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Kinetic theory of a normal quantum fluid. II. Transport properties

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Renormalized expressions for the transport coefficients of a normal quantum fluid are derived from a nonlocal kinetic equation. As was shown in the classical case by Forster and Martin and by Résibois, the expressions for the transport coefficients separate naturally into "kinetic" and "direct" parts. The kinetic terms are proportional to matrix elements of the inverse of the kinetic kernel, and have the same general structure as the "bare" transport coefficients obtained from a local Boltzmann equation. The direct terms are proportional to matrix elements of the kernel itself, and have no counterpart in calculations based on a local kinetic equation, as in the kinetic theory of gases. We evaluate the transport coefficients using the weak-coupling approximation to the kernel derived in a previous paper. Results are given, first, for both Bose and Fermi fluids at arbitrary temperature, and then for the Fermi fluid near $T = 0$, where complete solutions are obtained. It is found that the direct parts of the shear viscosity and thermal conductivity are of higher order in T than the kinetic parts, and are therefore negligible at very low temperature. The kinetic parts have the same leading temperature dependence as the predictions of the Landau theory. For the bulk viscosity, however, the direct and kinetic contributions begin at the same order (T^2) in the temperature.

I. INTRODUCTION

Linear-response theory provides an exact representation of the transport coefficients of a fluid in terms of the small wave-vector and frequency limit of hydrodynamic response functions.[1,2] Modern calculations of transport coefficients therefore focus on the problem of determining the response functions. Two general approaches are available. In the first, known as generalized hydrodynamics, a response function is obtained from an equation that involves configuration-space variables only. The equation contains another response function, the memory function, which is better behaved in the hydrodynamic limit than the original one, and therefore easier to approximate. If necessary, the scheme can be iterated, so that a hierarchy of memory-function equations is generated.[3]

In the second approach, the hydrodynamic response function is obtained from a phase-space correlation function. This approach could be called generalized kinetic theory, because the phase-space function is the solution of a kinetic equation related to the linearized Boltzmann equation. In place of the Boltzmann collision kernel, there appears a more complicated collision kernel which, in general, is the sum of two terms, one nonlocal in space and the other nonlocal in both space and time. For classical fluids, nonlocal equations of this type have been investigated extensively during the past decade. There has been a great deal of progress both in the construction of these equations and in their practical use for calculating response functions and transport coefficients.[4] The success of the classical theory now makes it attractive to apply this approach to the more difficult problems of quantum-mechanical fluids.

In a previous paper, we derived a nonlocal kinetic equation for a normal quantum fluid with Bose or Fermi statistics (neglecting spin) and evaluated the collision kernel explicitly to second order in the interparticle potential.[5] We showed that the classical limit of the kernel reduced to the weak-coupling kernel derived by Akcasu and Duderstadt[6] and that in the limit of small wave vector and frequency it reduced to the kernel of the linearized Uehling-Uhlenbeck equation[7] with the Born approximation for the cross section. In the present paper, we study the transport coefficients predicted by this nonlocal quantum kinetic equation.

Although we are concerned here with the hydrodynamic regime, it is important to note that such nonlocal kinetic equations are valid for all wave vectors and frequencies, and that a major part of the interest in them has to do with the calculation of correlations on an atomic scale. Such a calculation for normal liquid He^3 at low temperatures has recently been presented by Valls, Mazenko, and Gould,[8] who approximated the kernel of the nonlocal equation on a partly phenomenological basis and calculated the dynamic structure function and the dispersion relation for zero sound. Their investigation focused on the intermediate wave-vector and frequency regime, rather than on the hydrodynamic regime.

An important feature of these nonlocal kinetic equations is that they depend on only one time (or frequency) variable, the physical time displacement of the correlation function. This distinguishes them from the multiple-time equations that are the natural result of quantum mechanical Green's function techniques, such as the two-time generalized Boltzmann equation derived by Kadanoff and Baym.[9] Some advantages of the single-time formulation were noted by Valls *et al.*: it allows a natural separation of static and dynamic effects; the criteria for preserving conservation laws, sum rules, and symmetries are simpler; and physically reasonable approximations can often be introduced with greater ease. To these may be added the feature that is exploited in this paper, namely the straightforwardness of a low-order expansion in powers of the physical frequency, which is necessary for the study of the hydrodynamic regime.

It is clear that the wave-vector (k) and frequency (z) dependence of the collision kernel is essential for describing the structure and motion of a fluid on the atomic scale. It is less obvious, perhaps, that the k and z dependence of the kernel also has an important effect in the hydrodynamic regime. In fact, it has been shown in the classical case by Forster and Martin[10] and by Résibois[11] that there are contributions to the transport coefficients from terms through second order in the small-k and -z expansion of the kernel. The effect of the k- and z-dependent terms can be regarded as a renormalization of the transport parameters from the "bare" values given by the local limit of the kinetic equation, where k and z are set equal to zero in the kernel. The renormalization is what accounts for the effects of correlations due to the interparticle interactions. (Note that this renormalization is distinct from the renormalization that one may perform in deriving an expression for the kernel itself.) Such correlations are present even in the low-density classical gas, although their effects are small in that case.[12]

As we show in this paper, the transport coefficients of a quantum-mechanical fluid can be obtained from the nonlocal kinetic equation in essentially the same way as in the classical case. This is a consequence of the single-time form of the equation. The results for the renormalized transport coefficients—both classical and quantum—have the following basic structure. There are contributions from two types of terms, which may be called "kinetic" and "direct." The kinetic terms are proportional to matrix elements of the inverse of the kernel and in this respect are similar to the expressions for the bare transport coefficients. The direct terms are proportional to matrix elements of the kernel itself, and have no bare counterpart. In the zero density limit, the direct terms vanish, and the kinetic terms reduce to the result given by the Boltzmann equation. Little is currently known about the actual magnitudes of the various terms for dense fluids, but it is anticipated that as the density and interaction strength increase, the renormalization corrections will grow in importance relative to the bare term. In one case, liquid argon near the triple point, a calculation by Jhon and Forster based on a modeled kinetic equation indicates that the experimental viscosity and thermal diffusivity are mostly accounted for by the direct terms.[13]

We have already mentioned that our weak-coupling equation reduces to the Uehling-Uhlenbeck equation when the kernel is considered in the limit $k,\ z \to 0$. Another point of reference for a quantum kinetic theory is provided by the Landau theory of Fermi liquids.[14] Valls *et al.* have considered the correspondence between their kinetic equation and the Landau theory in some detail, and have used it in formulating their phenomenological model. In its usual form, the Landau equation has a wave-vector-dependent mean-field term, related to the static term of the nonlocal equation, but the wave-vector and frequency dependence of the Boltzmann-like collision term is neglected.[15] The Landau-theory transport coefficients calculated by Abrikosov and Khalatnikov[16] therefore do not contain any direct terms, and the mean-field term serves mainly to change the effective mass. Wave-vector and frequency dependence of the kernel has been included in calculations of higher terms in the temperature expansion of the Landau-theory transport coefficients,[17] but the methods of calculation used did not allow for the appearance of the direct terms. In view of the Jhon and Forster result for argon, it is interesting to see whether this can be justified. Briefly, our result for the weak-coupling kernel indicates that it is proper to neglect the direct terms of the shear viscosity and thermal diffusivity at low temperature, because they begin at higher order in the temperature than the kinetic terms. For the bulk viscosity, however, the contributions of the kinetic and direct terms begin at the same order in the temperature.

The outline of the body of the paper is as follows. In Sec. II A we review our previous results for the nonlocal kinetic equation, and introduce the additional definitions and notation that are needed for the discussion of the transport properties. In calculating the transport coefficients from the nonlocal kinetic equation, we follow the procedure developed by Forster and Martin in their study of the classical weak-coupling equation, and we refer to their paper for details of the argument. A heuristic outline of the calculation together with a discussion of the features of the renormalized expressions for the transport coefficients is given in Sec. II B. It is intended that this subsection will be accessible to the reader who does not wish to study the rest of the paper in detail. In Sec. III we turn to the weak-coupling equation and work out the transport equation expressions as far as possible for both Bose and Fermi fluids at arbitrary temperature. Finally, in Sec. IV, we specialize to the case of the Fermi fluid at low temperature, where complete solutions are obtained.

