High-order behavior in ϕ^3 field theories and the percolation problem

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The percolation problem is normally described in field theory by the $n \to 0$ limit of the n-component Potts model. This model has trilinear interactions which give rise to an ϵ expansion in $6-\epsilon$ dimensions. In contrast to positive values of n, the $n=0$ case is shown to have oscillatory growth at high orders in ϵ which permits Padé-Borel resummation. A new formulation of the $n=0$ limit is required to obtain this result. An improved estimate is given for the critical exponent ω which describes corrections to scaling.

I. INTRODUCTION

There are many models for critical phenomena where symmetry permits interactions trilinear in the order parameter ϕ. These models fall into two classes. In the first the order parameter corresponds to a physical fluctuating quantity; examples can be found in magnetic and displacive phase transitions[1] and in a model for the nematic fluctuations in a liquid crystal.[2] In the second, one is required to take a limit in which the number of components n of the field ϕ tends to zero. Examples are found in the percolation problem,[3] which can be described by the $n \to 0$ limit of the $(n+1)$-state Potts model,[4] and in the Edwards-Anderson model[5] for a spin-glass, in which the $n \to 0$ limit is introduced to handle the quenched nature of the randomness.[5,6]

Since interactions with the lowest powers of the field in general dominate the critical behavior,[7] one is led to neglect in the first instance ϕ^4 and higher interactions and to consider only the ϕ^3 terms. Dimensional analysis shows that the ϕ^3 coupling in the Hamiltonian has the form $g_0\Lambda^{(6-d)/2}$ where Λ is the microscopic momentum cutoff in the problem and d is the dimension of space. The natural ϵ expansion in these models is then around six dimensions. Within the framework of the renormalization group, the running coupling constant $g(\tau)$ (the effective coupling at momentum scale $e^{\tau}\Lambda$) is the solution of the differential equation

$$\frac{dg}{d\tau} = \beta(g) \tag{1}$$

with initial conditions $g(0) = g_0$. The function β can be calculated as a power series in g; it has the general structure

$$\beta(g) = -\tfrac{1}{2}(6-d)g + Ag^3 + Bg^5 + \cdots \; . \tag{2}$$

If $A<0$, the theory is asymptotically free in six dimensions and the ϵ expansion exists in $6+\epsilon$ dimensions. If $A>0$, there is an infrared-stable fixed point g^* of the renormalization group with $g^{*2} = \epsilon/2A + O(\epsilon^2)$, $\epsilon = 6-d$, and this permits a description of critical phenomena in $6-\epsilon$ dimensions. Explicit calculations[8] show that A can be positive both for models with physical fluctuating fields and for $n \to 0$ limits; the existence of an ϵ expansion in $6-\epsilon$ dimensions does not appear to discriminate between these two cases.

An obvious problem for these ϵ expansions is that one must set $\epsilon=3$ to obtain predictions in three dimensions; numerical results cannot be expected to be very reliable. However, the very nature of the ϵ expansion in $6-\epsilon$ dimensions presents a much more serious problem which we study in this paper.

Following the pioneering work of Langer[9] and Bender and Wu[10] there is now a standard technique[11,12] for obtaining the high-order behavior of Feynman-graph expansions. If λ is a coupling constant which orders the diagrams according to the number of loops, one finds that an N-point correlation function behaves like

$$G^{(N)} = \sum G_K^{(N)} \lambda^K \; , \tag{3}$$

where

$$G_K^{(N)} \sim K!\,a^K K^b c\,[1 + O(K^{-1})] \tag{4}$$

for large classes of boson field theories[13]; the number a depends only on the theory under study; b depends only on the theory and N; the first momentum dependence appears in c. The β function of Eq. (1) also has the structure (4), with $\lambda = g^2$, and the Kth term in the ϵ expansion of g^{*2} grows correspondingly.

The $K!$ factor in Eq. (4) shows that these perturbation theories are asymptotic expansions with zero radius of convergence. The crucial factor then is the sign of a. If $a < 0$ the series (3) oscillates and one may obtain numerical results which converge using Padé-Borel or other techniques.[14] If $a > 0$, resummation techniques cannot be applied directly to the series. We call these two cases benign and malignant growths ($a < 0$ and $a > 0$, respectively). The semiclassical techniques which yield the result (4) also show how these growths can be characterized.[15] A benign perturbation series arises when one does perturbation theory about a proper ground state of the system. (The case with tunneling between two or more degenerate minima is special.) Malignant growth occurs when one sets up a perturbation theory about a metastable ground state. That perturbation theory alone is incomplete in this case is clear, because one neglects the exponentially small tunneling effects out of the metastable state.[16]

These are important remarks as far as ϵ expansions in $6-\epsilon$ dimensions are concerned, since these expansions are dominated by ϕ^3 interactions whose classical potential is unbounded below. As far as critical phenomena are concerned, one may imagine that the ϕ^4 terms which are always present in the bare Hamiltonian may provide stability, and indeed they do. However both explicit calculation for the $n=2$ Potts model[17] and experiment[18] indicate that the system then goes through a first-order phase transition. At best the ϵ expansion in $6-\epsilon$ dimensions for theories of physical fluctuating fields may describe a multicritical region which is not easily accessible experimentally.

