PHYSICAL REVIEW B VOLUME 17, NUMBER 7 1 APRIL 1978

Response theory for superfluid ^{3}He

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(Received 1 November 1977)

We use an expansion in T_c/T_F to study the influence of strong-coupling effects on the linear response of superfluid ^{3}He. Our results indicate that the enhancement of the order parameter gives the major strong-coupling corrections to the magnetic susceptibilities, bending energies, and superfluid density. The additional "nontrivial" strong-coupling corrections depend sensitively on currently unknown details of the quasiparticle interactions. Hence, precise measurements of the response coefficients can provide additional information about the quasiparticle interactions.

I. INTRODUCTION

The purpose of this paper is to investigate the corrections to weak-coupling pairing theory for the response coefficients of superfluid ^{3}He. By weak-coupling theory we mean the BCS-type pairing theory for an interacting Fermi liquid discussed in detail in Ref. 1. The weak-coupling theory fails for the thermodynamic properties of superfluid ^{3}He: it cannot account for the stability of ^{3}He-A, and significantly underestimates the specific-heat discontinuity at high pressure. We call these and all other deviations from weak-coupling predictions strong-coupling effects.

The key theoretical idea for explaining the stability of the axial state is the feedback mechanism suggested by Anderson and Brinkman,[2] and first worked out in the spin-fluctuation model.[3-5] In Ref. 6 (hereafter referred to as I) we showed that using assumptions no more restrictive than those implicit in the previous spin-fluctuation theories, one can calculate the strong-coupling corrections to the unperturbed equilibrium free energy within the more general framework of Landau's microscopic Fermi-liquid theory. The basis of our calculation was an expansion in powers of the small parameter T_c/T_F, we found that strong-coupling corrections of order T_c/T_F result from the feedback mechanism and from the frequency dependence of the normal-state irreducible interactions, and can be expressed in terms of the normal-state quasiparticle scattering amplitude. Because the quasiparticle scattering amplitude is not fully known, we have no completely independent test of the applicability to ^{3}He of the T_c/T_F expansion scheme. With the s-p approximation for the scattering amplitude, our theory gives qualitatively correct results for the phase diagram and specific-heat discontinuities, but overemphasizes the strong-coupling corrections to the A-phase free energy by approximately a factor of 2. Given that these results are obtained with no adjustable parameters, and that the s-p approximation also overestimates the transport collision rates, we feel justified in using the T_c/T_F expansion to calculate additional strong-coupling effects in terms of the normal-state quasiparticle properties.

In the present paper we extend our scheme to calculate the strong-coupling corrections of order T_c/T_F to the linear response functions of a superfluid Fermi liquid. We distinguish two types of corrections to the response functions. The first, which we call trivial strong-coupling corrections, come from the strong-coupling corrections to the magnitude of the superfluid order parameter. Near T_c the trivial strong-coupling corrections can be accounted for by adjusting the magnitude of the order parameter to fit the measured specific-heat discontinuity. Our concern here is with the possibility of additional, nontrivial, strong-coupling effects.

In the Ginzburg-Landau region, the nontrivial strong-coupling effects appear as corrections to the couplings between external perturbations and the superfluid order parameter. These couplings are described by the coefficients in the second-order Ginzburg-Landau functional

$$\Omega_{\rm GL}[A]=\tfrac{1}{3}N(0)\left\{\frac{T-T_c}{T_c}A_{\rho l}A^*_{\rho l}+\sum_\rho\left[K_L\left|\left(\vec{\nabla}-\frac{2mi}{\hbar}\vec{v}\right)\cdot\vec{A}_\rho\right|^2+K_T\left|\left(\vec{\nabla}-\frac{2mi}{\hbar}\vec{v}\right)\times\vec{A}_\rho\right|^2-K_c\frac{4mi}{\hbar}(\vec{\nabla}\times\vec{v})\cdot(\vec{A}^*_\rho\times\vec{A}_\rho)\right]\right.$$
$$\left.+\tfrac{1}{2}g_z H_\mu A_{\mu j}H_\nu A^*_{\nu l}+\tfrac{1}{2}\bar{g}_z|\vec{H}|^2A_{\rho l}A^*_{\rho l}\right\}\ . \tag{1.1}$$

$A_{\rho l}$ is the complex 3×3 matrix order parameter for an $l=1$ superfluid, and $\vec{A}_\rho$ denotes a row vector $(\vec{A}_\rho)_l=A_{\rho l}$. For simplicity we have included in (1.1) only the two most important external perturbations, a magnetic field $\vec{H}$ and a velocity field $\vec{v}$. We have also neglected the A_1-A splitting term introduced by Ambegaokar and Mermin[7]; this contribution, which is outside the scope of our calculation, gives corrections to the weak-coupling free energy of order $(T_c/T_F)^2$. The superfluid mass-current density $\vec{j}(\vec{r})$ is related to $\Omega_{\rm GL}$ by

$$\vec{j}(\vec{r})=-[\delta\Omega_{\rm GL}/\delta\vec{v}(\vec{r})]|_{\vec{v}=0}\ . \tag{1.2}$$

The coupling constants g_z, $\bar{g}_z$, K_L, K_T, and K_C determine the magnetic susceptibility, the superfluid density, and the $\overleftrightarrow{C}$ tensor in the Ginzburg-Landau region

$$\chi^B_{\mu\nu}=[\chi^{\rm normal}-\tfrac{1}{3}N(0)(g_z+3\bar{g}_z)\langle\Delta(T)^2\rangle]\delta_{\mu\nu}\ , \tag{1.3}$$

$$\chi^A_{\mu\nu}=[\chi^{\rm normal}-N(0)\bar{g}_z\langle\Delta(T)^2\rangle]\delta-N(0)g_z\langle\Delta(T)^2\rangle(\hat{d})_\mu(\hat{d})_\nu\ , \tag{1.4}$$

$$\rho^B_S=(8N(0)m^2/3\hbar^2)(K_L+2K_T)\langle\Delta(T)^2\rangle\ , \tag{1.5}$$

$$(\rho^A_S)_{ij}=\frac{4N(0)m^2}{\hbar^2}\times\{2K_T(\hat{l})_i(\hat{l})_j+(K_L+K_T)\times[\delta_{ij}-(\hat{l})_i(\hat{l})_j]\}\langle\Delta(T)^2\rangle\ , \tag{1.6}$$

$$C^A_{ij}=\frac{4N(0)m^2}{\hbar^2}\times\{(K_T-2K_C)(\hat{l})_i(\hat{l})_j+(K_L-2K_C)\times[\delta_{ij}-(\hat{l})_i(\hat{l})_j]\}\langle\Delta(T)^2\rangle\ . \tag{1.7}$$

Here $\langle\Delta(T)^2\rangle$ is the average squared gap in the equilibrium state

$$\langle\Delta(T)^2\rangle=\tfrac{1}{3}{\rm Tr}(AA^\dagger)\ . \tag{1.8}$$

$\langle\Delta(T)^2\rangle$ is determined by the fourth-order Ginzburg-Landau coefficients β_i, and is proportional to the specific-heat jump at T_c,

$$\langle\Delta(T)^2\rangle=\tfrac{2}{3}\pi^2k_B^2T_c(T_c-T)\Delta C/C_N\ . \tag{1.9}$$

In the weak-coupling approximation, the coefficients in $\Omega_{\rm GL}$ are completely determined by the transition temperature and the Landau parameters:

$$g_z^{(0)}=\left(\frac{\gamma\hbar}{1+F_0^a}\right)^2\tfrac{7}{8}\zeta(3)\frac{1}{(\pi k_BT_c)^2}\ , \tag{1.10a}$$

$$\bar{g}_z^{(0)}=0\ , \tag{1.10b}$$

$$K_T^{(0)}=(\hbar v_F/\pi k_BT_c)^2\tfrac{7}{80}\zeta(3)\ , \tag{1.11a}$$

$$K_L^{(0)}=3K_T^{(0)}\ ,\quad K_C^{(0)}=K_T^{(0)}\ . \tag{1.11b}$$

The simplicity of (1.10b) and (1.11b) does not follow from general symmetry considerations, and hence must result from approximations in the weak-coupling theory. We call results such as these accidental weak-coupling symmetries. One motivation for the calculation described here was to determine which of the accidental weak-coupling symmetries are broken by strong-coupling corrections of order T_c/T_F.

We summarize our conclusions as follows. The nontrivial strong-coupling corrections to the Ginzburg-Landau functional are less than 10% of the weak-coupling contributions, and hence for many purposes can be safely neglected. Nevertheless, these small corrections are of interest because they carry information about the frequency dependence of the pairing interaction. Nontrivial strong-coupling effects enter the superfluid densities, the $\overleftrightarrow{C}$ tensor, the B-phase susceptibility, and the reduced eigenvalue of the A-phase susceptibility tensor. The large eigenvalues of the A-phase susceptibility and the compressibility of both phases are free of strong-coupling corrections through order T_c/T_F at all temperatures. The accidental weak-coupling symmetries $\bar{g}_z=0$ and $K_C=K_T$ survive, but $K_L=3K_T$ is violated to order T_c/T_F.

These specific results are derived and discussed more fully in Sec. IV. In Sec. II we develop our general strong-coupling response formalism, including the T_c/T_F expansion and Fermi-liquid renormalizations. In Sec. III we specialize these results to obtain the general form of the microscopic strong-coupling Ginzburg-Landau functional.

II. GENERAL THEORY

In this section, we present a general framework for calculating the response functions of a superfluid Fermi liquid. Our goal is to extend the conventional weak-coupling response theory to include leading-order strong-coupling effects, and to demonstrate that these strong-coupling corrections can be evaluated within Landau's Fermi-liquid theory.