II. NONLOCAL KINETIC THEORY

A. Kinetic equation and correlation functions

We begin with a brief summary of the notation and definitions used in our previous paper.[5] The basic quantity in our description is the quantum phase space or Wigner density operator

$$f(rpt)=(2\pi)^{-3}\int dr'\,e^{i\vec{p}\cdot\vec{r}'}\times\psi^\dagger(r-\tfrac{1}{2}r',t)\psi(r+\tfrac{1}{2}r',t)\,,\qquad(1)$$

where $\psi(rt)$ and $\psi^\dagger(rt)$ are the Heisenberg field operators satisfying the usual equal-time commutator or anticommutator relations for Bose or Fermi statistics. For simplicitly, we ignore spin and consider particles of unit mass which interact by a central potential $\hat{v}(r)=(2\pi)^{-3}\int dk\,e^{i\vec{k}\cdot\vec{r}}v(k)$. Except where otherwise noted, we take $\hbar=1$ and use the letters k, r and p to represent vector quantities. From $f(rpt)$ we form the anticommutator correlation function

$$F(rp,r'p'\,|\,t-t')=\langle\tfrac{1}{2}\{f(rpt),f(r'p't')\}\rangle-\langle f(rpt)\rangle\langle f(r'p't')\rangle\,,\qquad(2)$$

where the angular brackets indicate an average in the grand canonical ensemble with inverse temperature β and chemical potential μ. Transforms of $F(rp,r'p'\,|\,t-t')$ and similar quantities are given by the convention

$$F(kzpp')=-i\int_0^\infty dt\,e^{izt}\times\int d(r-r')\,e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')}F(rp,r'p'\,|\,t)\,,\qquad(3)$$

with the imaginary part of z positive. The spatial Fourier transform $F(kpp')$ of the equal-time anticommutator function also plays an important role. It is given by

$$F(kpp')=\delta(p-p')N(kp)+H(kpp')\,,\qquad(4)$$

where $H(kpp')$ is the connected part of the two-particle distribution function, and $N(kp)$ is the symmetrized combination

$$N(kp)=\tfrac{1}{2}n(p+\tfrac{1}{2}k)\tilde{n}(p-\tfrac{1}{2}k)+\tfrac{1}{2}\tilde{n}(p+\tfrac{1}{2}k)n(p-\tfrac{1}{2}k)\qquad(5)$$

of the one-particle momentum distribution $n(p)=\langle f(np)\rangle$ and its complement $\tilde{n}(p)=1+\epsilon(2\pi)^3n(p)$. ϵ is $+1$ for Bosons and -1 for Fermions. For free particles, $n(p)$ is

$$n_0(p)=(2\pi)^{-3}(e^{\beta(p^2/2-\mu)}-\epsilon)^{-1}\,.\qquad(6)$$

The factors of 2π come from the definition of $f(rpt)$ in Eq. (1), which corresponds to a classical normalization. Thus, momentum integrals are not accompanied by a factor of $(2\pi)^{-3}$, and $n(p)$ reduces in the classical limit to $n\phi(p)$, where n is the density and $\phi(p)$ is the Maxwellian. $F(kpp')$ and $F(kzpp')$ are of course closely related (after integration over the momenta) to the static and dynamic structure factors $S(k)$ and $S(k\omega)$ of the fluid.

The equation of motion of $F(kzpp')$ has the general form

$$(z-\vec{k}\cdot\vec{p})F(kzpp')=F(kpp')+\int d\bar{p}\,\Sigma(kzp\bar{p})\times F(kz\bar{p}p')\,.\qquad(7)$$

The function Σ, which we call the kinetic kernel, accounts for the effects of collisions as $F(kzpp')$ evolves from its initial condition $F(kpp')$. The kinetic kernel separates naturally into a sum of two terms, a frequency-independent or "static" term $\Sigma^{(s)}(kpp')$ and a frequency-dependent or "dynamic" term $\Sigma^{(d)}(kzpp')$, which vanishes like z^{-1} at large z. The static term describes instantaneous mean-field effects, and is the now-familiar generalization of the term in the Landau theory of Fermi liquids that couples an inhomogeneous distribution of excitations to the ground state distribution of quasiparticles.[15] The dynamic term describes the effects of collisions, and is the generalization of the Boltzmann collision kernel. To recover the linearized Boltzmann equation (either classical or quantum) from Eq. (7) one would first evaluate Σ at low density, where only binary collisions contribute, and then take the limit of zero wave vector and zero frequency ($k\to0$, $z\to0+i0^+$). The $k\to0$ limit does away with spatial correlations, and causes the mean-field term to vanish, while the $z\to0$ limit does away with memory effects. Thus in the limit k, $z\to0$

(the uniform Markovian limit), the kernel reduces to its local value

$$K^{(d)}(0, i0^{+}; pp') = iJ(pp')\,, \tag{8}$$

where $J(pp')$ is the standard collision kernel of the linearized Boltzmann equation. Here we have introduced the kernel $K(kzpp')$, which we obtain from $\Sigma(kpp')$ by convoluting it with the initial value $F(kpp')$:

$$K(kzpp') = \int d\bar{p}\, \Sigma(kzp\bar{p})\, F(k\bar{p}p')\,, \tag{9a}$$

$$K(kzpp') = K^{(s)}(kpp') + K^{(d)}(kzpp')\,. \tag{9b}$$

It is more convenient to deal with K than with Σ because, as discussed in our previous paper, the two quantities $[\vec{k}\cdot\vec{p}F(kpp') + K^{(s)}(kpp')]$ and $K^{(d)}(kzpp')$ are symmetric in p and p'.

This completes the summary of general results from our previous paper. Some further comments on the equation are given in Sec. III in connection with the explicit results for the weak-coupling approximation to the kernel.

As will be discussed in Sec. II B, the transport coefficients are obtained from the normalized density-density correlation function $G_{nn}(kz)$ and transverse current correlation function $G_t(kz)$ defined by[18]

$$G_{nn}(kz) = F_{nn}(kz)/F_{nn}(k)\,, \tag{10}$$

$$G_t(kz) = F_{xx}(kz)/F_{xx}(k)\,, \tag{11}$$

where

$$F_{nn}(kz) = \int dp\, dp'\, F(kzpp')\,, \tag{12}$$

$$F_{xx}(kz) = \int dp\, dp'\, p_x F(kzpp')\, p'_x\,, \tag{13}$$

and the normalization factors are the static correlation functions $F_{nn}(k) = \int dp\, dp'\, F(kpp')$, etc. Here we have taken k to point in the z direction. In the calculation of the transport coefficients, it turns out that only the $k=0$ values of the static functions $F_{nn}(k)$ and $F_{xx}(k)$ are needed. These are given by[2]

$$F_{nn}(0) = \frac{n}{\beta}\left(\frac{\partial n}{\partial P}\right)_{\beta}\,, \tag{14}$$

$$F_{xx}(0) = n/\beta\,. \tag{15}$$

Now we introduce a convenient operator notation, and define the remaining quantities needed for the discussion in Sec. II B. Let $\{|p\rangle\}$ be a set of vectors satisfying

$$\langle p | p' \rangle = F(kpp')\,, \tag{16a}$$

$$\int dp\, dp'\, |p\rangle F^{-1}(kpp') \langle p'| = 1\,, \tag{16b}$$

where $F^{-1}(kpp')$ is the solution to

$$\int d\bar{p}\, F^{-1}(kp\bar{p}) F(k\bar{p}p') = \delta(p-p')\,. \tag{17}$$

The vectors $|p\rangle$ depend implicitly on k as well as on the thermodynamic state of the system. The static inverse $F^{-1}(kpp')$ can be simply expressed in terms of a momentum-dependent generalization of the classical direct correlation function.[5] Given a function $A(kzpp')$, we define the matrix elements of the corresponding operator A by

$$\langle p | A | p' \rangle = \int d\bar{p}\, A(kzp\bar{p}) F(k\bar{p}p')\,. \tag{18}$$

With these definitions, the solution $F(kzpp')$ of the kinetic equation (7) can be written

$$F(kzpp') = \langle p | [z - \Omega(kz)]^{-1} | p' \rangle\,, \tag{19}$$

where the kinetic operator $\Omega(kz)$ is defined by

$$\Omega(kz) = \Omega^{(0)}(k) + \Omega^{(s)}(k) + \Omega^{(d)}(kz)\,, \tag{20}$$

with

$$\langle p | \Omega^{(0)}(k) | p' \rangle = \vec{k}\cdot\vec{p}\, F(kpp')\,, \tag{21a}$$

$$\langle p | \Omega^{(s)}(k) | p' \rangle = K^{(s)}(kpp')\,, \tag{21b}$$

$$\langle p | \Omega^{(d)}(kz) | p' \rangle = K^{(d)}(kzpp')\,. \tag{21c}$$

With the symmetries of K noted above, we see that the matrix elements of $\Omega^{(0)} + \Omega^{(s)}$ and of $\Omega^{(d)}$ are symmetric in p and p'. This is the reason for the inclusion of the weighting factor $F(kpp')$ in the operator definition (18).