In this paper we consider high-order behavior in ϕ^3 theories and its relationship to the stability of the ground state about which one does perturbation theory. In Sec. II we illustrate the general result that expansions in $6-\epsilon$ dimensions for all theories of physical fluctuating fields are malignant, with explicit reference to the n-component Potts model. The percolation problem, involving the "unphysical" limit $n \to 0$, may evade this result, but we show that the n-component Potts model is not a suitable vehicle for resolving the question definitively. In Sec. III we present an alternative field theory for the percolation problem which avoids the necessity of taking an $n \to 0$ limit. This new field theory is equivalent to the $n=0$ Potts model order by order in perturbation theory. In Sec. IV we show that the standard methods[12] may be applied to this new field theory; they show that perturbation theory and the ϵ expansion for the percolation problem are benign. The results are used to make an improved estimate of the correction to scaling exponent ω, which diverges particularly rapidly.

II. HIGH-ORDER BEHAVIOR WITH PHYSICAL FLUCTUATING FIELDS; THE POTTS MODEL

A. General formalism

We are interested in critical phenomena arising from a bare reduced Hamiltonian of the form

$$H = \int d^d x \left[\frac{1}{2}(\nabla \phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{1}{3!} g \Lambda^{(6-d)/2} d_{ijk} \phi_i \phi_j \phi_k \right] . \tag{5}$$

The order parameter ϕ_i $(i = 1, 2, \ldots, n)$ is taken to transform under an irreducible representation of some symmetry group; d_{ijk} is an invariant tensor allowing a ϕ^3 interaction (there may be more than one ϕ^3 invariant in general).

Many technical simplifications occur if one uses renormalization theory to set up the renormalization-group (RG) and ϵ expansions (Brézin *et al.*, Ref. 7). First, the RG action is closed in the space of ϕ^3 interactions; there is no need to introduce ϕ^4 or higher interactions in looking for a fixed point. Second, apart from the trivial term $\frac{1}{2}\epsilon g$, the remaining coefficients A, B, etc., in Eq. (2) are independent of ϵ and the mass if one uses the renormalization procedure of 't Hooft[19]; they are therefore obtainable from the ultraviolet divergences of the massless (critical) six-dimensional theory in this case.[20] This is important for our current ability to calculate high-order behavior analytically.[21]

The classic approach for estimating high-order behavior consists in evaluating the discontinuity of $G^{(N)}$ across cuts in the complex g plane by the method of steepest descents; this is the semiclassical method for calculating the tunneling phenomena which give rise to imaginary parts of $G^{(N)}$. Dispersion relations then enable one to obtain high-order behavior in the region of physical g. A calculation for ϕ^3 models in $6-\epsilon$ dimensions using this approach has been done by one of us [D.J.W. (unpublished)]; contours of integration can be identified (see also Langer[9]) and the small oscillations sum can be controlled conveniently by dimensional regularization. We do not report this rather lengthy calculation here because we have not succeeded in making the corresponding discussion of integration contours for the model of Secs. III and IV, which is our interest in this paper.

The discussion of integration contours may be side stepped, however, by picking out the Kth-order term in perturbation theory as follows: If we have

$$G^{(N)}(x_1, \ldots, x_N) = \langle \phi(x_1) \cdots \phi(x_N) \rangle \equiv \int D\phi\, \phi(x_1) \cdots \phi(x_N) e^{-H} , \tag{6}$$

then one writes

$$G_K^{(N)} = \int D\phi \frac{1}{2\pi i} \int \frac{dg}{g^{K+1}} \phi(x_1) \cdots \phi(x_N) e^{-H} . \tag{7}$$

To be specific, let us consider the partition function Z and pick out Z_K the coefficient of g^{2K} for the Hamiltonian (5):

$$Z_K = \int D\phi \frac{1}{2\pi i} \int \frac{dg}{g} e^{-(H+2K\ln g)} . \tag{8}$$

One now proceeds to evaluate this by steepest descents, which involves expanding around the saddle point of the "action" $H+2K\ln g$ as a function of the variables ϕ and g. The equations for the saddle point with the Hamiltonian (5) are (we take the $d=6$ massless case for reasons discussed previously)

$$\nabla^2\phi_i = \tfrac{1}{2} g d_{ijk}\phi_j\phi_k , \tag{9}$$

$$\frac{1}{3!}\int d^6x\, d_{ijk}\phi_i\phi_j\phi_k = \frac{-2K}{g} . \tag{10}$$

These equations decouple if we change from ϕ_i to $\psi_i = \frac{1}{2} g\phi_i$ and make the ansatz

$$\psi_i(x) = u_i\phi_c(x) , \tag{11}$$

where

$$u_i = d_{ijk}u_j u_k . \tag{12}$$

Then (9) and (10) become

$$\nabla^2\phi_c = \phi_c^2 \tag{13}$$

and

$$\frac{2}{3}\vec{u}\cdot\vec{u}\int d^6x\,\phi_c^3 = -Kg^2 . \tag{14}$$

The solutions of (13) (which give minimum action, and hence the saddle point of g^2 closest to $g=0$) are of the form