We base our calculations on the stationary free-energy functional formalism of I, generalized, as by Baym,[8] and DeDominicis and Martin,[9] to include a space and time-dependent external potential. We take the external perturbation to have the form

$$U(t)=\sum_{\vec{q}}\sum_{\vec{k}} u_{\alpha\beta}(\vec{k},\vec{q};t) \times a^{\dagger}_{\vec{k}+\vec{q}/2,\alpha}a_{\vec{k}-\vec{q}/2,\beta} \quad . \tag{2.1}$$

The bare vertex function $u_{\alpha\beta}(\vec{k},\vec{q};t)$ represents the coupling of the external field to an appropriate single-particle operator. For a magnetic field $\vec{H}(\vec{q},t)$ coupled to the magnetization fluctuation, we have

$$u_{\alpha\beta}(\vec{k},\vec{q};t)=-(\tfrac{1}{2}\gamma\hbar)\vec{H}(\vec{q},t)\cdot\vec{\sigma}_{\alpha\beta} \quad ;$$

for a velocity field $\vec{v}(\vec{q},t)$ coupled to the mass current we have

$$u_{\alpha\beta}(\vec{k},\vec{q};t)=\hbar\vec{v}(\vec{q},t)\cdot\vec{k}\,\delta_{\alpha\beta} \quad ,$$

etc. The appropriate stationary functional in the presence of this external perturbation can be written

$$\Omega[\hat{\Sigma},\hat{G};\hat{U}]=-\tfrac{1}{2}k_BT\,\mathrm{Tr}[(\hat{\Sigma}-\hat{U})\hat{G}+\ln(-\hat{G}_0^{-1}+\hat{\Sigma})]+\Phi[\hat{G}] \quad . \tag{2.2}$$

$\hat{\Sigma}(\vec{k},\epsilon_n;\vec{k}',\epsilon_{n'})$ and $\hat{G}(\vec{k},\epsilon_n;\vec{k}',\epsilon_{n'})$ are the 4×4 matrix self-energy and Green's function defined as in Eq. (2.9) of I, except that they now depend on two momenta and two discrete Fermion frequencies; we will find it convenient to take $\hat{\Sigma}$ and $\hat{G}$ as independent in (2.2). The matrix multiplications and the trace in (2.2) run over all arguments of the matrix functions. $\hat{U}$ is a 4×4 matrix representation of the perturbation. All the $\hat{U}$ dependence of $\Omega[\hat{\Sigma},\hat{G};\hat{U}]$ is explicit; in particular $\Phi[\hat{G}]$ is the same functional as in the unperturbed case. With a time-independent perturbation U, the functional Ω reduces at its stationary point to an appropriate thermodynamic free energy. With a time-dependent external potential, Ω no longer has a simple interpretation as a thermodynamic free energy, but its stationarity conditions with respect to independent variations of $\hat{\Sigma}$ and $\hat{G}$ determine the nonequilibrium Green's function and self-energy. Because we will only consider the linear response functions, we can expand the full functional $\Omega[\hat{\Sigma},\hat{G};\hat{U}]$ through second order in the perturbation $\hat{U}$ and in $\delta\hat{\Sigma}$ and $\delta\hat{G}$, the deviations from equilibrium of the single-particle self-energy and Green's function. In this case the natural arguments for $\delta\hat{G}$, $\delta\hat{\Sigma}$, and $\hat{U}$ are the average and difference of the outgoing and incoming momenta and frequencies. Specifically, we define $\delta\hat{G}(\vec{k},\epsilon_n;\vec{q},\omega_m)$ to be the component linear in U of the full Green's function

$$\hat{G}(\vec{k}+\tfrac{1}{2}\vec{q},\epsilon_n+\tfrac{1}{2}\omega_m;\vec{k}-\tfrac{1}{2}\vec{q},\epsilon_n-\tfrac{1}{2}\omega_m) \quad ,$$

with $\omega_m=4m\pi k_BT$. The function $\delta\hat{G}(\vec{k},\epsilon_n;\vec{q},\omega_m)$ can then be thought of as the linear response of $\hat{G}(\vec{k},\epsilon_n)$ to the component of the external potential carrying wave vector $\vec{q}$ and frequency ω_m. With this choice of variables, the second-order functional decomposes into a sum of terms which couple only external wave vectors $\vec{q}$ and $-\vec{q}$ and external frequencies ω_m and $-\omega_m$; in a compact notation the second-order free-energy-density functional is given by

$$\Delta\Omega_2[\delta\hat{\Sigma},\delta\hat{G};\hat{U}]=-(1/2V)k_BT \times\mathrm{Tr}[2(\delta\hat{\Sigma}-\hat{U})\delta\hat{G}-\delta\hat{\Sigma}\bar{R}\delta\hat{\Sigma}-\delta\hat{G}\bar{I}\delta\hat{G}] \tag{2.3}$$

where V is the volume and $\mathrm{Tr}\hat{A}\hat{B}$ is defined by

$$\mathrm{Tr}\hat{A}\hat{B}=\sum_{\vec{q},m}\sum_{\vec{k},n}\tfrac{1}{2}\mathrm{Tr}_4[\hat{A}(\vec{k},\epsilon_n;-\vec{q},-\omega_m) \times\hat{B}(\vec{k},\epsilon_n;\vec{q},\omega_m)] \tag{2.4}$$

$\bar{R}\delta\hat{\Sigma}$ stands for the matrix

$$\hat{G}_{\mathrm{eq}}(\vec{k}+\tfrac{1}{2}\vec{q},\epsilon_n+\tfrac{1}{2}\omega_m)\delta\hat{\Sigma}(\vec{k},\epsilon_n;\vec{q},\omega_m) \times\hat{G}_{\mathrm{eq}}(\vec{k}-\tfrac{1}{2}\vec{q},\epsilon_n-\tfrac{1}{2}\omega_m) \quad , \tag{2.5}$$

where $\hat{G}_{\mathrm{eq}}(\vec{k},\epsilon_n)$ is the equilibrium matrix Green's function in the absence of $\hat{U}$. The elements of the tensor $\bar{I}$ are the irreducible interactions connecting components of the two matrix Green's functions $\delta\hat{G}$ between which $\bar{I}$ stands in (2.3); the particle-hole irreducible interaction connects δG to δG, the particle-particle irreducible interaction connects δF to $\delta\bar{F}$, and the number nonconserving irreducible interactions, which vanish in the normal state, connect δF to δG, δF to δF, etc. For example, the particle-hole component of $\bar{I}\delta\hat{G}$ is

$$k_BT\sum_{n'}\sum_{\vec{k}'} I^{\mathrm{ph}}(\vec{k},\epsilon_n;\vec{k}',\epsilon_{n'};\vec{q},\omega_m) \times\delta G(\vec{k}',\epsilon_{n'};\vec{q},\omega_m) \quad . \tag{2.6}$$

Equation (2.6) should of course also include spin sums, but for clarity we have not shown these explicitly. The imaginary-frequency perturbation $\hat{U}$ can be written as a linear combination of matrices $\hat{U}_{\omega_m}$ which carry a single frequency $\omega_m=4m\pi k_BT$,

$$\hat{U}_{\omega_m}(\vec{k},\epsilon_n;\vec{q},\omega_l)=\delta_{\omega_m,\omega_l}\begin{pmatrix} u(\vec{k},\vec{q}) & 0 \\ 0 & -u^T(-\vec{k},\vec{q})\end{pmatrix} \quad . \tag{2.7}$$

The requirement that $\Delta\Omega_2[\delta\hat{\Sigma},\delta\hat{G};\hat{U}]$ be stationary with respect to variations of $\delta\hat{\Sigma}$ and $\delta\hat{G}$ leads to the two equations

$$\delta\hat{G} = \bar{R}\,\delta\hat{\Sigma} \ , \tag{2.8}$$

$$\delta\hat{\Sigma} = \hat{U} + \bar{I}\,\delta\hat{G} \ , \tag{2.9}$$

which are together equivalent to the Bethe-Salpeter equation. The solution of these two equations gives the physical Green's function

$$\delta\hat{G}(\vec{k},\epsilon_n;\vec{q},\omega_m) = \begin{pmatrix} \delta G(\vec{k},\epsilon_n;\vec{q},\omega_m) & \delta F(\vec{k},\epsilon_n;\vec{q},\omega_m) \\ \delta\bar{F}(\vec{k},\epsilon_n;\vec{q},\omega_m) & \delta\bar{G}(\vec{k},\epsilon_n;\vec{q},\omega_m) \end{pmatrix} \ , \tag{2.10}$$

defined by

$$\left.\begin{matrix} \delta G_{\alpha\gamma}(\vec{k},\epsilon_n;\vec{q},\omega_m) \\ \delta\bar{G}_{\alpha\gamma}(\vec{k},\epsilon_n;\vec{q},\omega_m) \\ \delta F_{\alpha\gamma}(\vec{k},\epsilon_n;\vec{q},\omega_m) \\ \delta\bar{F}_{\alpha\gamma}(\vec{k},\epsilon_n;\vec{q},\omega_m) \end{matrix}\right\} = \frac{1}{\beta}\int_0^\beta d\tau \int_0^\beta d\tau' \int_0^\beta d\tau'' \exp\{i[(\epsilon_n + \tfrac{1}{2}\omega_m)\tau - (\epsilon_n - \tfrac{1}{2}\omega_m)\tau' - \omega_m\tau'']\}$$

$$\times \begin{cases} \langle T_\tau[a_{\vec{k}+\vec{q}/2,\alpha}(\tau)\,a^\dagger_{\vec{k}-\vec{q}/2,\gamma}(\tau')\,U(\tau'')]\rangle \ , \\ \langle T_\tau[a^\dagger_{-\vec{k}-\vec{q}/2,\alpha}(\tau)\,a_{-\vec{k}+\vec{q}/2,\gamma}(\tau')\,U(\tau'')]\rangle \ , \\ \langle T_\tau[a_{\vec{k}+\vec{q}/2,\alpha}(\tau)\,a_{-\vec{k}+\vec{q}/2,\gamma}(\tau')\,U(\tau'')]\rangle \ , \\ \langle T_\tau[a^\dagger_{-\vec{k}-\vec{q}/2,\alpha}(\tau)\,a^\dagger_{\vec{k}-\vec{q}/2,\gamma}(\tau')\,U(\tau'')]\rangle . \end{cases} \tag{2.11}$$

In (2.11) the statistical averages are in the equilibrium grand canonical ensemble, and the time dependence of the operators is governed by the unperturbed Hamiltonian. When it is necessary to indicate the functional dependence of the physical Green's function on the perturbation, we will write this Green's function as $\delta\hat{G}[\hat{U}]$.