Vectors $|n\rangle$, $|u_i\rangle$, and $|\epsilon\rangle$ corresponding to the number, momentum, and kinetic-energy densities are defined by:

$$|n\rangle = [F_{nn}(k)]^{-1/2} \int dp\, |p\rangle\,, \tag{22}$$

$$|u_i\rangle = [F_{ii}(k)]^{-1/2} \int dp\, p_i |p\rangle\,, \tag{23}$$

$$|\epsilon\rangle = [F_{\epsilon\epsilon}(k)]^{-1/2} \int dp\, \epsilon(kp)\, |p\rangle\,, \tag{24}$$

where the kinetic-energy function $\epsilon(kp)$ is

$$\epsilon(kp) = \tfrac{1}{2}p^2 - \alpha_k\,, \tag{25}$$

with

$$\alpha_k = [F_{nn}(k)]^{-1} \int dp\, dp'\, \tfrac{1}{2}p^2 F(kpp')\,. \tag{26}$$

The term α_k is included in (25) in order to make $|\epsilon\rangle$ orthogonal to $|n\rangle$. (Both $|\epsilon\rangle$ and $|n\rangle$ are of course orthogonal to the $|u_i\rangle$.) In the classical limit, $\epsilon(kp)$ loses its k dependence, and reduces simply to $\frac{1}{2}p^2 - \frac{3}{2}\beta^{-1}$.

The vectors $|n\rangle$, $|u_i\rangle$, and $|\epsilon\rangle$ span the null space of the kinetic operator Ω. Specifically, we have

$$\langle n\,|\,\Omega^{(s)}(k)+\Omega^{(d)}(kz)\,|\,p'\rangle=0\,, \tag{27}$$

$$\langle u_i\,|\,\Omega(0,z)\,|\,p'\rangle=0\,, \tag{28}$$

$$\langle \epsilon\,|\,\Omega(0,i0^{+})\,|\,p'\rangle=0\,. \tag{29}$$

Equation (27), which is equivalent to the number-conservation law, says that the zeroth moment of the kinetic kernel vanishes for all k and z:

$$\int dp\,[K^{(s)}(kpp')+K^{(d)}(kzpp')]=0\,. \tag{30}$$

The moments corresponding to momentum and kinetic energy are more complicated. We define the momentum flux tensor $T_{ij}(p)=T_{ij}^{(s)}(p)+T_{ij}^{(d)}(p)$ by

$$\int dp\,p_i[\vec{\mathrm{k}}\cdot\vec{\mathrm{p}}F(kpp')+K^{(s)}(kpp')] =k_j T_{ij}^{(s)}(p')+\ldots\;, \tag{31a}$$

$$\int dp\,p_i K^{(d)}(kzpp')=k_j T_{ij}^{(d)}(p')+\ldots\;, \tag{31b}$$

and the kinetic energy flux vector $Q_i(p)=Q_i^{(s)}(p)+Q_i^{(d)}(p)$ by

$$\int dp\,\epsilon(p)[\vec{\mathrm{k}}\cdot\vec{\mathrm{p}}F(kpp')+K^{(s)}(kpp')] =k_i Q_i^{(s)}(p')+\ldots\;, \tag{32a}$$

$$\int dp\,\epsilon(p)K^{(d)}(kzpp')=k_i Q_i^{(d)}(p')+z\mathcal{E}(p')+\ldots\;, \tag{32b}$$

where $\epsilon(p)$ is the $k=0$ value of $\epsilon(kp)$, and the omitted terms in Eqs. (31) and (32) are of higher order in k and/or z (these are not needed). Note that the z dependence of the kernel plays a role only in Eq. (32b). It can be shown that conservation of momentum and total energy (kinetic plus potential) follow from the fact that Eqs. (31) and (32) have the indicated form.[10]

Specific results for the quantum T_{ij} and Q_i are given in later sections, but it may be remarked here that the static contributions $T_{ij}^{(s)}(p)$ and $Q_i^{(s)}(p)$ are very simple in the classical case. The combination $\vec{\mathrm{k}}\cdot\vec{\mathrm{p}}F(kpp')+K^{(s)}(kpp')$ appearing in their definition reduces to $\vec{\mathrm{k}}\cdot\vec{\mathrm{p}}n\phi(p)\delta(p-p')$, and so one has $T_{ij}^{(s)}(p)=p_ip_jn\phi(p)$ and $Q_i^{(s)}(p)=\frac{1}{2}p_i(p^2-3\beta^{-1})n\phi(p)$.

B. Transport coefficients

The transport coefficients are obtained by formally solving the kinetic equation for the density-density and transverse-current correlation functions, and comparing these functions with the results predicted by ordinary hydrodynamics. The calculation also gives the speed of sound and the ratio of specific heats. The argument is simpler for the transverse mode, so we consider it first.

From Eqs. (11) and (19), we see that the normalized transverse-current correlation function can be written as the matrix element of the inverse kinetic operator,

$$G_t(kz)=\langle u_x\,|\,[z-\Omega(kz)]^{-1}\,|\,u_x\rangle\,. \tag{33}$$

This is to be compared with the result given by hydrodynamics for small k and z,[2]

$$G_t^{\mathrm{hydro}}(kz)=(z+ik^2\eta/n)^{-1}\,, \tag{34}$$

where η is the shear viscosity. At first glance, one may be tempted to reach inside the matrix element in Eq. (33) and immediately take the $k,\ z\to 0$ limit of $\Omega(kz)$. In essence, this is what is done in the usual (local) kinetic theory: by setting $k,\ z=0$ in $\Sigma(kzpp')$ for the low-density gas (but keeping the $\vec{\mathrm{k}}\cdot\vec{\mathrm{p}}$ term) and then solving Eq. (33), one gets the Boltzmann equation result for the shear viscosity. Although this is a good approximation in some circumstances (the low-density gas, for example), it is clearly not valid in general because of the highly singular behavior of the correlation functions in the hydrodynamic limit. We must therefore rearrange Eq. (33) in such a way that the correct limit can be taken. This can be accomplished in the following way.[19]

Note that singularities in Eq. (33) can arise because in the $k,\ z\to 0$ limit, the operator Ω vanishes on the subspace spanned by the density, momentum, and kinetic energy vectors $|n\rangle$, $|u_i\rangle$, and $|\epsilon\rangle$. We can handle these singularities by separating the hydrodynamic subspace from the rest of function space, using the projection operator

$$P=|n\rangle\langle n|+\sum_i|u_i\rangle\langle u_i|+|\epsilon\rangle\langle\epsilon|\,, \tag{35}$$

and its complement $Q=1-P$. A straightforward calculation based on the identity

$$(z-\Omega)^{-1}=(z-\Omega Q)^{-1}+(z-\Omega Q)^{-1}\Omega P(z-\Omega)^{-1} \tag{36}$$

shows that the correlation function (33) can be written in the form

$$G_t(kz)=[z-\varphi_t(kz)]^{-1}\,, \tag{37}$$

where the transverse memory function $\varphi_t(kz)$ is given by

$$\varphi_t(kz)=\varphi_k'(kz)+\varphi_t''(kz)\,, \tag{38a}$$

with

$$\varphi_t'(kz)=\langle u_x\,|\,\Omega(kz)\,Q[z-Q\Omega(kz)Q]^{-1}Q\Omega(kz)\,|\,u_x\rangle\,, \tag{38b}$$

$$\varphi_t''(kz) = \langle u_x | \Omega(kz) | u_x \rangle . \tag{38c}$$

Because of the projection operator Q in Eq. (38b), we are now free to take the k, $z \to 0$ limit of φ_t. Comparison of Eqs. (34) and (37) shows that the shear viscosity is given by

$$\eta = \eta' + \eta'' = in \lim_{k \to 0} \frac{\varphi_t'(k0) + \varphi_t''(k0)}{k^2} . \tag{39}$$

As we mentioned in Sec. I, a result equivalent to Eq. (39) was obtained by Résibois using a different approach.[11]

In general, then, η separates into two parts, corresponding to the two terms of the memory function $\varphi_t(kz)$. The "direct" term η'', proportional to a matrix element of the collision operator, comes from the term of order $k^2 z^0$ in the small k, z expansion of the kernel, and therefore has no counterpart in calculations based on a local kinetic equation. Its occurrence is the most obvious manifestation of the renormalization produced by the nonlocal kernel. However, the "kinetic" part η', which is similar in form to the result of the local theory, also contains renormalization effects. To show this, we write out the expression for the kinetic part of the shear viscosity, from Eq. (38b), as

$$\eta' = \beta \int dp\, T_{xz}(p) g(p) , \tag{40}$$

where $T_{xz}(p)$ is the transverse-momentum flux obtained from the nonlocal kinetic equation via Eqs. (31), and $g(p)$ is the solution to[20]

$$\int dp'\, J(pp') g(p) = -T_{xz}(p) . \tag{41}$$

The local result η_0 comes from solving Eqs. (40) and (41) using the free-streaming momentum flux $T^0_{xz}(p) = p_x p_z n_0(p) \tilde{n}_0(p)$. The nonlocal contributions to η' enter through the remaining parts of the momentum flux—in particular, there are contributions from the potential-induced correlations in $F(kpp')$ and from the terms of order $k^1 z^0$ of the kernel.

