes place. A modified mean-field theory introduced in Ref. 1 provides an adequate description in all dimensions between two and four of the first-order superconducting transition in type-I systems as long as the Londern penetration depth at $t = t_c^-$ is much larger than the inverse momentum cutoff Λ^{-1}. Quantities for such systems, fluctuation effects become important. It is convenient to express quantities such as t_c and ψ_c in the type-II systems in terms of the same quantities in the type-I systems (cf. Fig. 5) since these quantities are known for type-I systems for all dimensions between two and four. Thus, we have $t_c^{\rm II}/t_c^{\rm I} = Z\kappa^{-2/\epsilon\nu}\kappa^{2(4-\epsilon)/\epsilon}$, where ν is the correlation length exponent for the xy transition in the absence of coupling to $\vec{A}$ and $|\psi_c|^2_{\rm II}/|\psi_c|^2_{\rm I} = Z_1\kappa^{4/\epsilon}$. We find $Z_1 = 2.01$ and $Z = 0.30$ at $\epsilon = 1$ if the experimental value of ν_H is used and $Z = 0.25$ if the first order in ϵ value of ν_H is used.

ACKNOWLEDGMENTS

We are grateful to B. I. Halperin for a number of helpful conversations and to J. Rudnick for providing us with a preliminary handwritten manusscript of Ref. 11. D. R. N. would like to acknowledge conversations with S. Weinberg in the early stages of this investigation. One of us (T. C. L.) is grateful to the Alfred P. Sloan Foundation for financial support. This work was supported in part by the NSF, MRL Program Grant No. DMR 76-00678. This work was supported in part by the NSF under Grant No. DMR 76-21703 and the Office of Naval Research under Grant No. N00014-76-C-0106. D. R. N. was supported by a Junior Fellowship, Harvard Society of Fellows and by the NSF under Grant No. DMR 77-10210.

APPENDIX A: SOLUTION OF RENORMALIZATION-RECURSION RELATIONS

In this Appendix we show how to solve Eqs. (2) and (3). Equation (3) is decoupled from the others and depends only on $f(l)$ whose solution is immediately

$$f(l) = \frac{e^{\epsilon l}f(0)}{1+(n/6\epsilon)f(0)C_d(e^{\epsilon l}-1)}. \quad (A1)$$

Since we choose $\eta_A = \epsilon$ such that q_0 remains constant in the recursion relations Eq. (A1) tells us

$$\mu(l) = \frac{e^{\epsilon l}\mu(0)}{1+(n/6\epsilon)4\pi\mu(0)q_0^2C_d(e^{\epsilon l}-1)}. \quad (A2)$$

To solve for $u(l)$, we first naively neglect $r(l)$ in Eq. (2c) and the remaining equation becomes

$$\frac{du(l)}{dl} = [\epsilon + 6f(l)C_d]u(l) - \tfrac{1}{2}(n+8)u^2(l)C_d - 6f^2(l)C_d + O(r(l)u^2, r(l)f(l)u(l)). \quad (A3)$$

Making the changes of variables $u(l) = e^{\epsilon l}\tilde{u}(l)$ and $f(l) = e^{\epsilon l}\tilde{f}(l)$, we obtain

$$\frac{d\tilde{u}(l)}{dl} = e^{\epsilon l}[6\tilde{f}(l)\tilde{u}(l) - \tfrac{1}{2}(n+8)\tilde{u}^2(l) - 6\tilde{f}(l)]C_d. \quad (A4)$$

and

$$\frac{d\tilde{f}(l)}{dl} = -e^{\epsilon l}\left(\frac{n}{6}\right) C_l \tilde{f}^2(l)\,. \tag{A5}$$

Using Eq. (A5) we can express the left-hand side of Eq. (A4) as

$$\frac{d\tilde{u}(l)}{dl} = \frac{d\tilde{u}(l)}{d\tilde{f}(l)} (e^{\epsilon l})\left(-\frac{n}{6} Cd\tilde{f}^2(l)\right). \tag{A6}$$