From the physical Green's function $\delta\hat{G}[\hat{U}]$, we can obtain the response linear in U of any single-particle operator U'. If U is a time-independent operator of the form (2.1), its static isothermal response is given by

$$\delta\langle U'\rangle_{\text{isothermal}} = (1/\beta)\,\mathrm{Tr}(\hat{U}'_{\omega_0}\,\delta\hat{G}[\hat{U}_{\omega_0}]) \ . \tag{2.12a}$$

The dynamical response $\delta\langle U'\rangle_\omega$ to a perturbation of real frequency ω is found from

$$\delta\langle U'\rangle_\omega = u'(\omega + i\eta) \ , \tag{2.12b}$$

where $u'(z)$ is the unique analytic function given at the imaginary frequency points $z = i\omega_m$ by

$$u'(i\omega_m) = (1/\beta)\,\mathrm{Tr}(\hat{U}'_{\omega_m}\,\delta\hat{G}[\hat{U}_{\omega_m}]) \ . \tag{2.13}$$

Although formally exact, the functional (2.3) (or the equivalent Bethe-Salpeter equation) is not a suitable starting point for explicit calculations of the ^{3}He response functions, because a calculation based on (2.3) requires as input the equilibrium Green's function and irreducible interactions at all momenta and frequencies. Unfortunately, the irreducible interactions are not directly related to any measurable quantities, and the single-particle Green's function, while in principle measurable in tunneling experiments, is in practice unknown except in the quasiparticle region $\epsilon_n \ll k_B T_F$, $|k - k_F| \ll k_F$. Furthermore, neither the Green's function nor the irreducible interactions can at present be satisfactorily calculated from first principles for a system as dense and strongly interacting as is liquid ^{3}He.

To progress further we need Landau's insight that for external frequencies $\omega \ll E_f/h$, external momenta $q \ll k_F$, and temperatures $T \ll T_F$, all nonquasiparticle contributions to the normal-state response functions can be absorbed into renormalized quasiparticle interactions and into renormalized couplings of the quasiparticles to the external perturbations.[10] The real part of the renormalized quasiparticle interaction then determines the normal-state static response functions; the normal-state dynamical response functions satisfy a quasiparticle Boltzmann equation whose collision operator comes from the imaginary part of the quasiparticle interaction. The extension of this renormalization procedure to weak-coupling BCS-superfluids is conceptually straightforward, though formally somewhat complicated; Leggett's treatment in Ref. 11 is closest in outlook to our own. In the present section we will generalize this approach to show that the strong-coupling effects of leading order in T_c/T_F can also be calculated in a quasiparticle theory.

The static response functions of a superfluid Fermi liquid are most naturally defined in terms of the coefficients in a Ginzburg-Landau-type free-energy functional. For this reason we choose to eliminate the nonquasiparticle contributions by requiring the free-energy functional to be stationary with respect to these variables. We emphasize, however, that this procedure is completely equivalent to the more usual one which begins from the Bethe-Salpeter equation and eliminates the nonquasiparticle parts by a partial summation.

A. T_c/T_F Classification

To classify terms in the general free-energy functional $\Delta\Omega_2[\delta\hat{\Sigma}, \delta\hat{G}; \hat{U}]$, Eq. (2.3), we use arguments similar to those discussed in detail in our study of the strong-coupling corrections to the equilibrium free energy. Hence we sketch only briefly the parts of our analysis which parallel arguments in I. We concentrate instead on the new problems in calculating the response functions, and especially on the elimination of the nonquasiparticle background.

As a formal device to facilitate the separation of quasiparticle from nonquasiparticle contributions, we introduce a cut-off energy E_0 (and the equivalent temperature $T_0 = E_0/k_B$) which divides high frequencies from low frequencies. The cutoff must be chosen such that $T_c << T_0 << T_F$, but is otherwise arbitrary, and will not appear in our final results for measurable quantities.

We first discuss the terms $2(\delta\hat{\Sigma} - \hat{U})\delta\hat{G} - \delta\hat{\Sigma}\vec{R}\delta\hat{\Sigma}$, in which the low and high frequencies are decoupled,

$$\mathrm{Tr}[2(\delta\hat{\Sigma} - \hat{U})\delta\hat{G} - \delta\hat{\Sigma}\vec{R}\delta\hat{\Sigma}] = \mathrm{Tr}[2(\delta\hat{\Sigma} - \hat{U})_{\text{low}}(\delta\hat{G})_{\text{low}} - (\delta\hat{\Sigma})_{\text{low}}(\vec{R})_{\text{low}}(\delta\hat{\Sigma})_{\text{low}} + 2(\delta\hat{\Sigma} - \hat{U})_{\text{high}}(\delta\hat{G})_{\text{high}} - (\delta\hat{\Sigma})_{\text{high}}(\vec{R})_{\text{high}}(\delta\hat{\Sigma})_{\text{high}}] \quad . \tag{2.14}$$

Here the low- and high-frequency parts of the matrix functions are defined by

$$\begin{aligned}[\hat{A}(\epsilon_n, \omega_m)]_{\text{low}} &= \hat{A}(\epsilon_n, \omega_m)\Theta(E_0 - |\epsilon_n|) \quad , \\ [\hat{A}(\epsilon_n, \omega_m)]_{\text{high}} &= \hat{A}(\epsilon_n, \omega_m)\,\Theta(|\epsilon_n| - E_0) \quad .\end{aligned} \tag{2.15}$$

To evaluate the kernel of $\vec{R}_{\text{low}}$ in (2.14) we can substitute for $\hat{G}_{\text{eq}}$ the quasiparticle Green's function

$$\hat{G}_{\text{qp}}(\vec{k}, \epsilon_n) = (1/Z)[i\epsilon_n - \xi_k\hat{\tau}_3 - \hat{\Sigma}_{\text{qp}}(\hat{k}, \epsilon_n)]^{-1} \quad . \tag{2.16}$$

Our notation for constant matrices in the 4×4 space is as follows: the $\tau_i(\sigma_i)$ are Pauli matrices in the 2×2 particle-hole (spin) space, and $\hat{\tau}_i$ is the 4×4 matrix $\tau_i 1_\sigma$. Because we want the free-energy functional through order T_c/T_F, the quasiparticle self-energy $\hat{\Sigma}_{\text{qp}}(\hat{k}, \epsilon_n)$ must comprise all self-energy terms through order T_c/T_F not already absorbed into Z and ξ_k in (2.16). The strong-coupling functional $\Delta\Phi[\hat{G} - \hat{G}_N]$ derived in I determines $Z^{-1}(\hat{\Sigma} - \hat{\Sigma}_N)$, the superfluid contribution to $\hat{\Sigma}_{\text{qp}}$, through the relation [Eq. (2.13) of I):

$$\hat{\Sigma}^T - \hat{\Sigma}_N^T = 2\delta\Delta\Phi/\delta(\hat{G} - \hat{G}_N) \quad . \tag{2.17}$$

In addition we must include in $\hat{\Sigma}_{\text{qp}}$ the normal-state quasiparticle lifetime term,

$$\frac{1}{Z}\Sigma_{N_{\text{qp}}}(\epsilon_n) = i\,\mathrm{sgn}\,\epsilon_n \frac{\pi}{16}\frac{\epsilon_n^2 - (\pi k_B T)^2}{v_F p_F} \times \langle |T^{(s)}(\theta, \phi)|^2 + 3|T^{(a)}(\theta, \phi)|^2\rangle \quad . \tag{2.18}$$

Here and in the following our notation for angular averages of the quasiparticle scattering amplitude is as in I.

To simplify the low-frequency part of $\Delta\Omega_2$, we assume that at the stationary point $\delta\hat{\Sigma}(\vec{k}, \epsilon_n; \vec{q}, \omega_m)$ varies with k on the scale of k_F; the consistency of this assumption can easily be verified at the end of our calculation. In the low-frequency terms of (2.14), $\delta\hat{\Sigma}$ always appears together with either $(\delta\hat{G})_{\text{low}}$ or $(\vec{R})_{\text{low}}$, both of which are sharply peaked for

$$|k - k_F| \lesssim (T_0/T_F)k_F << k_F \quad ,$$

as can be seen from (2.8) and (2.16). Hence in the low-frequency terms of (2.14) we can evaluate $\delta\hat{\Sigma}$ and $\hat{U}$ at $k = k_F$ and perform the k integral only over $(\delta\hat{G})_{\text{low}}$ and $(\vec{R})_{\text{low}}$; neglecting the k dependence of $\delta\hat{\Sigma}$ and $\hat{U}$ in these terms introduces errors of order $(T_c/T_F)^2$. In order to eliminate explicitly the irrelevant k dependences in (2.14), we define an integrated quasiparticle Green's function by

$$\delta\hat{g}(\hat{k}, \epsilon_n; \vec{q}, \omega_m) = \frac{Z}{N(0)}\int \frac{k^2\,dk}{2\pi^2}[\delta\hat{G}(\vec{k}, \epsilon_n; \vec{q}, \omega_m)]_{\text{low}} \tag{2.19}$$

Similarly, we define an integrated $\vec{R}$ tensor which operates on matrices $\hat{a}(\hat{k}, \epsilon_n; \vec{q}, \omega_m)$,

$$\vec{R}_{\text{qp}}\hat{a}(\hat{k}, \epsilon_n; \vec{q}, \omega_m) = \frac{Z^2}{N(0)}\int \frac{k^2 dk}{2\pi^2}\hat{G}_{\text{qp}}(\vec{k} + \tfrac{1}{2}\vec{q}, \epsilon_n + \tfrac{1}{2}\omega_m)\hat{a}(\hat{k}, \epsilon_n; \vec{q}, \omega_m)\hat{G}_{\text{qp}}(\vec{k} - \tfrac{1}{2}\vec{q}, \epsilon_n - \tfrac{1}{2}\omega_m) \quad . \tag{2.20}$$