Now we turn to the longitudinal modes. The analysis is similar to the transverse case, although the algebra is more involved. The density-density correlation function is obtained from the 3×3 matrix of correlation functions

$$G_{\alpha\beta}(kz) = \langle \alpha | [z - \Omega(kz)]^{-1} | \beta \rangle , \tag{42}$$

with $\alpha, \beta = 1, 2, 3$ representing the three longitudinal elements $|n\rangle$, $|u_z\rangle$, and $|\epsilon\rangle$, respectively. Using Eq. (36) as in the transverse case, we can express $G_{\alpha\beta}$ as the solution to

$$\sum_{\gamma=1}^{3} [z \delta_{\alpha\gamma} - \varphi_{\alpha\gamma}(kz)] G_{\gamma\beta}(kz) = \delta_{\alpha\beta} , \tag{43}$$

where the matrix elements of the memory kernel are given by

$$\varphi_{\alpha\beta}(kz) = \varphi'_{\alpha\beta}(kz) + \varphi''_{\alpha\beta}(kz) , \tag{44a}$$

with

$$\varphi'_{\alpha\beta}(kz) = \langle \alpha | \Omega(kz) Q [z - Q\Omega(kz)Q]^{-1} Q\Omega(kz) | \beta \rangle , \tag{44b}$$

$$\varphi''_{\alpha\beta}(kz) = \langle \alpha | \Omega(kz) | \beta \rangle . \tag{44c}$$

From the symmetries of the kernel,[5] it is straightforward to show that φ_{11} and φ_{13} vanish identically. By construction, $\varphi_{\alpha\beta}$ is symmetric in α and β. Now solving Eq. (43) for its (1, 1) matrix element, the normalized density-density correlation function, we obtain

$$G_{nn}(kz) = \frac{z^2 - (\varphi_{22} + \varphi_{33}) z + \varphi_{22}\varphi_{33} - (\varphi_{23})^2}{z^3 - (\varphi_{22} + \varphi_{33}) z^2 + (\varphi_{22}\varphi_{33} - \varphi_{23}^2 - \varphi_{12}^2) z + \varphi_{12}^2 \varphi_{33}} . \tag{45a}$$

It is to be emphasized that the φ matrix elements in this equation are still functions of k and z; Eq. (45a) is the general result for $G_{nn}(kz)$ determined by the nonlocal kinetic equation. To obtain the transport parameters, we compare the small k, z limit of Eq. (45a) with the result determined by the Navier-Stokes equations,[2] which can be written

$$G_{nn}^{\text{hydro}}(kz) = \frac{z^2 + ik^2(\Gamma + D_T) z - k^2 c^2 (1 - \gamma^{-1})}{z^3 + ik^2(\Gamma + D_T) z^2 - k^2 c^2 z - ik^4 c^2 D_T} , \tag{45b}$$

where D_T is the thermal diffusivity, and

$$\Gamma = (\gamma - 1) D_T + (1/n)(\tfrac{4}{3}\eta + \zeta) \tag{46}$$

is the sound-attenuation coefficient, which contains the shear viscosity, the thermal diffusivity, and the bulk viscosity ζ. The thermodynamic parameters in these equations are the adiabatic speed of sound c, and the ratio of specific heats $\gamma = C_p / C_v$.

To exhibit the k and z dependence of $G_{nn}(kz)$ for small k and z, we must now expand the matrix elements $\varphi_{\alpha\beta}(kz)$ appearing in Eq. (45a). Forster and Martin show that it is sufficient to keep terms through second order in k and z (that is, terms through order k^2, kz, and z^2). To this order, the matrix elements of the memory kernel are given by

$$\varphi_{12}(kz) = k a_{12} , \tag{47a}$$

$$\varphi_{22}(kz) = -ik^2 a_{22}\,, \tag{47b}$$

$$\varphi_{23}(kz) = k\,a_{23} + ikz\,b_{23}\,, \tag{47c}$$

$$\varphi_{33}(kz) = -ik^2 a_{33} - z b_{33} - iz^2 c_{33}\,, \tag{47d}$$

where a, b, and c denote the coefficients (chosen to be real) belonging to the terms of order z^0, z^1, and z^2, respectively. We now regroup the terms in the numerator and denominator of Eq. (45a) and compare the coefficients of powers of z with those of the hydrodynamic formula, Eq. (45b). To leading order in k^2 in each coefficient, we obtain[21]

$$c^2 = a_{12}^2 + \frac{a_{23}^2}{1+b_{33}}\,, \tag{48a}$$

$$\gamma = \frac{c^2}{a_{12}^2} = 1 + \frac{a_{23}^2}{a_{12}^2(1+b_{33})}\,, \tag{48b}$$

$$D_T = \frac{1}{c^2}\,\frac{a_{12}^2 a_{33}}{1+b_{33}}\,, \tag{48c}$$

$$\Gamma = a_{22} - \frac{2a_{23}b_{23}}{1+b_{33}} + \frac{a_{23}^2 c_{33}}{(1+b_{33})^2} + \frac{a_{23}^2 a_{33}}{c^2(1+b_{33})^2}\,. \tag{48d}$$

These formulas are general expressions for the renormalized hydrodynamic parameters determined by the nonlocal kinetic equation, as is Eq. (39) for the viscosity. There is a natural separation of the a, b, and c coefficients into direct and kinetic parts. In Sec. III, we show that in the weak-coupling case, the thermal diffusivity D_T and the bulk viscosity ζ, which is obtained from Γ, also separate in this way. The bare results for the transport parameters are given by Eqs. (48) with coefficients $a_{ij} = a_{ij}^0$ obtained from the k, $z \to 0$ limit of the kernel, and b_{23}, b_{33}, and c_{33} set equal to zero.

III. WEAK-COUPLING APPROXIMATION

In Sec. II B we derived general expressions for the transport parameters in terms of the kinetic kernel $K(kzpp')$ and the initial condition $F(kpp')$. We now carry out the calculation of the transport coefficients using the weak-coupling approximation for K and F. For convenience, we gather in the next paragraphs the pertinent results from our previous paper.

The nonsingular part of the initial condition, Eq. (4), is given to first order by

$$H_1(kpp') = -\beta[v(k) + \epsilon v(p-p')]\,N_0(kp)\,N_0(kp') \times \frac{\tanh(\frac{1}{2}\beta\vec{k}\cdot\vec{p}) - \tanh(\frac{1}{2}\beta\vec{k}\cdot\vec{p}\,')}{\frac{1}{2}\beta\vec{k}\cdot(\vec{p}-\vec{p}\,')}\,, \tag{49}$$

and the term $N(kp)$ multiplying the δ function in Eq. (4) is determined by

$$n_1(p) = -n_0(p)\tilde{n}_0(p') \times \int d\bar{p}\;\beta[v(0) + \epsilon v(p-\bar{p})]\,n_0(\bar{p})\,. \tag{50}$$

The first-order term of the static kernel is

$$K_1^{(s)}(kpp') = \Big(\beta[v(k) + \epsilon v(p-p')]\,M_0(kp) - \epsilon\,\delta(p-p') \times \int d\bar{p}\,\beta v(p-\bar{p}) M_0(k\bar{p})\Big) N_0(kp')\,, \tag{51}$$

where

$$M_0(kp) = \beta^{-1}[n_0(p-\tfrac{1}{2}k) - n_0(p+\tfrac{1}{2}k)]\,. \tag{52}$$

Note that $K_1^{(s)}$ is odd in k. This is true of $K^{(s)}$ in general. The first-order term (51) involves only one-particle functions, and can be obtained from a mean-field or Hartree-Fock approximation to the equation of motion.[9] It should be mentioned that higher-order terms of $K^{(s)}$ contain two- and three-particle correlations, and are not of the Hartree-Fock form. In our previous paper we evaluated $F(kpp')$ and $K^{(s)}$ to second order, but we do not reproduce those terms here.