Thus, we have

$$\frac{d\tilde{u}(l)}{d\tilde{f}(l)} = -\frac{6}{n}\left[6\frac{\tilde{u}}{\tilde{f}} - \tfrac{1}{2}(n+8)\left(\frac{\tilde{u}}{\tilde{f}}\right)^2 - 6\right], \tag{A7}$$

which can be expressed as

$$\tilde{f}\frac{dR(l)}{d\tilde{f}(l)} = \frac{3}{n}(n+8)R^2 - \frac{n+36}{n}R + \frac{36}{n}\,, \tag{A8}$$

where $R(l) = u(l)/f(l)$. The solution of Eq. (A8) is straightforward. For $n<n_c$, we have

$$R(l) = (1/A)\{B + |\Delta| \tan[\theta_0 - \theta(l)]\}, \tag{A9}$$

where

$$A = 6(n+8), \quad B = n+36, \quad \Delta = (n^2 - 360n - 2160)^{1/2}, \tag{A10a}$$

$$\theta_0 = \tan^{-1}\{[AR(0) - B]/|\Delta|\}, \tag{A10b}$$

$$\theta(l) = (|\Delta|/2n)\ln[f(0)/\tilde{f}(l)]. \tag{A10c}$$

For $n>n_c$, we have

$$R(l) = (1/A)\{B + \Delta\tanh[\theta_0']\}, \tag{A11}$$

where

$$\tanh\theta_0' = [AR(0) - B]/\Delta, \tag{A12}$$

and finally, for $n=n_c$, we have

$$R(l) = \frac{1}{A}\left(B + \frac{AR(0) - B}{1 + \{[AR(0) - B]/2n\}\ln[f(0)/\tilde{f}(0)]}\right). \tag{A13}$$

Equation (A12) is the limit of (A9) as n approaches n_c. Using the above result as a starting point for a perturbation solution of $u(l)$, we can show that, as long as $r(l)$ is less than one, the contribution of the neglected terms in Eq. (A3) is of order ϵ^2 and can be neglected.

To solve Eq. (2a) we write $1/[1+r(l)]$ as

$$1/[1+r(l)] = 1 - r(l) + r^2(l)/[1+r(l)].$$

Equation (2a) then becomes

$$\begin{aligned}\frac{dr(l)}{dl} &= [2 + 3f(l)C_d - \tfrac{1}{2}(n+2)u(l)C_d]r(l) \\ &\quad + 3f(l)C_d + \tfrac{1}{2}(n+2)u(l)C_d \\ &\quad \times[-3f(l) + \tfrac{1}{2}(n+2)u(l)]C_d\frac{r^2(l)}{1+r(l)}\,. \end{aligned}\tag{A14}$$

Now let $r(l) = P(l)\tilde{r}(l)$, where

$$P(l) = \exp\left(2l + 3\int_0^l f(l')C_d\,dl' - \tfrac{1}{2}(n+2)\int_0^l C_d u(l')\,dl'\right). \tag{A15}$$

The equation for $\tilde{r}(l)$ is then

$$\frac{d\tilde{r}(l)}{dl} = P^{-1}(l)\left(3f(l)C_d + \tfrac{1}{2}(n+2)u(l)C_d - 3f(l)\frac{C_d r^2(l)}{1+r(l)} + \tfrac{1}{2}(n+2)\frac{u(l)C_d r^2(l)}{1+r(l)}\right). \tag{A16}$$

Equation (A16) can be integrated by parts. Keeping terms up to order ϵ, we obtain

$$\begin{aligned}\tilde{r}(l) &= \tilde{r}(0) + \{-\tfrac{3}{2}f(l)C_d - \tfrac{1}{4}(n+2)u(l)C_d \\ &\quad - \tfrac{3}{2}f(l)C_d r(l)\ln[1+r(l)] \\ &\quad + \tfrac{1}{4}(n+2)u(l)C_d r(l)\ln[1+r(l)]\}P^{-1}(l). \end{aligned}\tag{A17}$$

Thus, we have

$$\begin{aligned} r(l) &= -\tfrac{3}{2}f(l)C_d - \tfrac{1}{4}(n+2)u(l)C_d + [\tfrac{1}{4}(n+2)u(l) - \tfrac{3}{2}f(l)]C_d r(l)\ln[1+r(l)] \\ &\quad + p(l)\{r(0) + \tfrac{3}{2}f(0)C_d + \tfrac{1}{4}(n+2)u(0)C_d - [\tfrac{1}{4}(n+2)u(0) - \tfrac{3}{2}f(0)]C_d r(0)\ln[1+r(0)]\}. \end{aligned}\tag{A18}$$

This equation is more conveniently expressed as

$$t(l) = p(l)t(0), \tag{A19}$$

where

$$\begin{aligned} t(l) &= r(l) + \tfrac{3}{2}f(l)C_d + \tfrac{1}{4}(n+2)u(l)C_d \\ &\quad - [\tfrac{1}{4}(n+2)u(l) - \tfrac{3}{2}f(l)]C_d(l)\ln[1+r(l)]. \end{aligned}\tag{A20}$$