Finally, we introduce a renormalized low-frequency self-energy and perturbation,

$$\delta\hat{\sigma}(\hat{k}, \epsilon_n; \vec{q}, \omega_m) = (1/Z)[\delta\hat{\Sigma}(k_F\hat{k}, \epsilon_n; \vec{q}, \omega_m)]_{\text{low}}, \quad \hat{u}(\hat{k}; \vec{q}, \omega_m) = \hat{U}(k_F\hat{k}, \epsilon_n; \vec{q}, \omega_m) \quad . \tag{2.21}$$

In terms of these functions, our approximation for the contribution to $\Delta\Omega_2$ from the low-frequency part of (2.14) is

$$-\frac{1}{2V}(k_B T)\,\mathrm{Tr}\{2[(\delta\hat{\Sigma})_{\mathrm{low}}-\hat{U}](\delta\hat{G})_{\mathrm{low}}-(\delta\hat{\Sigma})_{\mathrm{low}}(\overline{R})_{\mathrm{low}}(\delta\hat{\Sigma})_{\mathrm{low}}\}\simeq-\tfrac{1}{2}N(0)\,\mathrm{Tr}\left(k_B T\sum_n[2(\delta\hat{\sigma}-Z^{-1}\hat{u})\delta\hat{g}-\delta\hat{\sigma}\overline{R}_{\mathrm{qp}}\delta\hat{\sigma}]\right) . \tag{2.22}$$

The new trace operation introduced in (2.22) is defined by

$$\mathrm{Tr}(\hat{a}\hat{b})=\sum_{\vec{q},m}\int\frac{d\Omega}{4\pi}\frac{1}{2}\mathrm{Tr}_4[\hat{a}(\hat{k},\epsilon_n;-\vec{q},-\omega_m)\hat{b}(\hat{k},\epsilon_n;\vec{q},\omega_m)] . \tag{2.23}$$

In the high-frequency part of $\delta\hat{\Sigma}\overline{R}\delta\hat{\Sigma}$ we can replace $\overline{R}$ by its normal-state value, to leading order in T_c/T_F and T_c/T_0:

$$\overline{R}_{\mathrm{high}}\simeq\overline{R}_{\mathrm{high}}^{\mathrm{normal}} . \tag{2.24}$$

The superfluid propagators modify the diagonal part of $(\overline{R})_{\mathrm{high}}$ (GG and $F\overline{F}$ products) in order $(T_c/T_0)^2$. The GF products in the off-diagonal kernel are of order T_c/T_0, but we will see below that the off-diagonal kernel always enters the physical response functions either quadratically or together with a number-nonconserving irreducible interaction. Since the number-nonconserving interactions are themselves of order T_c/T_F, the superfluid contributions to $(\overline{R})_{\mathrm{high}}$ all enter the physical response functions in terms of order $(T_c/T_0)^2$ or smaller, which we neglect. We also set $\omega=0$ and $q=0$ in $(\overline{R})_{\mathrm{high}}^{\mathrm{normal}}$, an approximation borrowed from the microscopic theory of normal Fermi liquids, and justified by the observation that $(\overline{R})_{\mathrm{high}}^{\mathrm{normal}}$ varies with ω on the scale of E_F, and varies with q on the scale of k_F.[10] The important ω and q dependences of the response functions all come from $\overline{R}_{\mathrm{qp}}$, which varies with ω on the scale of $k_B T_c$, qv_F, and τ^{-1}, and varies with q on the scale of $k_B T_c/\hbar v_F$.

The remaining term in $\Delta\Omega_2$, which involves the interaction $\overline{I}$, couples the low-frequency and high-frequency Green's functions:

$$\begin{aligned}\mathrm{Tr}(\delta\hat{G}\overline{I}\delta\hat{G})=\mathrm{Tr}[&(\delta\hat{G})_{\mathrm{low}}\overline{I}_{\mathrm{low,low}}(\delta\hat{G})_{\mathrm{low}}\\&+(\delta\hat{G})_{\mathrm{high}}\overline{I}_{\mathrm{high,high}}(\delta\hat{G})_{\mathrm{high}}\\&+(\delta\hat{G})_{\mathrm{low}}\overline{I}_{\mathrm{low,high}}(\delta\hat{G})_{\mathrm{high}}\\&+(\delta\hat{G})_{\mathrm{high}}\overline{I}_{\mathrm{high,low}}(\delta\hat{G})_{\mathrm{low}}] ,\end{aligned} \tag{2.25}$$

where the interactions $\overline{I}_{\mathrm{low,low}}$, $\overline{I}_{\mathrm{low,high}}$, etc. are defined by a straightforward extension of (2.15). Superfluid corrections to $\overline{I}_{\mathrm{low,high}}$, $\overline{I}_{\mathrm{high,low}}$, and $\overline{I}_{\mathrm{high,high}}$ first enter the response functions in order $(T_c/T_F)^2$, and hence for our purposes these interactions can be approximated by the corresponding normal-state interactions,

$$\begin{aligned}\overline{I}_{\substack{\mathrm{high,low}\\(\mathrm{low,high})}}&\simeq\overline{I}^{\mathrm{normal}}_{\substack{\mathrm{high,low}\\(\mathrm{low,high})}} ,\\ \overline{I}_{\mathrm{high,high}}&\simeq\overline{I}^{\mathrm{normal}}_{\mathrm{high,high}} .\end{aligned} \tag{2.26}$$

The argument for this approximation parallels that given for $(\overline{R})_{\mathrm{high}}$: all diagrams for $\overline{I}_{\mathrm{high,high}}$ or $\overline{I}_{\mathrm{high,low}}$ containing superfluid Green's functions are of order $(T_c/T_F)^2$, except for the number-nonconserving diagrams with exactly one F function; and these number-nonconserving interactions always enter either quadratically or together with $(\mathrm{GF})_{\mathrm{high}}$. We also neglect the weak dependence of $\overline{I}^{\mathrm{normal}}_{\mathrm{high,low}}$ and $\overline{I}^{\mathrm{normal}}_{\mathrm{high,high}}$ on the external frequency and momentum, as in the theory of normal Fermi liquids.

In $\overline{I}_{\mathrm{low,low}}$ we finally encounter strong-coupling corrections to the irreducible interactions which enter the response functions in order T_c/T_F. To leading (zeroth) order in T_c/T_F, $\overline{I}_{\mathrm{low,low}}$ is given by the normal-state irreducible interaction evaluated with all frequencies equal to zero. Corrections to $\overline{I}_{\mathrm{low,low}}$ of order T_c/T_F come from all diagrams with a two-quasiparticle cut. These diagrams are precisely those generated by twice differentiating the strong-coupling $\Delta\Phi$ functional of I with respect to the superfluid quasiparticle Green's functions. The diagrams for the strong-coupling irreducible interactions of order T_c/T_F are shown in Fig. 1. The demonstration that these diagrams include *all* T_c/T_F corrections follows closely the argument given in I for the $\Delta\Phi$ functional itself; in particular we again emphasize that in order to avoid double-counting, one must subtract from diagrams 1(a), 1(b), and 1(d) their normal-state, zero-frequency limit, since these parts are already included in the zeroth-order interactions.

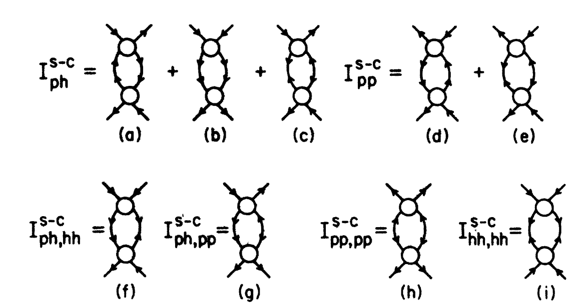

FIG. 1. Diagrams for the strong-coupling corrections to the low-frequency irreducible interactions. Open circles represent the normal-state quasiparticle scattering amplitude, and the Green's-function lines represent quasiparticle propagators.

We assume that the irreducible interactions are slowly varying functions of momentum, and therefore in (2.25) we replace the rapidly-varying function $(\delta\hat{G})_{\rm low}$ by the integrated Green's function $\delta\hat{g}$, and restrict to the Fermi surface all momentum arguments which the irreducible interactions share with a $(\delta\hat{G})_{\rm low}$.

B. Elimination of the high-frequency parts

By using the results of the preceding section to evaluate Eq. (2.3) for $\Delta\Omega_2$, we obtain the strong-coupling functional needed to calculate the response functions of a superfluid Fermi liquid through order T_c/T_F. The high-frequency parts of the Green's functions, self-energies, and interactions still appear explicitly in this formulation, however. To eliminate these unknown, nonquasiparticle quantities, we require that $\Delta\Omega_2$ be stationary with respect to variations of $(\delta\hat{\Sigma})_{\rm high}$ and $(\delta\hat{G})_{\rm high}$. This condition implies that

$$(\delta\hat{G})_{\rm high}=(\overline{R})^{\rm normal}_{\rm high}(\delta\hat{\Sigma})_{\rm high}\ , \qquad (2.27)$$

and

$$(\delta\hat{\Sigma})_{\rm high}=\hat{U}+\overline{T}^{\rm normal}_{\rm high,high}(\delta\hat{G})_{\rm high}+\overline{T}^{\rm normal}_{\rm high,low}(\delta\hat{G})_{\rm low}\ . \qquad (2.28)$$

We can formally solve these equations for $(\delta\hat{\Sigma})_{\rm high}$,

$$(\delta\hat{\Sigma})_{\rm high}=[1-\overline{T}^{\rm normal}_{\rm high,high}(\overline{R})^{\rm normal}_{\rm high}]^{-1}\times[\hat{U}+\overline{T}^{\rm normal}_{\rm high,low}(\delta\hat{G})_{\rm low}] \qquad (2.29)$$