The dynamic part of the kernel begins with the second-order term, which can be written in the symmetric form

$$K_2^{(d)}(kzpp') = \tfrac{1}{8}(2\pi)^{-3}\int dp_1\,dp_2\,dp_3\,dp_4\,W(1234,p)\,W(1234,p') \times [\hat{A}_0(1234\,|\,k,z) - \hat{A}_0(1234\,|\,-k,-z)]\,, \tag{53}$$

where

$$\begin{aligned} W(1234,p) = {} & [v(p_1-p_3) + \epsilon v(p_1-p_4)]\,\delta(p_1-p) + [v(p_2-p_4) + \epsilon v(p_2-p_3)]\,\delta(p_2-p) \\ & -[v(p_3-p_1) + \epsilon v(p_3-p_2)]\,\delta(p_3-p) - [v(p_4-p_2) + \epsilon v(p_4-p_1)]\,\delta(p_4-p)\,, \end{aligned} \tag{54}$$

$$\hat{A}_0(1234\,|\,k,z) = \delta(p_1+p_2-p_3-p_4+k)\,\frac{A_0(p_1+\frac{1}{2}k,\,p_2+\frac{1}{2}k,\,p_3-\frac{1}{2}k,\,p_4-\frac{1}{2}k)}{z - E(p_1+\frac{1}{2}k,\,p_2+\frac{1}{2}k,\,p_3-\frac{1}{2}k,\,p_4-\frac{1}{2}k)}\,, \tag{55}$$

$$A_0(p_1p_2p_3p_4) = \tfrac{1}{2}[n_0(p_1)n_0(p_2)\tilde{n}_0(p_3)\tilde{n}_0(p_4) + \tilde{n}_0(p_1)\tilde{n}_0(p_2)n_0(p_3)n_0(p_4)]\,, \tag{56}$$

$$E(p_1p_2p_3p_4) = \tfrac{1}{2}(p_1^2 + p_2^2 - p_3^2 - p_4^2)\,. \tag{57}$$

In the local limit, this reduces to $K_2^{(d)}(0, i0^+; pp') = iJ_2(pp')$, where

$$J_2(pp') = -\tfrac{1}{2}(2\pi)^{-3}\int dp_2\, dp_3\, dp_4 \{2\pi[v(p-p_3)+\epsilon v(p-p_4)]^2 n_0(p) n_0(p_2)\tilde{n}_0(p_3)\tilde{n}_0(p_4)\delta(p+p_2-p_3-p_4)$$
$$\times\, \delta(E(pp_2p_3p_4))[\delta(p-p')+\delta(p_2-p')-\delta(p_3-p')-\delta(p_4-p')]\}. \qquad (58)$$

J_2 is the kernel of the linearized quantum Boltzmann equation with the transition probability evaluated in the Born approximation.

A. Shear viscosity

The two terms of the shear viscosity, η' and η'', are obtained from Eqs. (38) and (39). With the weak-coupling approximation for the kernel, the appropriate value of the density in Eq. (39) is the one that occurs in the initial value of the transverse-current correlation function, namely, n expanded to second order.[22]

The result for the kinetic part of the shear viscosity, η', is given directly by Eqs. (40) and (41) in terms of the off-diagonal element $T_{xz}(p)$ of the momentum flux. (Since this is evaluated at $k=0$ all the off-diagonal elements are equal.) The static part of the momentum flux is

$$T^{(s)}_{ij}(p) = p_i p_j N_0(0,p) + \frac{p_i p_j}{p} \times \frac{d}{dp}\left(N_0(0,p)\int d\bar{p}\,[v(0)+\epsilon v(p-\bar{p})]n_0(\bar{p})\right) \qquad (59)$$

plus a second-order contribution which we do not write out. The zero-order term in Eq. (59) is the free streaming momentum flux, as noted in Sec. II. The first-order corrections come from both $\vec{k}\cdot\vec{p}F(kpp')$ and $K^{(s)}(kpp')$. In classical mechanics, the static kernel does not contribute at all to the shear viscosity. The dynamical part of the momentum flux is found to be

$$T^{(d)}_{ij}(p) = (2\pi)^{-3}\mathcal{P}\int dp_2\, dp_3\, dp_4\, \delta(p+p_2-p_3-p_4)\,[E(pp_2p_3p_4)]^{-1}A_0(pp_2p_3p_4)$$
$$\times[v(p-p_3)+\epsilon v(p-p_4)]\,[u_{ij}(p-p_3)+\epsilon u_{ij}(p-p_4)], \qquad (60)$$

where

$$u_{ij}(p) = \frac{p_i p_j}{p}\frac{dv(p)}{d|p|} + \delta_{ij}v(p) \qquad (61)$$

and $\mathcal{P}$ indicates the principal value of the integral.

We now consider the direct part of the shear viscosity η''. As in the classical case, only the dynamical part of the kernel contributes. From Eq. (39), we have

$$\eta'' = \lim_{k\to 0}\frac{in}{k^2}\langle u_x|\Omega^{(d)}(k, i0^+)|u_x\rangle, \qquad (62)$$

which reduces to

$$\eta'' = \beta(32\pi^2)^{-1}\int dp_1\, dp_2\, dp_3\, dp_4 A_0(p_1p_2p_3p_4)$$
$$\times\,\delta(p_1+p_2-p_3-p_4)\delta(E(p_1p_2p_3p_4))$$
$$\times[u_{xz}(p_1-p_3)+\epsilon u_{xz}(p_1-p_4)]^2. \qquad (63)$$

Obviously, η'' is positive. As noted in Sec. III, it comes from a term of order k^2z^0 of the kernel, and therefore does not appear in a local theory. It can be seen that η'' begins at second order in the density, since at low density $A_0(p_1p_2p_3p_4)$ reduces to $\frac{1}{2}n^2[\phi(p_1)\phi(p_2)+\phi(p_3)\phi(p_4)]$.

It is interesting to express η'' in a form that allows us to examine the classical limit. The inverse of the change of variables used to obtain Eq. (4.21) in Ref. 5 gives

$$\eta'' = \beta\int\frac{dk\, dp\, dp'}{8(2\pi)^2}\,\frac{N_0(kp)N_0(kp')}{\cosh(\frac{1}{2}\beta\hbar\vec{k}\cdot\vec{p})}\,\delta(\vec{k}\cdot(\vec{p}-\vec{p}'))$$
$$\times\left[u_{xz}(k)+\epsilon u_{xz}\left(\frac{p-p'}{\hbar}\right)\right]^2. \qquad (64)$$

In the classical limit $N_0(kp)$ becomes $n_0\phi(p)$, the exchange term ϵu_{xz} vanishes, and we recover the expression derived by Forster and Martin. The classical η'' has a simple density and temperature dependence, but this does not carry over to the quantum result.

B. Speed of sound and C_p/C_v

In the weak-coupling approximation, the expression (48a) for the square of the speed of sound can be reduced to

$$c^2 = n_{(2)}\Big/\left[n\left(\frac{\partial n}{\partial P}\right)_\beta\right]_{(2)} + \frac{(\beta/n)A^2}{B}, \qquad (65)$$

where the subscript (2) indicates expansion through second order, and

$$A = \int dp\,[T^{(s)}_{33}(p)+T^{(d)}_{33}(p)]\epsilon(p), \qquad (66a)$$

$$B = F_{\epsilon\epsilon}(0) - \int dp\,\mathcal{E}(p)\tfrac{1}{2}p^2. \qquad (66b)$$

The relation (65), without the implied expansions, also applies in general. The first term on the right-hand side of Eq. (65) comes from the initial conditions $F_{nn}(k=0)$, Eq. (14), and $F_{zz}(k=0) = F_{xx}(k=0)$, Eq. (15). The momentum flux elements $T_{33}(p)$ are given by Eqs. (59) and (60), while the weak-coupling result for the function $\mathcal{E}(p)$ defined in Eq. (32b) is

$$\begin{aligned}\mathcal{E}(p_1) = -(2\pi)^{-3}\mathcal{P}\int dp_2\, dp_3\, dp_4 A_0(p_1p_2p_3p_4) \\ \times[E(p_1p_2p_3p_4)]^{-1} \\ \times\delta(p_1+p_2-p_3-p_4) \\ \times[v(p_1-p_3)+\epsilon v(p_1-p_4)]^2 .\end{aligned} \tag{67}$$

Equation (65) is to be compared with the hydrodynamic result for the adiabatic speed of sound, $c^2 = (\partial P/\partial n)_S$, expressed in the form

$$c^2 = \left(\frac{\partial P}{\partial n}\right)_\beta + \frac{\beta}{n}\left[-\left(\frac{\partial P}{\partial \beta}\right)_n\right]^2 \Big/ \frac{nC_v}{\beta^2} . \tag{68}$$

One can quickly verify that the first terms on the right-hand sides of Eqs. (65) and (68) agree to second order. The remaining terms are more difficult. We might begin by asking whether the numerators and denominators agree separately, that is, whether it is true that

$$\int dp\,[T_{33}^{(s)}(p)+T_{33}^{(d)}(p)]\,\epsilon(p) \overset{?}{=} -\left(\frac{\partial P}{\partial \beta}\right)_n \tag{69a}$$

and

$$F_{\epsilon\epsilon}(0) - \int dp\, \mathcal{E}(p)\tfrac{1}{2}p^2 \overset{?}{=} \frac{nC_v}{\beta^2} . \tag{69b}$$

Although these conditions do hold classically,[23,24] it turns out that neither of them is true in the quantum case. This is seen by a comparison of the perturbation expansions of the various terms. Of course, the disagreement vanishes in the classical limit.