$$\phi_c(x) = -24\lambda^2/[\lambda^2(x-x_0)^2+1]^2 . \tag{15}$$

The parameters λ and x_0 reflect the dilatation and translational invariances of Eq. (13). The integral in Eq. (14) is independent of these parameters, of course; with ϕ_c given by Eq. (15) one finds a classical saddle point at $g=g_c$,

$$g_c^2 = 3.2^8(\vec{u}\cdot\vec{u})\pi^3/5K . \tag{16}$$

The value of the integrand of (8) at the saddle point (15),(16) gives the leading behavior of Z_K for K large,

$$\begin{aligned} Z_K &\sim \exp[-H(\phi_c,g_c) - 2K\ln g_c] \\ &= e^{-K}[5K/3.2^8(\vec{u}\cdot\vec{u})\pi^3]^K \\ &\sim K!a^K , \end{aligned}$$

where

$$a = 5/3.2^8(\vec{u}\cdot\vec{u})\pi^3 . \tag{17}$$

The terms involving b and c in expression (4) come from the determinant of the small oscillations about the classical solution (15),(16) and terms of order $1/K$ from the anharmonic perturbations.

For all theories in which u is a real vector, a in Eq. (17) is clearly positive. The correlation functions, therefore, have malignant growth in six dimensions as foreseen in the Introduction and this result carries over also to the behavior of the ϵ expansion.

B. The case of the Potts model

Since we are interested in studying the percolation problem we consider specifically the n-component Potts model. The invariant tensor d_{ijk} is conveniently written in terms of $n+1$ vectors e_i^α ($\alpha=1,2,\ldots,n+1$, $i=1,2,\ldots,n$) which obey

$$\sum_{\alpha=1}^{n+1} e_i^\alpha = 0 , \tag{18a}$$

$$\sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha = (n+1)\delta_{ij} , \tag{18b}$$

$$\sum_{i=1}^{n} e_i^\alpha e_i^\beta = (n+1)\delta^{\alpha\beta} - 1 . \tag{18c}$$

Geometrically, these are vectors to the $n+1$ vertices of the tetrahedron in n dimensions. For the Potts model one may write

$$d_{ijk} = \sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha e_k^\alpha . \tag{19}$$

In order to obtain the solutions u_i of Eq. (12) take a basis of the first n vectors e_i^α ($\alpha=1,2,\ldots,n$),

$$u_i = \sum_{\alpha=1}^{n} a_\alpha e_i^\alpha .$$

Substituting the equation into (12) and using the identities (18) gives the equation

$$a_\alpha = (n+1)^2 a_\alpha^2 - 2(n+1)\left(\sum_{j=1}^{n} a_j\right) a_\alpha . \tag{20}$$

The solutions of this equation have all nonzero a's the same. Hence, up to equivalence (by permutation of the vectors e^α), there are n solutions u_i^r ($r=1,2,\ldots,n$) of Eq. (12) of the form

$$u_i^r = a^r \sum_{\alpha=1}^{r} e_i^\alpha , \tag{21a}$$

where

$$a^r = 1/(n+1)(n+1-2r) . \tag{21b}$$

These n solutions of Eq. (12) give n inequivalent saddle points. The obvious prescription for choosing the correct saddle point is to take that corresponding to the minimum $\bar{u}\cdot\bar{u}$. This is the saddle point in g closest to $g=0$ and gives the maximum growth according to Eq. (17). (A proper justification requires analysis of the steepest-descent contours of integration in ϕ and g space.) One readily finds

$$\bar{u}^r\cdot\bar{u}^r=\frac{(n+1)r-r^2}{(n+1)^2(n+1-2r)^2},\quad r=1,2,\ldots,n\ . \tag{22a}$$

For fixed n this has its minimum value for $r=1$ and $r=n$ ($u_i^1=e_i^1$ and $u_i^n=\sum_{\alpha=1}^{n}e_i^\alpha\equiv-e_i^1$),

$$\bar{u}^1\cdot\bar{u}^1=n/(n+1)^2(n-1)^2\ . \tag{22b}$$

The percolation problem is obtained by taking the $n\to0$ limit of the Potts model.[3] The solution, Eq. (22b), which is relevant for physical values of n ($n\geq2$: $d_{ijk}\equiv0$ for $n=1$) clearly does not permit a description of the $n\to0$ case, because the solution for g_c^2, Eq. (16), vanishes in this limit. The parameter a, Eq. (17), has correspondingly no limit.

There seem to be two possibilities at this point. Either the techniques developed so far are simply not applicable to the percolation problem or one must refine the prescription for choosing the correct saddle point for the steepest descent. However, it is clear that it is extremely difficult to do the latter for the $n\to0$ limit, since this would involve a discussion of steepest-descent contours in a zero-dimensional integral!