$(\delta\hat{G})_{\rm high}$ is then given by (2.27) and (2.29). Substituting these results into (2.3), we obtain a quasiparticle functional, $\Delta\Omega_2^{\rm qp}$, defined on the low-frequency variables $\delta\hat{\sigma}$ and $\delta\hat{g}$ alone,

$$\Delta\Omega_2^{\rm qp}[\delta\hat{\sigma},\delta\hat{g};\hat{u}]=\tfrac{1}{2}N(0)\,{\rm Tr}\left\{\delta\hat{g}_0(\overline{A}^{\rm normal}_{\rm qp}+\overline{V}^{\rm normal}_{\rm qp})\delta\hat{g}_0+2\hat{u}^{\rm normal}_{\rm qp}\delta\hat{g}_0+\hat{u}\overleftrightarrow{\chi}^{\rm normal}\hat{u}+k_BT\sum_n(\delta\hat{g}\overline{T}^{s-c}_{\rm qp}\delta\hat{g}+\delta\hat{\sigma}\overline{R}_{\rm qp}\delta\hat{\sigma}-2\delta\hat{\sigma}\delta\hat{g})\right\}\ . \qquad (2.30)$$

The quasiparticle Green's function enters the first two terms of $\Delta\Omega_2^{\rm qp}$ only through

$$\delta\hat{g}_0(\hat{k};\vec{q},\omega_m)=k_BT\sum_n\delta\hat{g}(\hat{k},\epsilon_n;\vec{q},\omega_m)\ , \qquad (2.31)$$

because the interaction $(\overline{A}^{\rm normal}_{\rm qp}+\overline{V}^{\rm normal}_{\rm qp})$ and the renormalized perturbation $\hat{u}^{\rm normal}_{\rm qp}$ are independent of both the internal and external frequencies. In $\overline{A}^{\rm normal}_{\rm qp}+\overline{V}^{\rm normal}_{\rm qp}$ we collect the zero-frequency Fermi surface limit of $\overline{T}^{\rm normal}_{\rm low,low}$ and of the kernel of the terms quadratic in $(\delta\hat{G})_{\rm low}$ generated by eliminating the high-frequency parts. $\overline{A}^{\rm normal}_{\rm qp}$ denotes the particle-hole part of this quasiparticle interaction, and $\overline{V}^{\rm normal}_{\rm qp}$ denotes the particle-particle part. We emphasize that through order T_c/T_F, all contributions to $\overline{A}^{\rm normal}_{\rm qp}$ and $\overline{V}^{\rm normal}_{\rm qp}$ can be evaluated in the normal-state limit. For convenience we have included a factor $N(0)/Z^2$ in $\overline{A}^{\rm normal}_{\rm qp}+\overline{V}^{\rm normal}_{\rm qp}$, and also in the strong-coupling interaction, $\overline{T}^{s-c}_{\rm qp}$, given by Fig. 1. $\overline{A}^{\rm normal}_{\rm qp}$ can be expressed in terms of the normal-state Landau parameters,

$$\overline{A}^{\rm normal}_{\rm qp}\delta\hat{g}_0=\int\frac{d\Omega'}{4\pi}\sum_{l=0}^{\infty}P_l(\hat{k}\cdot\hat{k}')\{\tfrac{1}{4}A_l^s[\hat{1}\,{\rm Tr}_4\delta\hat{g}_0(\hat{k}')+\hat{\tau}_3{\rm Tr}_4\hat{\tau}_3\delta\hat{g}_0(\hat{k}')]+\tfrac{1}{4}A_l^a[\hat{1}\vec{\sigma}\,{\rm Tr}_4\hat{1}\vec{\sigma}\delta\hat{g}_0(\hat{k}')+\hat{\tau}_3\vec{\sigma}\,{\rm Tr}_4\hat{\tau}_3\vec{\sigma}\delta\hat{g}_0(\hat{k}')]\}\ . \qquad (2.32)$$

The corresponding interaction in the particle-particle channel has the form

$$\overline{V}^{\rm normal}_{\rm qp}\delta\hat{g}_0=\int\frac{d\Omega'}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\hat{k}\cdot\hat{k}')\,V_l[\delta\hat{g}_0(\hat{k}')]_{\rm off-diag}\ , \qquad (2.33)$$

where $(\delta\hat{g}_0)_{\rm off-diag}$ denotes the off-diagonal part of $\delta\hat{g}_0$. We call the angular momentum components V_l pairing pseudointeractions; unlike the $A_l^{s,a}$, which are cutoff independent, the V_l depend logarithmically on the cutoff E_0. This cutoff dependence of the V_l just compensates for the cutoff dependence of the nonconvergent low-frequency sums in the particle-particle channel, and hence assures that E_0 drops out of all observable quantities. The term $\hat{u}^{\rm normal}_{\rm qp}\delta\hat{g}_0$, which represents the coupling of the external field to a renormalized quasiparticle vertex, contains the low-frequency part $Z^{-1}\hat{u}\delta\hat{g}$ from (2.22) and all contributions linear in $\hat{U}$ generated by eliminating the high-frequency parts. The final ϵ_n-independent term in (2.30),

$$\tfrac{1}{2}N(0)\,{\rm Tr}(\hat{u}\overleftrightarrow{\chi}^{\rm normal}\hat{u})\ ,$$

consists of all the contributions quadratic in $\hat{U}$; $\overleftrightarrow{\chi}^{\rm normal}$

is a generalized normal-state isothermal static susceptibility tensor. We again emphasize that for calculating the superfluid response functions through order T_c/T_F, $\hat{u}_{\rm qp}^{\rm normal}$ and $\overleftrightarrow{\chi}^{\rm normal}$ can be evaluated at zero external frequency and momentum using the normal-state Landau theory.

From the strong-coupling, quasiparticle functional, Eq. (2.30), we can calculate the linear response coefficients of a superfluid Fermi liquid through first order in T_c/T_F. The familiar weak-coupling theory for the static response coefficients is obtained by omitting the term containing $\overline{T}_{\rm qp}^{s-c}$ and approximating the equilibrium Green's functions in $\overline{R}_{\rm qp}$ by the weak-coupling equilibrium Green's functions; the latter approximation amounts to keeping only the weak-coupling part of $\Delta\Phi$ in (2.17).

For the explicit calculations that follow, we find it convenient to isolate all the strong-coupling effects in a single term of the free-energy functional. To this end, we introduce a new strong-coupling kernel defined by the following implicit equation:

$$\overline{R}_{\rm qp}^{s-c} = \overline{R}_{\rm qp} + \overline{R}_{\rm qp}\overline{T}_{\rm qp}^{s-c}\overline{R}_{\rm qp}^{s-c} \quad . \tag{2.34}$$

Here the pre- and post-multiplying $\overline{R}_{\rm qp}$'s again allow us to take $\overline{T}_{\rm qp}^{s-c}$ on the Fermi surface. We use $\overline{R}_{\rm qp}^{s-c}$ to define a new stationary functional,

$$\Delta\overline{\Omega}_2^{\rm qp}[\delta\hat{\sigma}_0, \delta\hat{g}_0;\hat{u}] = \tfrac{1}{2}N(0)\,{\rm Tr}[\delta\hat{g}_0(\overline{A}_{\rm qp}^{\rm normal} + \overline{V}_{\rm qp}^{\rm normal})\delta\hat{g}_0 + \delta\hat{\sigma}_0\overline{K}\delta\hat{\sigma}_0 - 2(\delta\hat{\sigma}_0 - \hat{u}_{\rm qp}^{\rm normal})\delta\hat{g}_0 + \hat{u}\overleftrightarrow{\chi}^{\rm normal}\hat{u}] \quad . \tag{2.35}$$

The strong-coupling effects enter this functional through the operator $\overline{K}$, which is obtained from $\overline{R}_{\rm qp}^{s-c}$ by summing over all the internal frequencies. $\overline{\Omega}_2^{\rm qp}$ is defined on the variables $\delta\hat{\sigma}_0(\hat{k};\vec{q},\omega_m)$ and $\delta\hat{g}_0(\hat{k};\vec{q},\omega_m)$, which have no internal-frequency dependence. The stationarity conditions for $\Delta\overline{\Omega}_2^{\rm qp}$ are

$$\delta\hat{g}_0 = \overline{K}\delta\hat{\sigma}_0 \quad , \tag{2.36}$$

$$\delta\hat{\sigma}_0 = \hat{u}_{\rm qp}^{\rm normal} + (\overline{A}_{\rm qp}^{\rm normal} + \overline{V}_{\rm qp}^{\rm normal})\delta\hat{g}_0 \quad . \tag{2.37}$$

At the respective stationary points of $\Delta\Omega_2^{\rm qp}$ and $\Delta\overline{\Omega}_2^{\rm qp}$ these functionals have the same value; the stationary points $(\delta\hat{\sigma}, \delta\hat{g})$ and $(\delta\hat{\sigma}_0, \delta\hat{g}_0)$ are themselves related by

$$\delta\hat{g} = \overline{R}_{\rm qp}^{s-c}\delta\hat{\sigma}_0 \quad , \tag{2.38}$$

$$\delta\hat{\sigma} = \delta\hat{\sigma}_0 + \overline{T}_{\rm qp}^{s-c}\delta\hat{g} \quad . \tag{2.39}$$

The new functional $\Delta\overline{\Omega}_2^{\rm qp}[\delta\hat{\sigma}_0, \delta\hat{g}_0;\hat{u}]$ has the practical and conceptual advantage that $\delta\hat{\sigma}_0$ is ϵ_n independent, and hence represents a more convenient choice for the order parameter than does the ϵ_n-dependent self-energy $\delta\hat{\sigma}$.

Equations (2.34)–(2.37) form the basis of our strong-coupling theory for the response functions of superfluid ^{3}He. These equations hold for long-wavelength, low-frequency perturbations, and include all effects through first order in the expansion parameter T_c/T_F. Depending on the specific problem of interest, it may be more convenient to work either from the stationarity conditions (2.36) and (2.37) or directly from the generating functional (2.35). Calculations of the dynamical response functions begin from the stationarity equations, analytically continued to real frequencies; for the normal state, Eqs. (2.36) and (2.37) are then equivalent to the quasiparticle Boltzmann equation derived in Ref. 12.