In order for the two expressions for c^2 to agree, though, it is necessary only that

$$\frac{A^2}{B} = \frac{\beta^2}{nC_v}\left[\left(\frac{\partial P}{\partial \beta}\right)_n\right]^2 \tag{70}$$

be true. We have verified this to first order in perturbation theory: the two stronger conditions (69) fail at this order but the errors compensate when the quotient is formed. The perturbation expansion of the right-hand side is extremely laborious, unfortunately, and we have not carried it out to higher order. We conjecture that Eq. (70) is correct in general.

With the expressions (48a) and (65) for c^2, the value of γ given by Eq. (48b) is seen to be the square of the ratio of the adiabatic and isothermal speeds of sound, which is, of course, identical to the ratio C_P/C_V.

C. Thermal diffusivity

From Eq. (48c), we write the thermal diffusivity as

$$D_T = a_{33}/\gamma(1+b_{33}) . \tag{71}$$

As usual, this breaks up into $D_T = D_T' + D_T''$. Each of these terms can be seen to contain an overall factor of $(\gamma B)^{-1}$. In the classical limit this factor becomes $\beta^2/\gamma n C_V = \beta^2/nC_P$, and one can rearrange the formulas to obtain the thermal conductivity λ from the hydrodynamic formula $D_T = \lambda(nC_P)^{-1}$. A similar procedure is not evident in the quantum-mechanical case, however, since one no longer has the condition (69b) to give the appropriate value of the specific heat, and therefore we work with D_T itself.

The contribution D_T'' can be reduced to

$$\begin{aligned}D_T'' = \frac{1}{\gamma B}\int \frac{dp_1\, dp_2\, dp_3\, dp_4}{24(2\pi)^2} A_0(p_1p_2p_3p_4)\delta(p_1+p_2-p_3-p_4)\delta(E(p_1p_2p_3p_4)) \\ \times\left((p_1+p_2)_i[v(p_1-p_3)+\epsilon v(p_1-p_4)] - \frac{1}{2}\frac{\partial}{\partial p_{1i}}[v(p_1-p_3)(p_2^2-p_4^2)+\epsilon v(p_1-p_4)(p_2^2-p_3^2)]\right)^2 ,\end{aligned} \tag{72}$$

which is positive, and which has the correct classical limit.

The remaining contribution is given by an integral equation like the one for η' discussed in Sec. II B. It is

$$D_T' = (\gamma B)^{-1}\int dp\, \chi^D(p) g^D(p) , \tag{73}$$

where g^D satisfies

$$\int dp'\, J(pp')g^D(p') = -\chi^D(p) , \tag{74}$$

with

$$\chi^D(p) = Q_z(p) - \beta A n^{-1}\int dp'\, F(0,pp')p_z' . \tag{75}$$

The term subtracted from $Q_z(p)$ ensures the orthogonality of $\chi^D(p)$ to the null space of J. The leading term in the static part of the energy flux is

$$Q_i^{(s)}(p) = \epsilon(p)p_i\left[N_0(0,p) + \frac{1}{p}\frac{d}{dp}\left(N_0(0,p)\int d\bar{p}\,[v(0)+\epsilon v(p-\bar{p})]\,n_0(\bar{p})\right)\right] . \tag{76}$$

The dynamic part of the energy flux is

$$Q_i^{(d)}(p_1) = -\int \frac{dp_2\,dp_3\,dp_4}{2(2\pi)^3}\,[v(p_1-p_3)+\epsilon v(p_1-p_4)]^2\delta(p_1+p_2-p_3-p_4)\left(\frac{\partial}{\partial p_{1i}}-\frac{\partial}{\partial p_{2i}}-2\frac{\partial}{\partial p_{3i}}\right)A_0(p_1p_2p_3p_4)$$
$$-\mathcal{P}\int\frac{dp_2\,dp_3\,dp_4}{(2\pi)^3}\,[v(p_1-p_3)+\epsilon v(p_1-p_4)]\delta(p_1+p_2-p_3-p_4)\frac{A_0(p_1p_2p_3p_4)}{E(p_1p_2p_3p_4)}$$
$$\times\left(-(p_1+p_2)_i[v(p_1-p_3)+\epsilon v(p_1-p_4)]+\frac{1}{2}\frac{\partial}{\partial p_{1i}}[v(p_1-p_3)(p_2^2-p_4^2)+\epsilon v(p_1-p_4)(p_2^2-p_3^2)]\right). \qquad (77)$$

In the classical limit this expression for $Q^{(d)}$ simplifies greatly. The first group of terms, which is due essentially to the k dependence of $n_0(p+\frac{1}{2}\hbar k)$, vanishes, while the principal-value integral reduces to the Forster and Martin result.

D. Bulk viscosity

We obtain the bulk viscosity ζ from the sound attenuation coefficient Γ, Eq. (46), by subtracting the contributions from the thermal diffusivity and shear viscosity. From Eq. (48d), we have

$$\frac{1}{n}\left(\frac{4}{3}\eta+\zeta\right)=a_{22}-\frac{2a_{23}b_{23}}{1+b_{33}}+\frac{a_{23}^2c_{33}}{(1+b_{33})^2}\,. \qquad (78)$$

In combination with the expressions for η obtained above, this gives a result for $\zeta=\zeta'+\zeta''$ that can be put into a manifestly positive form. For ζ'', we obtain

$$\zeta''=\beta\int\frac{dp_1\,dp_2\,dp_3\,dp_4}{8(2\pi)^2}A_0(p_1p_2p_3p_4)$$
$$\times\delta(p_1+p_2-p_3-p_4)\delta(E(p_1p_2p_3p_4))$$
$$\times\left(u(p_1-p_3)+\epsilon u(p_1-p_4)\right.$$
$$\left.-\frac{A}{B}[v(p_1-p_3)+\epsilon v(p_1-p_4)]\right)^2, \qquad (79)$$

where [see Eq. (61)]

$$u(p)=\tfrac{1}{3}u_{ii}(p)=\tfrac{1}{3}\vec{p}\cdot\vec{\nabla}v(p)+v(p)\,. \qquad (80)$$

The remaining contribution is given by

$$\zeta'=\beta\int dp\,\chi^{\zeta}(p)g^{\zeta}(p)\,, \qquad (81)$$

where $g^{\zeta}(p)$ satisfies

$$\int dp'\,J(pp')g^{\zeta}(p')=-\chi^{\zeta}(p)\,, \qquad (82)$$

with

$$\chi^{\zeta}(p)=\tfrac{1}{3}T_{ii}(p)+\frac{A}{B}\mathcal{E}(p)$$
$$-\int dp'\left(\frac{n}{\beta F_{nn}}+\frac{A}{B}\epsilon(p')\right)F(0,pp')\,. \qquad (83)$$

In contrast to χ^{η} for the shear viscosity and χ^{D} for the thermal diffusivity, the inhomogeneity χ^{ζ} vanishes for a noninteracting system. At low density it is of order n^2, and therefore ζ' also begins at order n^2.

IV. LOW-TEMPERATURE FERMI SYSTEM

The weak-coupling expressions of Sec. III can be applied to either a Bose or a Fermi normal fluid, at arbitrary temperature. In this section, we consider the special case of the Fermi fluid in the low temperature limit, $\beta\mu\to\infty$.