To obtain some guidance one may turn to low-order perturbation theory. Computations of multiplicities of all three-point graphs with up to three loops shows that they have the sign $g(-g^2)^K$. This suggests that the theory really is benign for $n=0$ and that the problem arises in taking the $n\to0$ limit in the Potts model. In Sec. III we show how to write a field theory for the percolation problem which avoids this $n\to0$ limit.

III. ALTERNATIVE FIELD THEORY FOR THE PERCOLATION PROBLEM

In this section we show that the two- and three-point correlation functions of the $n=0$ Potts model are equivalent order by order in perturbation theory to the ϕ correlation functions of the Hamiltonian

$$H=\int d^dx\,[\tfrac{1}{2}(\nabla\phi)^2-\tfrac{1}{2}(\nabla\psi)^2+\tfrac{1}{2}r_0(\phi^2-\psi^2)+(1/3!)g\Lambda^{(6-d)/2}(\phi+\psi)^3]\ , \tag{23}$$

with the additional rule that only graphs which are connected by ϕ lines are to be included in these correlation functions. Slightly more complicated conditions on the "ϕ connectivity" are required for higher correlation functions.

Expression (23) means that the ψ propagator is $-(q^2+r_0)^{-1}$, ψ is a ghost field.[22] The replacement $\psi\to i\psi$ gives an interaction $(\phi+i\psi)^3$; the appearance of ig_0 as a coupling indicates[23] that oscillatory behavior may indeed occur order by order in g^2. We shall formulate the ϕ-connectivity condition in functional terms which permit this result to be established.

First, we establish the equivalence of (23) with the $n=0$ Potts model. Consider the contraction, in the Wick expansion of two fields ϕ_i and ϕ_l from interaction vertices $\sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha\phi_i\phi_j\phi_k$ and $\sum_\beta e_l^\beta e_m^\beta r_n^\beta\phi_l\phi_m\phi_n$. Since the contraction gives δ_{il} there is a factor $e_i^\alpha e_i^\beta=(n+1)\delta_{\alpha\beta}-1$. Thus, in the Feynman rules for the Potts model one has vertices labeled by $\alpha,\beta,\gamma,\ldots$ (summed from 1 to $n+1$) and propagators connecting vertices labeled by α and β of the form $(n+1)\delta^{\alpha\beta}-1$. Further, when an external line ϕ_i contracts into a vertex, say γ, a factor e_i^γ is left.

In the limit $n\to0$, each of the sums on α,β, etc., contain only one term. The term $(n+1)\delta^{\alpha\beta}$ in the propagator can be represented by a one-component field [ϕ in Eq. (23)]. The term -1 in the propagator can be represented by a ghost field ψ. The factor $(\phi+\psi)^3$ ensures that the net propagator is a sum of ϕ and ψ propagators[24] in Eq. (23).

The only feature which remains to be clarified concerns the factors e_i^α, etc., left over from contractions with the external legs of the graphs. Consider a given graph for the two- or three-point function, and imagine expanding the propagator factors $\prod_{\langle\alpha\beta\rangle}[(n+1)\delta^{\alpha\beta}-1]$. The term $\prod_{\langle\alpha\beta\rangle}[(n+1)\delta^{\alpha\beta}]$ is represented in the $n\to0$ limit by graphs with only ϕ internal lines, the terms with -1 once are represented by all graphs with one ψ internal line, etc. At some point in this procedure one begins to see graphs which are not connected by ϕ lines. Examples are shown in Fig. 1. The graphs in which the external legs are not connected to one another by ϕ lines give zero because we are left with factors $\sum_{\alpha=1}^{n+1}e_i^\alpha\equiv0$. If the external legs are connected by ϕ lines this means that there are sufficient "$\delta^{\alpha\beta}$" factors to ensure that the α labels must be the same; the sum on these α labels then simply reproduces the tensorial factors for the particular vertex function, e.g., $\sum_{\alpha=1}^{n}e_i^\alpha e_j^\alpha=(n+1)\delta_{ij}$. For four-point correlation functions, the vertex proportional to $d_{ijkl}\equiv\sum_{\alpha=1}^{n}e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha$ is represented by the set of graphs in which all external legs are connected by ϕ lines. The vertex proportional to $\delta_{ij}\delta_{kl}$ is represented by graphs in which external legs are ϕ-connected in pairs only, etc.

These remarks establish the result stated at the beginning of this section. It remains to give a functional formalism to represent the ϕ-connectivity rule. Let us introduce an external source $J(x)$ for the ϕ fields, and define the functional $W(\psi,J,g)$ by

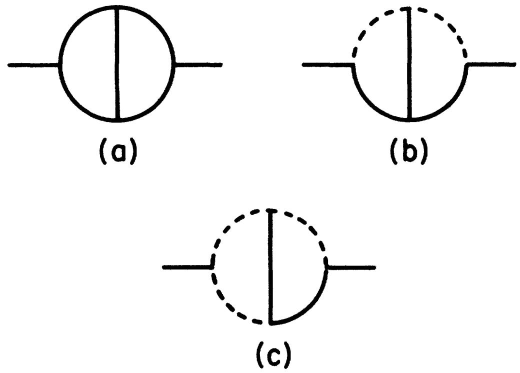

FIG. 1. Graph (a) contributes to the two-point function. In graphs (b) and (c), solid lines represent ϕ propagators [factors of $(n+1)\delta^{\alpha\beta}$] and dashed lines represent ψ propagators (factors of -1). Graph (c) is not ϕ connected and gives zero.