To calculate the static response functions, we find it useful to work directly with the free-energy functional. This procedure avoids subtleties connected with the $q \to 0$ limit of the stationarity equations.

The parameters which enter our theory are: (i) the equilibrium quasiparticle Green's functions generated by the strong-coupling functional of I, (ii) the normal-state Landau parameters A_l^s and A_l^a, (iii) the pairing pseudointeractions V_l, (iv) the normal-state isothermal "susceptibility tensor" $\overleftrightarrow{\chi}^{\rm normal}$, (v) the normal-state renormalized perturbation $\hat{u}_{\rm qp}^{\rm normal}$, and (vi) the normal-state quasiparticle scattering amplitudes $T^{(s)}$ and $T^{(a)}$, which determine the quasiparticle lifetimes and the strong-coupling corrections. Fortunately these parameters are not all independent. The Landau parameters $A_l^{s,a}$ fix the forward-scattering limit of $T^{(s),(a)}$, and conservation laws lead to relations among the normal-state quasiparticle quantities (Ref. 11 contains a careful discussion of this point). For example, for any conserved single-particle operator U we have

$$\hat{u}_{\rm qp}^{\rm normal} = (1 - \overline{A}_{\rm qp}^{\rm normal})\hat{u} \quad , \tag{2.40}$$

and

$$\overleftrightarrow{\chi}^{\rm normal}\hat{u} = \hat{u}_{\rm qp}^{\rm normal} \quad . \tag{2.41}$$

These important relations allow one to express most of the interesting normal-state susceptibilities and renormalized quasiparticle operators in terms of the Landau parameters.

III. GINZBURG-LANDAU FUNCTIONAL

As a first application of the scheme developed in Sec. II, we will study the strong-coupling corrections to the Ginzburg-Landau free-energy functional. To obtain the Ginzburg-Landau functional, one first makes the static ($\omega_m = 0$) free-energy functional stationary with respect to all variables except for the superfluid order parameter. Expanding in powers of the order parameter then yields the Ginzburg-Landau functional, whose arguments are the order parameter and the static external perturbations. The physical, fully stationary free energy is finally obtained by minimizing with respect to the order parameter; this is equivalent to solving the Ginzburg-Landau equation for the order parameter. Proceeding in this way from our strong-coupling functional, Eq. (2.35), we do not directly arrive at the full Ginzburg-Landau functional given in (1.1) but instead at the functional obtained by expanding (1.1) to second order in the external perturbations and in $\delta A_{\mu l}$, the deviation of the order parameter from its unperturbed uniform equilibrium value $A^0_{\mu l}$:

$$\Delta\Omega_{\rm GL}=\tfrac{1}{3}N(0)\left\{\frac{T-T_c}{T_c}\delta A_{\mu l}\delta A^*_{\mu l}+K_L\left[(\partial_l\delta A_{\mu l})(\partial_l\delta A^*_{\mu l})+\left(\frac{2m}{\hbar}\right)^2(v_lA^0_{\mu l})(v_lA^0_{\mu l}{}^*)-\frac{2im}{\hbar}[v_lA^0_{\mu l}(\partial_l\delta A^*_{\mu l})-(\partial_l\delta A_{\mu l})v_lA^0_{\mu l}{}^*]\right]+\cdots+\tfrac{1}{2}g_ZH_\mu A^0_{\mu l}H_\nu A^0_{\nu l}{}^*+\cdots\right\}. \quad (3.1)$$

Here, for brevity, we have given only a few characteristic terms of $\Delta\Omega_{\rm GL}$. From our microscopic derivation of (3.1), we obtain the coefficients K_L, K_T, K_C, g_Z, and $\bar{g}_Z$. These coefficients suffice to determine the full Ginzburg-Landau functional, which applies even in some circumstances when the assumption of small deviations from uniform equilibrium breaks down.

The functional $\Delta\overline{\Omega}{}^{\rm qp}_2$, Eq. (2.35), depends on the 4×4 matrix variables $\delta\hat{\sigma}_0$ and $\delta\hat{g}_0$. We wish to retain as explicit variables only the off-diagonal components of the self-energy. We introduce a partial-wave expansion with respect to $\hat{k}$ for $[\delta\hat{\sigma}_0(\hat{k};\vec{q})]_{\text{off-diag}}$,

$$(\delta\hat{\sigma}_0)_{\text{off-diag.}}=\delta\hat{\Delta}=\sum_{l=0}^{\infty}\delta\hat{\Delta}_l\ , \quad (3.2)$$

and represent the $l=1$ order parameter appropriate to superfluid ^{3}He by

$$\delta\hat{\Delta}(\hat{k};\vec{q})_{l=1}=\delta A_{\mu l}(\vec{q})(\hat{k})_l i\sigma_\mu\sigma_2\tau^+ - \delta A^*_{\mu l}(-\vec{q})(\hat{k})_l i\sigma_2\sigma_\mu\tau^-\ . \quad (3.3)$$

To eliminate $\delta\hat{g}_0$ and $(\delta\hat{\sigma}_0)_{\text{diag.}}$ we use the stationarity conditions with respect to these variables:

$$\delta\hat{g}_0=(\overline{A}{}^{\rm normal}_{\rm qp}+\overline{V}{}^{\rm normal}_{\rm qp})^{-1}\times(\delta\hat{\sigma}_0-\hat{u}^{\rm normal}_{\rm qp}) \quad (3.4)$$

from stationarity with respect to $\delta\hat{g}_0$, and

$$(\delta\hat{g}_0)_{\text{diag.}}=\overline{K}_{\succeq\succeq}(\delta\hat{\sigma}_0)_{\text{diag.}}+\overline{K}_{\succeq\Delta}\delta\hat{\Delta} \quad (3.5)$$

from stationarity with respect to $(\delta\hat{\sigma}_0)_{\text{diag.}}$. In writing (3.5) we have used the following decomposition of $\overline{K}$:

$$\overline{K}=\overline{K}_{\succeq\succeq}+\overline{K}_{\succeq\Delta}+\overline{K}_{\Delta\succeq}+\overline{K}_{\Delta\Delta}\ , \quad (3.6)$$

where $\overline{K}_{\succeq\succeq}$ connects two diagonal self-energies, $\overline{K}_{\succeq\Delta}$ and $\overline{K}_{\Delta\succeq}$ connect one diagonal and one off-diagonal self-energy, and $\overline{K}_{\Delta\Delta}$ connects two off-diagonal self-energies.

We now substitute (3.2) and (3.4)–(3.6) into (2.35) and expand to second order in the order parameter, noting that for $\omega_m=0$, $\overline{K}_{\succeq\succeq}$ is of order Δ^2 and $\overline{K}_{\succeq\Delta}$ is of order Δ. This yields the Ginzburg-Landau functional, for small deviations from equilibrium, in the form

$$\begin{aligned}\Delta\Omega_{\rm GL}[\delta\hat{\Delta};\hat{u}]=N(0)\sum_{\vec{q}}\int\frac{d\Omega}{4\pi}\int\frac{d\Omega'}{4\pi}\tfrac{1}{4}{\rm Tr}_4\Bigg[&\delta\hat{\Delta}(\hat{k};-\vec{q})\left(-\frac{3(\hat{k}\cdot\hat{k}')}{V_1}+\overline{K}_{\Delta\Delta}(\hat{k},\hat{k}';\vec{q})\right)\delta\hat{\Delta}(\hat{k}';\vec{q})\\&+\delta\hat{\Delta}(\hat{k};-\vec{q})\overline{K}_{\Delta\succeq}(\hat{k},\hat{k}';\vec{q})\hat{u}^{\rm normal}_{\rm qp}(\hat{k}';\vec{q})\\&+\hat{u}^{\rm normal}_{\rm qp}(\hat{k};-\vec{q})\overline{K}_{\succeq\Delta}(\hat{k},\hat{k}';\vec{q})\delta\hat{\Delta}(\hat{k}';\vec{q})\\&+\hat{u}^{\rm normal}_{\rm qp}(\hat{k};-\vec{q})\overline{K}_{\succeq\succeq}(\hat{k},\hat{k}';\vec{q})\hat{u}^{\rm normal}_{\rm qp}(\hat{k}';\vec{q})\Bigg]\ .\end{aligned} \quad (3.7)$$

To calculate the strong-coupling $\overline{K}$ kernels in (3.7) we must solve Eq. (2.34) for $\overline{R}{}^{\rm s\text{-}c}_{\rm qp}$. Through first order in T_c/T_F, we can solve this equation by a perturbation expansion in $\overline{T}{}^{\rm s\text{-}c}_{\rm qp}$ and in the strong-coupling corrections to the equilibrium quasiparticle self-energy, $\hat{\Sigma}_{\rm qp}$, discussed following Eq. (2.16),

$$\overline{R}{}^{\rm s\text{-}c}_{\rm qp}=\overline{R}{}^{\rm w\text{-}c}_{\rm qp}+\delta\overline{R}_{\rm qp}+\overline{R}{}^{\rm w\text{-}c}_{\rm qp}\overline{T}{}^{\rm s\text{-}c}_{\rm qp}\overline{R}{}^{\rm w\text{-}c}_{\rm qp}+o(T_c/T_F)\ . \quad (3.8)$$

Here $\overline{R}{}^{\rm w\text{-}c}_{\rm qp}$ denotes the weak-coupling kernel

$$\overline{R}{}^{\rm w\text{-}c}_{\rm qp}\hat{a}=\int_{-\infty}^{\infty}d\xi\left[i\epsilon_n-\left(\xi+v_F\frac{\hbar\hat{k}\cdot\vec{q}}{2}\right)\hat{\tau}_3-\hat{\Delta}_0(\hat{k})\right]^{-1}\hat{a}(\hat{k},\epsilon_n;\vec{q})\times\left[i\epsilon_n-\left(\xi-v_F\frac{\hbar\hat{k}\cdot\vec{q}}{2}\right)\hat{\tau}_3-\hat{\Delta}_0(\hat{k})\right]^{-1}\ , \quad (3.9)$$

and $\delta\vec{R}_{\rm qp}$ denotes the strong-coupling kernel containing the T_c/T_F corrections to the equilibrium weak-coupling self-energy. For the Ginzburg-Landau functional, $\vec{T}_{\rm qp}^{s-c}$ is given by diagram (1d) alone.[13]

In the next section, we use (3.7)–(3.9) to calculate g_Z, $\bar{g}_Z$, K_L, K_T, and K_C through first order in T_c/T_F.