We begin with the direct contribution to the viscosity, η''. It is convenient to put the integrand in Eq. (63) in invariant form, by replacing the final factor with $\frac{1}{10}[\bar{u}_{ij}(p_1-p_3)-\bar{u}_{ij}(p_1-p_4)]^2$, where $\bar{u}_{ij}$ is the traceless part of u_{ij} and a sum over Cartesian indices is implied. Reducing this factor, we have

$$\eta''=\beta\int\frac{dp_1\,dp_2\,dp_3\,dp_4}{80(2\pi)^2}n_0(p_1)n_0(p_2)\tilde{n}_0(p_3)$$
$$\times\tilde{n}_0(p_4)\delta(p_1+p_2-p_3-p_4)$$
$$\times\delta(E(p_1p_2p_3p_4))w_{\eta}(p_1-p_3,p_1-p_4)\,, \qquad (84)$$

where

$$w_{\eta}(pp')=\tfrac{2}{3}p^2[v'(p)]^2+\tfrac{2}{3}p'^2[v'(p')]^2$$
$$+\tfrac{2}{3}pp'v'(p)v'(p')-\frac{2(\vec{p}\cdot\vec{p}')^2}{pp'}v'(p)v'(p')\,. \qquad (85)$$

The right-hand side of Eq. (84) has the same structure as the collision integral appearing in the Landau theory, and can be evaluated (to leading order in the temperature) by the method of Abrikosov and Khalatnikov.[16] We define three dimensionless energies

$$t=\beta(\tfrac{1}{2}p_1^2-\mu),\quad x=\beta(\tfrac{1}{2}p_3^2-\mu),\quad y=\beta(\tfrac{1}{2}p_4^2-\mu), \qquad (86)$$

and three angles: θ and ϕ_2 are the polar and azimuthal angles of p_2 with respect to p_1, and ϕ is the angle between the planes determined by (p_1,p_2) and (p_3,p_4). After the change of variables, the low-temperature limit of Eq. (84) becomes

$$\eta'' = \frac{p_F}{40(2\pi)^7\beta^2}\int_{-\infty}^{\infty} dt\,dx\,dy\,h(t)\,h(x+y-t)[1-h(x)][1-h(y)]\int_0^{\pi} d\theta\int_0^{\pi} d\phi\int_0^{2\pi} d\phi_2 \sin(\tfrac{1}{2}\theta) w_\eta(\theta\phi)\,, \tag{87}$$

where $h(t)=(e^t+1)^{-1}$, $p_F=(2\mu)^{1/2}$ is the Fermi momentum, and $w_\eta(\theta\phi)$ is $w_\eta(p_1-p_2, p_1-p_4)$ with all the momenta restricted to the Fermi surface, namely,

$$w_\eta(\theta\phi)=\tfrac{8}{3}p_F^2\bar{w}_\eta(\tfrac{1}{2}\theta,\tfrac{1}{2}\phi)\,, \tag{88}$$

$$\begin{aligned}\bar{w}_\eta(\alpha,\alpha') &= \sin^2\alpha\sin^2\alpha'[v'(2p_F\sin\alpha\sin\alpha')]^2\\ &\quad+\sin^2\alpha\cos^2\alpha'[v'(2p_F\sin\alpha\cos\alpha')]^2\\ &\quad+\sin^2\alpha\sin\alpha'\cos\alpha'\; v'(2p_F\sin\alpha\sin\alpha')\\ &\quad\times v'(2p_F\sin\alpha\cos\alpha')\,.\end{aligned} \tag{89}$$

Since w_η depends only on θ and ϕ, the ϕ_2 integration in Eq. (87) is trivial. The integrals involving t, x, and y, though not trivial, can also be done exactly,[25] and give $\frac{2}{3}\pi^2$. Thus, the final result for η'' is

$$\eta''=\frac{p_F^3}{360\pi^4\beta^2}\int_0^{\pi/2} d\alpha\int_0^{\pi/2} d\alpha'\sin\alpha\,\bar{w}_\eta(\alpha,\alpha')\,. \tag{90}$$

Note that this vanishes like T^2 and, as we shall see, is therefore not the dominant contribution to the low-temperature shear viscosity.

A similar analysis gives the direct contributions to the thermal diffusivity and bulk viscosity. The end results are

$$D_T''=\frac{p_F^2}{12\pi^4}\int_0^1 du\int_0^{\pi/2} d\alpha\, u(1-u^2)^{1/2}w(u,\alpha)\,, \tag{91}$$

$$\begin{aligned}\zeta''=\frac{p_F^3}{216\pi^4\beta^2}&\int_0^1 du\int_0^{\pi/2} d\alpha\, u(1-u^2)^{-1/2}\\ &\times\left(u\sin\alpha\, v'(2p_F u\sin\alpha)-u\cos\alpha\, v'(2p_F u\cos\alpha)+\frac{1}{2p_F}[w(u,\alpha)]^{1/2}\right)^2,\end{aligned} \tag{92}$$

where

$$w(u,\alpha)=[v(2p_F u\sin\alpha)-v(2p_F u\cos\alpha)]^2\,. \tag{93}$$

In evaluating these quantities we have used expressions for γB and A/B appropriate to the low-temperature free gas; thus, Eqs. (91) and (92) are correct to order v^2. Since the free-particle specific heat is $(nC_p)_0=p_F/6\beta$ at low temperatures, the thermal conductivity corresponding to D_T is proportional to T.

We now turn to the kinetic contributions to the transport coefficients. The task is to construct the appropriate inhomogeneities and to solve the three integral equations (41), (74), and (82), using the low-temperature limit of the local kernel $J(p,p')$. It is evident that $J(p,p')$ has the same form as the collision integral that appears in the Landau theory. Exact solutions to the Landau theory transport equations have been obtained by Sykes and Brooker,[26] and Højgård Jensen, Smith, and Wilkins.[27] The same methods can be used to obtain exact solutions in our problem also. In the next paragraph, we collect the pertinent tools.

First, we need the leading term of the local kernel, which is proportional to T^2. Following Sykes and Brooker, we see that to this order the action of J on the product of a spherical harmonic and any function of $t'=\beta(\frac{1}{2}p'^2-\mu)$ is given by

$$\int dp'\,J(p_1,p')\,Y_{lm}(\hat{p}')\psi(t')=-\frac{\bar{w}}{4(2\pi)^6\beta^2}\,Y_{lm}(\hat{p}_1)e(t)\int_{-\infty}^{\infty} dt'\,K(t,t')\,[\psi(t)-\lambda_{ls}\psi_s(t')-\lambda_{la}\psi_a(t')]\,, \tag{94}$$

where ψ_s and ψ_a are the symmetric and antisymmetric parts of ψ, and

$$e(t)=[(e^t+1)(e^{-t}+1)]^{-1}\,, \tag{95}$$

$$K(t,t')=\frac{\cosh\frac{1}{2}t}{\cosh\frac{1}{2}t'}\;\frac{\frac{1}{2}(t-t')}{\sinh\frac{1}{2}(t-t')}\,. \tag{96}$$

The real quantities $\bar{w}$, λ_{ls}, and λ_{la} are angular integrals of the scattering probability on the Fermi surface. In particular, we need

$$\begin{bmatrix} \bar{w} \\ \bar{w}\lambda_{2s} \\ \bar{w}\lambda_{1a} \end{bmatrix} = \frac{4}{\pi}\int_0^1 du \int_0^{\pi/2} d\alpha\, \frac{uw(u,\alpha)}{(1-u^2)^{1/2}} \begin{bmatrix} 1 \\ 1-3u^4\sin^2 2\alpha \\ 3-4u^2 \end{bmatrix} \begin{matrix} (97a) \\ (97b) \\ (97c) \end{matrix}$$

with $w(u, \alpha)$ given by Eq. (93). Next, we need the integrals

$$\begin{bmatrix} I_\eta \\ I_D \\ I_\xi \end{bmatrix} = \int_{-\infty}^{\infty} dt\, e(t) \begin{bmatrix} q_\eta(t) \\ tq_D(t) \\ (t^2-\frac{1}{3}\pi^2)q_\xi(t) \end{bmatrix}, \begin{matrix} (98a) \\ (98b) \\ (98c) \end{matrix}$$

where $q_\eta, q_\xi,$ and q_D are solutions to

$$\int_{-\infty}^{\infty} dt'\, K(t,t') \begin{bmatrix} q_\eta(t) - \lambda_{2s}q_\eta(t') \\ q_D(t) - \lambda_{1a}q_D(t') \\ q_\xi(t) - q_\xi(t') \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ t^2-\frac{1}{3}\pi^2 \end{bmatrix}, \begin{matrix} (99a) \\ (99b) \\ (99c) \end{matrix}$$

with $q_\eta(t)$ and $q_\xi(t)$ even and $q_D(t)$ odd in t. The integrations in Eqs. (98) remove any nonuniqueness in the solutions, in accord with the remarks in Sec. III. (In this case, q_η and q_D are unique, while q_ξ is unique except for an additive constant.) In terms of the Sykes and Brooker functions

$$c(\lambda) = \frac{1-\lambda}{4}\sum_{n=0}^{\infty}\frac{(4n+3)}{(n+1)(2n+1)[(n+1)(2n+1)-\lambda]} \tag{100}$$