$$e^{W(\psi,J,g)}=\int D\phi \exp\left[-H(\phi,\psi,g)+\int J(x)\phi(x)\right] . \tag{24}$$

Then W contains all ϕ-connected graphs with $\psi(x)$ and $J(x)$ as external fields. What we wish is the set of ϕ-connected graphs in the functional $G(Jg)$ defined by

$$\begin{aligned} e^{G(Jg)} &= \int D\psi\, D\phi \exp\left[-H+\int d^dx\, J(x)\phi(x)\right] \\ &= \int D\psi\, e^{W(\psi,J,g)} . \end{aligned} \tag{25}$$

Differentiating with respect to the source $J(x)$ gives

$$\frac{\delta G}{\delta J(x)}e^{G}=\int D\psi \frac{\delta W}{\delta J(x)}e^{W(\psi,J,g)} , \tag{26}$$

$$\begin{aligned} &\left[\frac{\delta^2 G}{\delta J(x)\delta J(y)}+\frac{\delta G}{\delta J(x)}\frac{\delta G}{\delta J(y)}\right]e^{G} \\ &\quad=\int D\psi\left[\frac{\delta^2 W}{\delta J(x)\delta J(y)}+\frac{\delta W}{\delta J(x)}\frac{\delta W}{\delta J(y)}\right]e^{W} . \end{aligned} \tag{27}$$

The second term on the left-hand side contains only disconnected graphs. On the right-hand side only the first term has graphs in which the two external legs are ϕ connected. Therefore, one sees

$$G^{(2)}_{\phi-c}(x,y)=\int D\psi \frac{\delta^2 W}{\delta J(x)\delta J(y)}e^{W-G} , \tag{28}$$

where the subscript $\phi-c$ denotes ϕ-connected graphs.

Expression (28) can be written in a more useful form for subsequent calculations by using the definition (24) of W. After explicit differentiation of W one may set $J=0$ to obtain

$$\begin{aligned} \frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)}e^{W} &= \int D\phi\; \phi(x_1)\phi(x_2)e^{-H}-\frac{\delta W}{\delta J(x_1)}\frac{\delta W}{\delta J(x_2)}e^{W} \\ &= \int D\phi\; \phi(x_1)\phi(x_2)e^{-H} \\ &\quad -\int D\phi_1\; \phi_1(x_1)e^{-H(\phi_1,\psi,g)}\int D\phi_2\; \phi_2(x_2)e^{-H(\phi_2,\psi,g)}e^{-W(\psi,0,g)} . \end{aligned} \tag{29}$$

Hence, one has

$$\begin{aligned} G^2_{\phi-c}(x_1,x_2) &= \int D\psi\, D\phi\, \phi(x_1)\phi(x_2)e^{-H(\phi,\psi,g)} \\ &\quad -\int D\psi\, D\phi_1\, D\phi_2\, \phi_1(x_1)\phi_2(x_2)\exp[-H(\phi_1,\psi,g)-H(\phi_2,\psi,g)-W(\psi,0,g)-G(0,g)] . \end{aligned} \tag{30}$$

One finds similarly for the three-point correlation function

$$\begin{aligned} G^3_{\phi-c}(x_1,x_2,x_3) &= \int D\psi\, D\phi\, \phi(x_1)\phi(x_2)\phi(x_3)e^{-H(\phi,\psi,g)} \\ &\quad -\left[\int D\psi\, D\phi_1\, D\phi_2\, \phi_1(x_1)\phi_2(x_2)\phi_2(x_3)\exp[-H(\phi,\psi,g)-H(\phi_2,\psi,g)-W(\psi,0,g)-G(0,g)]\right. \\ &\quad \left. +\text{"}x_1\leftrightarrow x_2\text{"}+\text{"}x_1\leftrightarrow x_3\text{"}\right]-\int D\psi\, D\phi_1\, D\phi_2\, D\phi_3\, \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \\ &\quad \times\exp\left[-\sum_{i=1}^{3}H(\phi_i,\psi,g)-2W(\psi,0,g)-G(0,g)\right] . \end{aligned} \tag{31}$$

Similar expressions for higher correlation functions require a corresponding increase in the number of ϕ fields.