IV. SOME SPECIFIC RESULTS

In the preceding sections we have derived the strong-coupling response theory for a superfluid Fermi liquid through order T_c/T_F. Our general result, valid for all temperatures, is contained in Eqs. (2.34) and (2.35). We saw in Sec. III that for static perturbations and for temperatures near T_c these equations can be considerably simplified to yield the Ginzburg-Landau functional given in Eq. (3.7). Except for a few temperature-independent exact results, we will concentrate on the Ginzburg-Landau region. Extending our calculations to lower temperatures, while conceptually straightforward, would in general require extensive numerical computation.

A. Compressibility

In the weak-coupling limit the compressibility of a Fermi liquid is unchanged by the superfluid transition. We will show that this result holds through order T_c/T_F. We first observe that the tensor $\vec{R}_{\rm qp}^{\rm w-c}$ given by Eq. (3.9) annihilates $\hat{\tau}_3$,

$$\vec{R}_{\rm qp}^{\rm w-c}\,\hat{\tau}_3=0 \tag{4.1}$$

We next consider the effect of replacing $\vec{R}_{\rm qp}^{\rm w-c}$ in (4.1) by the full $\vec{R}_{\rm qp}$ tensor, including the order T_c/T_F-corrections to the equilibrium self-energy, as specified by (2.17) and (2.18). These corrections introduce no new ξ_k dependence, compared with $\vec{R}_{\rm qp}^{\rm w-c}$; they introduce an ϵ_n-dependent correction to the diagonal self-energy, which can be included in (2.16) by the replacement $\epsilon_n \rightarrow \tilde{\epsilon}(\epsilon_n)$. From this one easily sees that (4.1) also holds for the full $\vec{R}_{\rm qp}$,

$$\vec{R}_{\rm qp}\hat{\tau}_3=0 \ , \tag{4.2}$$

and from (4.2) and (2.34) we see immediately that

$$\vec{R}_{\rm qp}^{s-c}\,\hat{\tau}_3=0 \ . \tag{4.3}$$

It follows from (2.32) that $\vec{A}_{\rm qp}^{\rm normal}$ acting on a function proportional to $\hat{\tau}_3$ gives another function proportional to $\hat{\tau}_3$. Together with (2.36), (2.38), (2.40), and (4.3), this implies that if $\hat{u}(\hat{k};\vec{q})$ is proportional to $\hat{\tau}_3$ and represents the coupling of a static external field to a conserved quantity, then

$$\delta\hat{g}(\hat{k};\vec{q})=0 \ . \tag{4.4}$$

For a static change in the chemical potential, we have

$$\hat{u}(\hat{k};\vec{q})=\delta\mu(\vec{q})\hat{\tau}_3 \ , \tag{4.5}$$

and hence the superfluid corrections to the compressibility vanish through order T_c/T_F at all temperatures.

B. Magnetic susceptibility

For a static, homogeneous, external magnetic field $\vec{H}$ coupled to the magnetization, the renormalized perturbation is

$$\hat{u}_{\rm qp}^{\rm normal}=-[\tfrac{1}{2}\gamma\hbar/(1+F_0^a)]\vec{H}\cdot\hat{\vec{\sigma}} \ , \tag{4.6}$$

where

$$\hat{\vec{\sigma}}=\tfrac{1}{2}[(1+\hat{\tau}_3)\,\vec{\sigma}-(1-\hat{\tau}_3)\,\vec{\sigma}^{\,T}] \ .$$

From (3.9) we find [for a unitary $\Delta_0(\hat{k})$]

$$\vec{R}_{\rm qp}^{\rm w-c}\vec{H}\cdot\hat{\vec{\sigma}}=\tfrac{1}{2}\pi\left(\frac{[\vec{\Delta}_0(\hat{k})\cdot\vec{H}][\vec{\Delta}_0{}^*(\hat{k})\cdot\hat{\vec{\sigma}}]+[\vec{\Delta}_0{}^*(\hat{k})\cdot\vec{H}][\vec{\Delta}_0(\hat{k})\cdot\hat{\vec{\sigma}}]}{[\epsilon_n^2+|\Delta_0(\hat{k})|^2]^{3/2}}\right.$$
$$\left.+\frac{2i\epsilon_n[\vec{H}\cdot\vec{\Delta}_0(\hat{k})\hat{\tau}_+-\vec{H}\cdot\vec{\Delta}_0{}^*(\hat{k})\hat{\tau}_-]i\sigma_2}{[\epsilon_n^2+|\vec{\Delta}_0(\hat{k})|^2]^{3/2}}\right) \ . \tag{4.7}$$

When $\vec{H}$ is perpendicular to the equilibrium order parameter,

$$\vec{H}\cdot\vec{\Delta}_0(\hat{k})=0 \ , \tag{4.8}$$

the right-hand side of (4.7) vanishes. If (4.8) holds for all $\hat{k}$, then from (2.34)–(2.37) we see that the free-energy difference $\Omega_S-\Omega_N$ is independent of $\vec{H}$, and the corresponding components of the susceptibility tensor are unchanged by the superfluid condensation, through order T_c/T_F. In particular this means that through order T_c/T_F the maximum A-phase susceptibility is equal to the normal-state susceptibility. Furthermore, by comparing (4.7) with the general Ginzburg-Landau functional, Eq. (1.1), we see that $\bar{g}_z=0$ through order T_c/T_F.

The coefficient g_z carries nonvanishing strong-coupling corrections of order T_c/T_F. From (3.7)–(3.9) we find

$$g_z=g_z^{(0)}+g_z^{(1)}+o(T_c/T_F) \ , \tag{4.9}$$

where $g_z^{(0)}$ is the weak-coupling coefficient,

$$g_z^{(0)} = \frac{(\frac{1}{2}\gamma\hbar)^2}{(1+F_0^a)^2} \int \frac{d\Omega}{4\pi} \frac{1}{2} \mathrm{Tr}_4 \left[\vec{H}\cdot\hat{\vec{\sigma}} \times k_B T_c \sum_n \bar{R}_{qp}^{w,c} \vec{H}\cdot\hat{\vec{\sigma}} \right] \times \left[\int \frac{d\Omega}{4\pi} [\vec{H}\cdot\vec{\Delta}_0(\hat{k})][\vec{H}\cdot\vec{\Delta}_0^*(\hat{k})] \right]^{-1}$$

$$= \left[\frac{\gamma\hbar}{1+F_0^a}\right]^2 \frac{7}{8}\zeta(3)\frac{1}{(\pi k_B T_c)^2} . \tag{4.10}$$

We decompose the strong-coupling correction into three parts,

$$g_z^{(1)} = g_z^{(\Sigma)} + g_z^{(\Delta)} + g_z^{(I)} . \tag{4.11}$$

$g_z^{(\Sigma)}$ and $g_z^{(\Delta)}$ come from the term $\delta\bar{R}_{qp}$ of Eq. (3.8); $g_z^{(\Sigma)}$ contains the order T_c/T_F corrections to the diagonal self-energy, while $g_z^{(\Delta)}$ contains the corrections to the off-diagonal self-energy. $g_z^{(I)}$ gives the strong-coupling corrections from the term of (3.8) containing $\bar{T}_{qp}^{s,c}$. We evaluate these contributions using the techniques described in I and in the appendix to perform the spin traces and to integrate over all variables except those in the quasiparticle scattering amplitudes. In this way we find

$$g_z^{(\Sigma)} = g_z^{(0)} \left[0.39 \frac{k_B T_c}{v_F p_F} \langle |T^{(s)}(\theta,\phi)|^2 + 3|T^{(a)}(\theta,\phi)|^2 \rangle \right] , \quad 0.39 \simeq \frac{3\pi^2}{28\zeta(3)}(1-\tfrac{1}{12}\pi^2)\tfrac{1}{4}\pi^2 , \tag{4.12}$$

$$g_z^{(\Delta)} = g_z^{(0)} \left[-2.23 \frac{k_B T_c}{v_F p_F} \langle [T^{(s)}(\theta,\phi)T^{(s)}(\theta',\phi') + T^{(a)}(\theta,\phi)T^{(a)}(\theta',\phi')][\cos^2(\tfrac{1}{2}\theta) + \sin^2(\tfrac{1}{2}\theta)\cos\phi] \rangle \right] ,$$

$$2.23 \simeq \frac{8\pi^2}{7\zeta(3)} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{2(n-m)}{(2m+1)(2n+1)^3} , \tag{4.13}$$

$$g_z^{(I)} = g_z^{(0)} \left[7.98 \frac{k_B T_c}{v_F p_F} \langle [T^{(s)}(\theta,\phi)T^{(s)}(\theta',\phi') - 3T^{(a)}(\theta,\phi)T^{(a)}(\theta',\phi')][\cos^2(\tfrac{1}{2}\theta) + \sin^2(\tfrac{1}{2}\theta)\cos\phi] \rangle \right] ,$$

$$7.98 \simeq \frac{\pi^2}{7\zeta(3)} \sum_{m=-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} \frac{\mathrm{sgn}\{(2n+1)[2(n+m)+1]\}}{(2n+1)^2} \right]^2 . \tag{4.14}$$

To estimate the size of these corrections, we evaluate the averages of the scattering amplitude in the *s-p* approximation.[14] We take A_0^s, A_0^a, and A_1^s from Ref. 15, and fix A_1^a by the forward scattering sum rule. At melting pressure this gives approximately 20, 1, and −11 for the averages in (4.12), (4.13), and (4.14), respectively, and implies that g_z is reduced by about 10% of its weak-coupling value. The average in (4.12) can also be obtained from the normal-state relaxation time $\tau(0)$,

$$\langle |T^{(s)}(\theta,\phi)|^2 + 3|T^{(a)}(\theta,\phi)|^2 \rangle = \frac{16\hbar T_F}{\pi^3 k_B T^2 \tau(0)} . \tag{4.15}$$

Using the value $T^2\tau(0) \simeq 0.26$ estimated from spin and orbital relaxation experiments, one finds $\langle |T^{(s)}|^2 + 3|T^{(a)}|^2 \rangle \simeq 15$.[16] The accuracy of the *s-p* approximation for the more complicated averages in $g_z^{(\Delta)}$ and $g_z^{(I)}$ is difficult to assess. A rough guess, based on the errors in the *s-p* approximation for the normal-state transport coefficients and for the *A*-phase specific-heat discontinuity, would be that the *s-p* results for $g_z^{(\Delta)}$ and $g_z^{(I)}$ are two times too large, and hence that the total strong-coupling corrections to g_z are approximately 5% of $g_z^{(0)}$. Some further discussion of the implications of this result can be found in Ref. 17. We note that the dominant contribution to $g_z^{(1)}$ comes from $g_z^{(I)}$, which is determined by the *p*-wave component of the leading frequency-dependent correction to the singlet pairing interaction.