APPENDIX B: CROSSOVER FUNCTION FOR THE SPECIFIC HEAT

We first discuss the free energy, which satisfies the following modified homogeneity relation:

$$\begin{aligned} F(r,u,f) &= \int_0^l \tfrac{1}{2}C_d\Big(n\eta(l) + 3\eta_A(l) \\ &\quad + n\{\ln[1+r(l)] - \tfrac{1}{2}\}e^{-dl} \\ &\quad + 3\{\ln[4\pi u(l)]^{-1} - \tfrac{1}{2}\}e^{-dl}\Big)dl \\ &\quad + e^{-dl}F(r(l),u(l),f(l)). \end{aligned}\tag{B1}$$

The trajectory integral represents the contribution to the free energy coming from rescaling and tracing over the degrees of freedom in an infinitesimal momentum shell at each iteration. We will evaluate Eq. (B1) at a particular l, $l=\min(l_1^*, l_2^*)$ such that $F(r(l),u(l),f(l))$ may be calculated by Landau theory. Taking into account the first fluctuation correction, we find

$$F(r(l),u(l),f(l))=\frac{n}{2}C_d\int_0^l q^3\ln[r(l)+q^2]dq+\tfrac{3}{2}C_d\int_0^l[\ln K(l)q^2]q^3\,dq\,. \qquad \text{(B2)}$$

The specific heat $C(r,u,f)$ is then obtained by differentiating Eq. (B1) with respect to temperature

$$C(r,u,f)=-2\,\frac{\partial^2F(r,u,f)}{\partial r^2} \qquad \text{(B3)}$$

$$=\frac{6nC_4}{r^2}\int_0^l\frac{f(l)C_dr^2(l)}{[1+r(l)]^3}dl+\frac{nC_4}{r^2}\int_0^l\frac{r^2(l)e^{-dl}}{[1+r(l)]^2}dl+\frac{nC_4}{r^2}r^2(l)e^{-dl}\int_0^1\frac{q^3dq}{[r(l)+q^2]^2}\,. \qquad \text{(B4)}$$

The first two integrals in Eq. (B4) can be evaluated by integration by parts. The third integral is standard. After evaluating these integrals $C(r,u,f)$ decomposes into a regular part and a singular part:

$$C(r,u,f)=C_r(r,u,f)+C_s(r,u,f), \qquad \text{(B5)}$$

where

$$C_r(r,u,f)=\frac{nC_4}{2r^2}\left(\frac{r}{1+r}-r+r^2\ln(1+r)\right)+\frac{3}{2}\frac{C_4}{r^2}\left(f(0)C_d\frac{1+2r}{(1+r)^2}-\frac{1+2r(l)}{[1+r(l)]^2}f(l)C_4\right), \qquad \text{(B6a)}$$

$$C_s(r,u,f)=\frac{nC_4}{t^2}\int_0^l e^{-dl}t^2(l)dl-\frac{nC_4}{2r^2}e^{-dl}t^2\ln t(l)\,. \qquad \text{(B6b)}$$

Explicitly differentiating C_s with respect to l, we find

$$\frac{\partial C_s(r,u,f)}{\partial l}=\frac{nC_4}{r^2}e^{-dl}r^2(l)-\frac{nC_4}{2r^2}r^2(l)e^{-dl}\frac{2r(l)}{r(l)}+O(\epsilon)=0+O(\epsilon).$$

This shows C_s is independent of the precise choice of the matching condition in leading order.

From Eq. (B6b) we are able to calculate the effective exponent for the specific heat

$$\alpha_{\text{eff}}=-\frac{\partial\ln C_s}{\partial\ln t}\,. \qquad \text{(B7)}$$

If $t_c<t<t_1$, $l=l_2$ is independent of t, and

$$\alpha_{\text{eff}}=\tfrac{1}{2}e^{-dl}p^2(l)\times\left(\int_0^l e^{-dl}p^2(l)dl-\tfrac{1}{2}e^{-dl}p^2(l)\ln t(l)\right)^{-1}, \qquad \text{(B8a)}$$

as $t>t_1$, $l=l_1$, $t(l)=1$,

$$\alpha_{\text{eff}}=\tfrac{1}{2}e^{-dl}p^2(l)\left(\int_0^l e^{-dl}p^2(l)\,dl\right)^{-1} \qquad \text{(B8b)}$$

as $t=t_1$, $l_1=l_2$, $t(l)=1$. Equation (B8a) coincides with Eq. (B8b). Thus α_{eff} is continuous at $t=t_1$. Note α_{eff} is positive in the high-temperature phase. Unfortunately, the integral in Eqs. (B6a) and Eq. (B8) cannot be integrated analytically.

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