We complete this section with three remarks. Although the functional W still appears in expressions (30) and (31), these are sufficiently explicit to apply the steepest-descents method, as we shall see. Second, the function $G(0,g)$ is in fact independent of g because of the special form $(\psi+\phi)^3$ of the interaction. To see this change to fields $S_\pm=2^{-1/2}(\phi\pm\psi)$. The only nonzero propagator is now $\langle S_+S_-\rangle$ and the only interaction is S_+^3: there are no graphs at any order in perturbation theory. This is just a reflection of the extensive factor n ($\equiv 0$ here) in the free energy of the n-component Potts model. Similar arguments show that the first terms on the right-hand sides of expresions (30) and (31) do not contribute beyond tree diagrams. One may simply neglect them as far as high-order behavior is concerned. Finally, we remark that the strange field theories (30) and (31) are to be interpreted at this level only in the context of perturbation theory; if we can show that their perturbation expansions are benign then this result must also hold for the $n=0$ Potts model to which they are equivalent order by order in perturbation theory.

IV. HIGH-ORDER BEHAVIOR IN THE PERCOLATION PROBLEM

A. Two-point function

Following Eq. (7) *et seq.* we pick out order g^{2K} for the two-point function (30) by[25]

$$G_K^{(2)}=\int D\psi\,D\phi_1\,D\phi_2\frac{1}{2\pi i}\times\int\frac{dg}{g}\phi_1\phi_2\exp[-H(\phi_1,\psi,g)-H(\phi_2,\psi,g)-W(\psi,0,g)-G(0,g)-2K\ln g]\ . \qquad (32)$$

The equations for the saddle points are now (we consider the massless theory again)

$$\nabla^2\phi_1=\tfrac{1}{2}g(\psi+\phi_1)^2\ , \qquad (33a)$$

$$\nabla^2\phi_2=\tfrac{1}{2}g(\psi+\phi_2)^2\ , \qquad (33b)$$

$$2\nabla^2\psi+\tfrac{1}{2}g(\psi+\phi_1)^2+\tfrac{1}{2}g(\psi+\phi_2)^2+\frac{\partial W}{\partial\psi}=0\ , \qquad (33c)$$

$$\frac{1}{3!}\int d^6x\,[(\psi+\phi_1)^3+(\psi+\phi_2)^3]+\frac{\partial W}{\partial g}+\frac{2K}{g}=0\ . \qquad (33d)$$

These equations still involve the functional W defined in Eq. (24). After the usual rescaling of the fields by g to decouple the equations, we see that the solution for g_c^2 behaves as K^{-1}, and vanishes as $K\to\infty$, as in Eq. (14). Hence $\partial W/\partial\psi$ and $\partial W/\partial g$ in Eqs. (33) can also be evaluated by steepest descents. One has

$$\frac{\partial W}{\partial\psi}=\int D\phi_0\,[-\nabla^2\psi-\tfrac{1}{2}g(\psi+\phi_0)^2]e^{-H(\phi_0,\psi,g)}\times\left[\int D\phi_0 e^{-H(\phi_0,\psi,g)}\right]^{-1}=[-\nabla^2\psi-\tfrac{1}{2}g(\psi+\phi_0)^2][1+O(g^2)]\ , \qquad (34)$$

where ϕ_0 in this last equation is a solution of the classical field equation

$$\nabla^2\phi_0=\tfrac{1}{2}g(\psi+\phi_0)^2\ . \qquad (35)$$

Similarly for small g one has

$$\frac{\partial W}{\partial g}=-\frac{1}{3!}g\int d^6x\,(\psi+\phi_0)^3\ , \qquad (36)$$

where ϕ_0 is again a solution of Eq. (35).

Combining Eqs. (33)–(36) and rescaling fields according to $\frac{1}{2}g\psi=\Psi$, $\frac{1}{2}g\phi_i=\Phi_i$ gives

$$\nabla^2\Phi=(\Psi+\Phi_i)^2,\quad i=0,1,2 \qquad (37a)$$

$$\nabla^2\Psi=-(\Psi+\Phi_1)^2-(\Psi+\Phi_2)^2+(\Psi+\Phi_0)^2\ , \qquad (37b)$$

and

$$\frac{2}{3}\int d^6x\,[(\Psi+\Phi_1)^3+(\Psi+\Phi_2)^3-(\Psi+\Phi_0)^3]=-Kg_c^2\ . \qquad (37c)$$

We denote again the solution for g^2 by g_c^2. The obvious ansatz for the solution of these equations is $\Psi=v\phi_c(x)$, $\Phi_i=u_i\phi_c(x)$, where $\phi_c(x)$ is the solution (15) of Eq. (13). Equations (37) then become algebraic equations for the coefficients v and u_i. Some simple algebra gives

$$u_i=(v+u_i)^2,\quad i=0,1,2 \qquad (38a)$$

$$v=-u_1-u_2+u_0\ , \qquad (38b)$$

$$Kg_c^2=-(3.2^8\pi^3/5)v\ . \qquad (38c)$$

The quadratic equation (38a) gives two roots for each of u_1, u_2, and u_0 as functions of v,

$$u^{(\pm)}=-\tfrac{1}{2}(2v-1)\pm\tfrac{1}{2}(1-4v)^{1/2}\ . \qquad (39)$$

Substituting into (38b) and taking all combinations of roots one has (i) $u_1=u_3$ or $u_2=u_0$. Then all v and u_i are zero; (ii) $u_1=u_2=u^+,u_0=u^-$. Equation (38b) does not permit this solution; and (iii) $u_i=u_2=u^-$, $u_0=u^+$. This gives the only acceptable solution,

$$\nu = \tfrac{2}{9} \ . \tag{40}$$

Therefore, g_c^2 in Eq. (38c) is a negative quantity and following Eqs. (16) *et seq.* one finds

$$G_K^2 \sim -K!a^K \ , \tag{41}$$

where

$$a = -15/2^9\pi^3 \ . \tag{42}$$

It is amusing to note that this corresponds to the value $n=0$, $r=-1$ or 2 in expression (22a). These are integer values of r which give a negative $\vec{u}^r \cdot \vec{u}^r$ with the smallest magnitude.