and

$$H(\lambda) = \frac{3-\lambda}{4}\sum_{n=0}^{\infty}\frac{(4n+5)}{(n+1)(2n+3)[(n+1)(2n+3)-\lambda]}, \tag{101}$$

the results are

$$I_\eta = [2/\pi^2(1-\lambda_{2s})]\,c(\lambda_{2s}), \tag{102}$$

$$I_D = [2/(3-\lambda_{1a})]\,H(\lambda_{1a}), \tag{103}$$

$$I_\xi = \tfrac{8}{27}\pi^2. \tag{104}$$

The calculations for the kinetic parts of the transport coefficients are now straightforward. For the weak-coupling approximation, we retain only the leading term in the potential expansion of the inhomogeneities. At low temperature, the inhomogeneity for the shear viscosity is then

$$T_{xz}(p_1) = (2\pi)^{-3}p_{1x}p_{1z}e(t). \tag{105}$$

With Eq. (94) for J, Eqs. (40) and (41) determining η' reduce to the system (98a) and (99a). We then have $\eta' = 64\pi\beta^2\mu^2 p_F I_\eta/15\bar{w}$, or

$$\eta' = \tfrac{8}{45}\beta^2 p_F^5\, c(\lambda_{2s}) \times\left(\int du\, d\alpha\, u^5(1-u^2)^{-1/2}\sin^2 2\alpha\, w(u,\alpha)\right)^{-1}. \tag{106}$$

This has the same T^{-2} dependence and general form as the viscosity calculated in the Landau theory. If the potential is scaled as $\mathcal{E}\hat{v}(r)$, then we have $\eta' \sim T^{-2}\mathcal{E}^{-2}$, since λ_{2s} is independent of the strength of the potential.

Similarly, the inhomogeneity for the thermal diffusivity is

$$\chi^D(p_1) = (2\pi)^{-3}\beta^{-1}p_{1z}t\,e(t). \tag{107}$$

Note that in the low-temperature limit, the second term on the right-hand side of Eq. (75) does not contribute. The system determining D_T' reduces to Eqs. (98b) and (99b), and we obtain $D_T' = 64\pi\beta^2\mu I_D/\bar{w}$, or

$$D_T' = 4\pi^2\beta^2 p_F^2 H(\lambda_{1a})\left(\int du\, d\alpha\, u^3(1-u^2)^{-1/2}w(u,\alpha)\right)^{-1}. \tag{108}$$

Again, we have the characteristic T^{-2} dependence of the Landau theory result. (With the free gas specific heat, the thermal conductivity goes like T^{-1}.)

The corresponding contribution to the bulk viscosity is considerably more difficult to evaluate since its inhomogeneity vanishes in a noninteracting system. After a rather lengthy calculation, we find that to leading order in perturbation theory it is

$$\chi_1^\xi(p) = \tfrac{1}{3}\beta N_0(0,p)\int dp'\, dp''\left(\delta(p-p') - \frac{N_0(0,p')}{N_0} - \frac{\epsilon_0(p)\epsilon_0(p')N_0(0,p')}{F_{\epsilon\epsilon}^0(0)}\right) \times\bar{v}(p'-p'')N_0(0,p'')(p''^2 - \vec{p}'\cdot\vec{p}''), \tag{109}$$

where $\bar{v}(p) = v(0) + \epsilon v(p)$ and $N_0 = \int dp\, N_0(0,p)$. Equation (109) holds for either statistics. The low-temperature limit of this in a Fermi system is, after much reduction,

$$\chi_1^{\zeta}(p_1) = [p_F^3 \Xi / 3(2\pi)^5 \beta^2 \mu^2] e(t)(t^2 - \tfrac{1}{3}\pi^2), \qquad (110)$$

with

$$\Xi = \frac{1}{2}\int_0^1 du\, u[(1-2u^2)\bar{v}(2p_F u) - p_F u(1-u^2)v'(2p_F u) + 2p_F^2 u^4 v''(2p_F u)]. \qquad (111)$$

With this result, the equations for ζ' reduce to the system (98c) and (99c). The final result for this part of the bulk viscosity is then $\zeta' = 4p_F^3 \Xi^2 I_\zeta / 9\pi^3\beta^2\mu^2\overline{w}$, or

$$\zeta' = \frac{32\Xi^2}{243 p_F \beta^2}\left[\int du\, d\alpha\, u(1-u^2)^{-1/2} w(u,\alpha)\right]^{-1}, \qquad (112)$$

which vanishes at T^2.

Summarizing the results of this section, we note that the dominant contributions to the shear viscosity and the thermal diffusivity are the kinetic terms. These diverge at zero temperature like T^{-2} and have the same form as the results predicted by the Landau theory. The direct contributions to the shear viscosity and thermal diffusivity are proportional to T^2 and T^0, respectively. The bulk viscosity is down by a factor of T^4 with respect to the shear viscosity and therefore contributes negligibly to sound attenuation. An interesting point, though, is that the kinetic and direct contributions to the bulk viscosity have the same T^2 temperature dependence.

V. DISCUSSION

In conclusion, we present a qualitative discussion of the renormalization corrections to the transport coefficients, emphasizing the idea that these corrections will eventually dominate as the density and interaction strength of the fluid increase. As we have noted, the general form of the quantum transport coefficients is the same as for classical fluids, so most of the following comments apply to both cases. Let us suppose that the kernel scales roughly like σ, which is some combination of density and interaction strength. From Eqs. (38) and (39), we see that the kinetic part of the shear viscosity has a term which begins at order σ^{-1}. This is the bare result, which would be obtained if we calculated the viscosity directly from a local kinetic equation as in the kinetic theory of gases.[28] However, the renormalized kinetic part also has terms of order σ^0 and σ^1, coming from collisional contributions to the stress tensor. The direct part of the shear viscosity, Eq. (62), clearly goes like σ^1. Thus, for sufficiently low densities or weak interactions, where σ is very small, only the bare σ^{-1} term of the kinetic part will be important, while for very large σ, the direct part and the σ^1 term of the kinetic part will dominate. In the Fermi liquid at low temperature, however, the effective strength of the interaction is curtailed by the exclusion principle. This is of course the basis of the Landau theory. If σ is measured by the quasiparticle scattering rate, it is proportional to T^2 at low temperature. Thus, no matter how large the interaction, there is always a temperature low enough so that the bare term dominates the shear viscosity. It is also known from the calculations of Emery and of Dy and Pethick,[17] based on the Landau theory, that the leading corrections to the shear viscosity come from the kinetic part rather than the direct part, and this also is what we expect. (However, this oversimplified argument would lead us to guess that the correction is proportional to T^0 rather than to T^{-1} as is in fact the case.)

The thermal diffusivity behaves like the shear viscosity as a function of interaction strength, but the bulk viscosity is qualitatively different because of the fact that the noninteracting part of the inhomogeneity χ^{ζ} vanishes. Thus, there is no bare contribution to the bulk viscosity, in accord with a result from the kinetic theory of monatomic gases. The direct term goes like σ^{-1} as usual. For the weakly coupled Fermi liquid, we found that the kinetic term begins at order v^0 and the direct term at order v^2.

Finally, it is interesting to ask whether the kinetic and direct parts of the transport coefficients reflect different physical mechanisms. We have pointed out that the kinetic part reduces to the bare result in the low-density or weak-interaction limit, where the dominant transport mechanism is the streaming of free particles, and the role of collisions is mainly to establish local equilibrium. In a dense fluid, in which free-particle motion is greatly suppressed, the dominant transport mechanism is the collisional transfer of momentum and energy. It is tempting to speculate that collisional transfer is mainly accounted for by the direct term, and that the kinetic term describes the contribution of particle transport, including the effects of collisions on the motion. This correlates with the result of Jhon and Forster, which we have already quoted, that for liquid argon near the triple point, the direct terms are much larger than the kinetic terms. It should be remembered, though, that there are renormalization contributions to both the direct and the kinetic terms, and it is possible that there is no clean separation of the different mechanisms. We suggest that systematic studies of the density and interaction strength dependence of the direct and kinetic terms would be very worthwhile.

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where

$$B_0(p_1p_2p_3p_4) = n_0(p_1)n_0(p_2)\tilde{n}_0(p_3)\tilde{n}_0(p_4) - \tilde{n}_0(p_1)\tilde{n}_0(p_2)n_0(p_3)n_0(p_4).$$

Also, the factor $\delta(p-p')$ should be deleted from the second term on the right-hand side of Eq. (4.7); in Eq. (4.13), the argument of coth( ) should be $\frac{1}{4}\beta(p^2-2\mu)$; Eq. (4.10) should read $n_1(p) = n_0(p)\sigma_1(p)\tilde{n}_0(p)$.

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