B. Higher correlation functions

The analysis of the second term in expression (31) for the three-point function follows exactly that of the two-point function and one obtains the same growth as in expression (42). Following the previous arguments the saddle-point equations for the high-order behavior of the third term in (31) are

$$\nabla^2\phi_i = \tfrac{1}{2}g(\psi+\phi_i)^2, \quad i=0,1,2,3 \tag{43a}$$

$$\nabla^2\psi = -\frac{1}{2}\sum_{i=1}^{3}(\psi+\phi_i)^2 + g(\psi+\phi_0)^2 \ , \tag{43b}$$

$$\frac{1}{3!}\int d^6x \left[\sum_{i=1}^{3}(\psi+\phi_i)^2 - 2(\psi+\phi_0)^2\right] + \frac{2K}{g} = 0 \ . \tag{43c}$$

The resulting algebraic equations for the coefficients ν and u_i $(i=0,1,2,3)$ are

$$u_i = (\nu + u_i)^2 \ , \tag{44a}$$

$$\nu = -u_1 - u_2 - u_3 + 2u_0 \ , \tag{44b}$$

$$Kg_c^2 = (-3.2^8\pi^3/5)\,\nu \ . \tag{44c}$$

The roots of (44a) are as before Eq. (39). The various combinations of $u^{\pm}$ are analyzed as follows: (i) $u_0 = u^-$. There are no solutions of Eq. (43b) with $\nu \neq 0$; (ii) $u_1 = u_2 = u_3 = u^-, u_0 = u^+$. This gives the solution

$$\nu = \tfrac{6}{25} > \tfrac{2}{9} \ ; \tag{45}$$

and (iii) $u_1 = u_0 = u^+, u_2 = u_3 = u^-$ and permutations of (1,2,3). This gives

$$\nu = \tfrac{2}{9} \ . \tag{46}$$

There are no other solutions with $\nu \neq 0$.

The naive prescription adopted previously suggests that the leading growth at high order is controlled by the solution (46), giving the same growth as the two-point function and the first term above. However, we suspect that there may be subtleties in the definition of the integration contours in this case which require the steepest descent contour to pass through the saddle point (45) with $\nu = \frac{6}{25}$. This would imply a growth exponentially smaller than the first term above, and another singularity in the Borel transform of the perturbation series again on the negative axis but further from the origin.

Our reasons for making the remark are: (i) ϕ_1, ϕ_2, and ϕ_3 are dummy integration variables to represent $(\delta W/\delta J)^3$. Since they all represent the same function, one expects the contour of functional integration in all of them to be the same. Hence, the solution (45) which has $u_1 = u_2 = u_3$ seems favored; and (ii) The small oscillations about the saddle point (45) have one bound state which appropriately cancels the factor of i in $(2\pi i)^{-1}$. For the saddle point (46) there are two bound states and two factors of i. Clearly a proper discussion of integration contours is required to clarify this problem.

The nature of the algebraic equations for higher correlation functions is clear from Eqs. (38b) and (44b). At most one has N quantities $u_1, u_2, \ldots, u_N$ for the N point function and ν obeys

$$\nu = -u_1 - u_2 \cdots - u_N + (N-1)u_0 \ .$$

Again, the ambiguities of choice of saddle point remain to be clarified by elucidating the steepest-descent contours.

C. Renormalization-group β function

The function β in Eq. (1) determines the behavior of the running coupling constant $g(\tau)$. The high-order terms in $\beta(g)$ are determined by the high-order behavior of the bare three-point vertex function; the wave-function renormalization produces terms smaller by $O(1/K)$. The function $\beta(g)$ is, therefore, also benign. Extension of the previous calculations to pick out also the factor K^b correctly yields

$$\beta(g) = -\frac{\epsilon g}{2} + \frac{7}{2^8\pi^3}g^3 + \sum_{K=2}^{\infty}\beta_K g^{2K+1} \ , \tag{47}$$

where

$$\beta_K = c(-15/2^9\pi^3)^K\Gamma(K+\tfrac{11}{2})[1+O(1/K)]$$
$$= K!(-15/2^9\pi^3)^K K^{9/2} c[1+O(1/K)] \ . \tag{48}$$

The small-oscillations determinant must be calculated to obtain the factor c; the form of the factors is unaffected by the discussion following Eqs. (45) and (46). The lowest-order terms are from explicit calculation. In (46), g is the renormalized coupling constant.

Renormalization-group functions which give critical exponents η, ν, etc., can be similarly calculated following Brézin *et al.*[12] They are all benign. We do not list them here because for all of these critical exponents governing leading singularities, the expected oscillatory behavior has not set in at the orders currently available in the ϵ expansion. (See Amit[8] and Priest and Lubensky.[8]) It is therefore meaningless to attempt to obtain improved values for these exponents using the information (48). Only for the β function and its derivative $\omega(g)=\beta'(g)$ which controls the corrections to scaling (see, e.g., Brézin *et al.*[7]) do the low-order terms follow the expected high-order oscillations.

The high-order behavior of ω in ϵ expansion can be obtained from expression (47). Straightforward differentiation of (47) gives

$$\omega(g)=-\frac{\epsilon}{2}+\frac{21}{28\pi^3}g^2+\sum_{K=2}^{\infty}(2K+1)\beta_K g^{2K} \quad . \tag{49}$$

The asymptotic series (47) is easily solved for the fixed point g^* as a power series in ϵ,

$$g^{*2}=\frac{2^7\pi^3}{7}+\sum_{K=2}^{\infty}f_K\epsilon^K \quad , \tag{50}$$

where

$$f_K=\Gamma(K+\tfrac{11}{2})(-\tfrac{15}{28})^K c'[1+O(1/K)] \quad . \tag{51}$$

Finally, substituting (50) into (48) gives

$$\omega=\omega(g^*)=\epsilon-\sum_{2}\omega_K\epsilon^K \quad , \tag{52}$$

where

$$\omega_K=\Gamma(K+\tfrac{13}{2})(-\tfrac{15}{28})^K c''[1+O(1/K)] \quad . \tag{53}$$

Only two terms are known explicitly in the ϵ expansion for ω (Amit[8]),

$$\omega=\epsilon-\frac{671}{7^2 3^2 2}\epsilon^2+O(\epsilon^3) \quad . \tag{54}$$

The second term is much larger than the first for physically interesting values of ϵ, and ω seems to become negative. Since the positivity of ω is essential for the stability of the fixed point, it is clearly important to obtain more reliable estimates. Table I lists results for ω in 2, 3, 4, and 5 dimensions using three resummation procedures:

(a) Straightforward Padé,

$$\omega=\epsilon\left[1+\frac{671}{7^2 3^2 2}\epsilon\right]^{-1} \quad ,$$

(b) Standard Padé-Borel using the representation

$$K!=\int_0^\infty e^{-t}t^K\,dt \quad ,$$

and (c) Generalized Padé-Borel using

$$\Gamma(K+\tfrac{13}{2})=\int_0^\infty e^{-t}t^{K+11/2}\,dt \quad .$$

All three results are very similar and indicate $\omega\simeq 1$ in three dimensions.

TABLE I. Values of the correction to scaling exponent ω in 2, 3, 4, and 5 dimensions using (a) Padé method, (b) standard Padé-Borel, and (c) generalized Padé-Borel.

d	ω (a)	ω (b)	ω (c)
5	0.568	0.614	0.582
4	0.793	0.925	0.831
3	0.914	1.13	0.973
2	0.989	1.28	1.07

V. CONCLUSION

The relevance of any ϕ^3-dominated field theory for physics has always been questionable because of the apparent absence of a lower bound for the Hamiltonian. This problem was recognized in the field-theory formulation of the percolation problem in terms of the $n\rightarrow 0$ Potts model. However, it is difficult to discuss meaningfully the thermodynamic potential in a model in which the order parameter has zero components.

In this paper we have presented a concrete reformulation of the percolation problem in terms of a set of one-component fields. This field theory is certainly of interest in itself and we believe should be the basis of a controlled discussion of the percolation problem in field theory.

As a first step in this program we have shown how the high orders in perturbation theory do grow in a benign way, which indicates stability of the ground state and permits Padé-Borel resummation to be applied to perturbative results. The correction to scaling exponent ω which is particularly badly behaved in perturbation theory has been estimated using this method.

Many aspects of the model remain to be clarified. Because of the ϕ-connectivity constraint, the usual proofs of cancellation of subdivergences in the renormalization program are not obviously applicable. the meaning of the functional integrals beyond perturbation theory should also be examined in terms of the functional-integral contours. It would be interesting to make a direct derivation of the field theory from the lattice-percolation theory: the closest starting point may be the "two-component" formulation of Fisher and Essam.[26] Finally, it may be possible to reformulate other $n\rightarrow 0$ models, such as the Edwards-

Anderson model, in analogous forms using ghost fields.

Note added in proof: Professor J. W. Essam has pointed out to us that the perturbation expansion for the unrenormalized 2-point function alternates in signs. Each contributing one particle irreducible graph carries sign $+1(-1)$ depending on whether there are an even (odd) number of loops in the graph. The proof is recursive and is based on the idea of inclusion and exclusion well known in graph theory. Details of the proof can be found in D. K. Arrowsmith and J. W. Essam, J. Math. Phys. 18, 235 (1977).

Borel summability of the series for the renormalized 2-point function however does not follow *a priori.*

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