C. Superfluid density

In order to find the strong-coupling corrections to the Ginzburg-Landau bending energies and to the superfluid momentum density, we will calculate the free-energy functional to first order in an external velocity field $\vec{v}(\vec{r})$. It is important to understand the connection between these terms and the pure gradient terms in the Ginzburg-Landau functional. Only two of the three gradient terms allowed by symmetry have independent physical significance; the third can always be eliminated by a partial integration. In contrast, the three distinct terms linear in $\vec{v}(\vec{r})$ are all independent and physically significant, as can be seen from Eqs. (1.6) and (1.7). Hence, one can find both the gradient terms and the superfluid current density from a microscopic calculation of the coupling between $\vec{v}(\vec{r})$ and the order parameter; one cannot, however, be sure of obtaining the correct superfluid current density from a microscopic calculation of the gradient terms alone, combined with Galilean invariance.

Because the total momentum density $\vec{g}(\vec{r})$ is a conserved quantity, the quasiparticle operator corresponding to the perturbation

$$-\int d^3r\, \vec{v}(\vec{r})\cdot\vec{g}(\vec{r})$$

is

$$\hat{u}_{\rm qp}^{\rm normal}(\hat{k};\vec{q}) = -[p_F/(1+\tfrac{1}{3}F_1^s)][\vec{v}(\vec{q})\cdot\hat{k}]\hat{1}$$
$$= -mv_F[\vec{v}(\vec{q})\cdot\hat{k}]\hat{1}\ . \quad (4.16)$$

Using (3.9), and keeping only terms linear in $\hat{\Delta}_0(\hat{k})$, we find

$$\overline{R}_{\rm qp}^{\rm w-c}[\vec{v}(\vec{q})\cdot\hat{k}]\hat{1} = \frac{i\pi\,\mathrm{sgn}(\epsilon_n)}{\epsilon_n^2+(\hbar v_F\hat{k}\cdot\frac{1}{2}\vec{q})^2}\hat{\Delta}_0(\hat{k})[\vec{v}(\vec{q})\cdot\hat{k}] - \frac{\pi\hbar v_F\hat{k}\cdot\frac{1}{2}\vec{q}}{|\epsilon_n|[\epsilon_n^2+(\hbar v_F\hat{k}\cdot\frac{1}{2}\vec{q})^2]}\hat{\Delta}_0(\hat{k})\hat{\tau}_3[\vec{v}(\vec{q})\cdot\hat{k}]\ . \quad (4.17)$$

After expanding (4.17) to first order in q and substituting into (3.7), we find that the weak-coupling limit of the coupling between $\delta\hat{\Delta}(\hat{k};-\vec{q})$ and $\vec{v}(\vec{q})$ is

$$N(0)\frac{\hbar m v_F^2}{2}\pi k_BT_c\sum_n\frac{1}{|\epsilon_n|^3}\int\frac{d\Omega}{4\pi}(\vec{q}\cdot\hat{k})[\vec{v}(\vec{q})\cdot\hat{k}]\tfrac{1}{2}\mathrm{Tr}_4[\delta\hat{\Delta}(\hat{k};-\vec{q})\hat{\Delta}_0(\hat{k})\hat{\tau}_3] \quad (4.18)$$

$$=\tfrac{1}{3}N(0)\frac{2m}{\hbar}K_T^{(0)}[\vec{q}\cdot\vec{v}(\vec{q})\delta_{ij}+q_iv_j(\vec{q})+q_jv_i(\vec{q})][A_{\mu i}^{0}{}^*\delta A_{\mu j}(-\vec{q})-A_{\mu i}^{0}\delta A_{\mu j}{}^*(\vec{q})]\ , \quad (4.19)$$

where $K_T^{(0)}$ is given by Eq. (1.11). From (4.19) and (3.1) we immediately deduce the weak-coupling relations

$$K_L^{(0)}=3K_T^{(0)}\ ,\quad K_C^{(0)}=K_T^{(0)} \quad (4.20)$$

These relations follow from the particularly simple form of the coupling in (4.18) between $\vec{v}(\vec{q})$ and the order parameter fluctuations, and hence are accidental weak-coupling symmetries which may be broken by strong-coupling corrections. We will see that the strong-coupling self-energy corrections modify the overall coupling strength while preserving the relations in (4.20), but the strong-coupling corrections from $\overline{T}_{\rm qp}^{\rm s-c}$ violate the first relation in (4.20). The calculation of the strong-coupling corrections to K_T, K_L, and K_C follows the same lines as the calculation for the magnetic field energy. We write

$$K_T=K_T^{(0)}+K_T^{(1)}$$
$$K_T^{(1)}=K_T^{(\Sigma)}+K_T^{(\Delta)}+K_T^{(I)}\ , \quad (4.21)$$

and similarly for K_L and K_C. The only additional result we need is the off-diagonal component of $\overline{R}_{\rm qp}^{\rm w-c}\delta\hat{\Delta}$ (the diagonal component does not contribute because $\overline{R}_{\rm qp}^{\rm w-c}[\vec{v}(\vec{q})\cdot\hat{k}]\hat{1}$ is purely off-diagonal through first order in $\hat{\Delta}_0$, and $\overline{T}_{\rm qp}^{\rm s-c}$ is number-conserving in the Ginzburg-Landau limit),

$$(\overline{R}_{qp}^{\rm w-c}\delta\hat{\Delta})_{\rm off-diag} = -\frac{\pi}{\epsilon_n^2+(\hbar v_F\hat{k}\cdot\frac{1}{2}\vec{q})^2}\times\left(|\epsilon_n|\delta\hat{\Delta}(\hat{k};\vec{q})+i\frac{\hbar v_F\hat{k}\cdot\vec{q}}{2}\times\mathrm{sgn}(\epsilon_n)\delta\hat{\Delta}(\hat{k};\vec{q})\hat{\tau}_3\right)\ . \quad (4.22)$$

By substituting (4.22), (4.17), and (3.8) into (3.7), and integrating out all variables except those in the quasiparticle scattering amplitudes, we find

$$K_T^{(\Sigma)}=K_T^{(0)}\left(0.39\frac{k_BT_c}{v_Fp_F}\langle|T^{(s)}(\theta,\phi)|^2+3|T^{(a)}(\theta,\phi)|^2\rangle\right)\ , \quad (4.23a)$$

$$K_L^{(\Sigma)}=3K_T^{(\Sigma)},\quad K_C^{(\Sigma)}=K_T^{(\Sigma)}\ , \quad (4.23b)$$

$$K_T^{(\Delta)}=K_T^{(0)}\left(-2.23\frac{k_BT_c}{v_Fp_F}\langle[T^{(s)}(\theta,\phi)T^{(s)}(\theta',\phi')+T^{(a)}(\theta,\phi)T^{(a)}(\theta',\phi')][\cos^2(\tfrac{1}{2}\theta)+\sin^2(\tfrac{1}{2}\theta)\cos\phi]\rangle\right)\ , \quad (4.24a)$$

$$K_L^{(\Delta)}=3K_T^{(\Delta)},\quad K_C^{(\Delta)}=K_T^{(\Delta)}\ , \quad (4.24b)$$

$$K_T^{(I)}=K_T^{(0)}\left(\frac{7.98}{2}\frac{k_BT_c}{v_Fp_F}\langle[T^{(s)}(\theta,\phi)T^{(s)}(\theta',\phi')+T^{(a)}(\theta,\phi)T^{(a)}(\theta',\phi')]\{3[\cos^2(\tfrac{1}{2}\theta)+\sin^2(\tfrac{1}{2}\theta)\cos\phi]^2-1\}\rangle\right)\ , \quad (4.25a)$$

$$K_L^{(l)} = 3K_T^{(l)} + K_T^{(0)} \left[5 \left(\frac{7.98}{2} \right) \frac{k_B T_c}{v_F p_F} \langle [T^{(s)}(\theta,\phi) T^{(s)}(\theta',\phi') + T^{(a)}(\theta,\phi) T^{(a)}(\theta',\phi')] \times \{1 - [\cos^2(\tfrac{1}{2}\theta) + \sin^2(\tfrac{1}{2}\theta)\cos\phi]^2\} \rangle \right] , \tag{4.25b}$$

$$K_C^{(l)} = K_T^{(l)} . \tag{4.25c}$$

The constants 0.39, 2.23, and 7.98 have the same origins here as in (4.12)–(4.14).

Estimating these contributions with the *s*-*p* approximation, we find at melting pressure that $K_T^{(1)}/K_T^{(0)} = K_C^{(1)}/K_C^{(0)} \simeq 0.02$, $K_L^{(1)}/K_L^{(0)} \simeq 0.06$, and $(K_L - 3K_T)/K_T^{(0)} \simeq 0.12$. For most purposes these corrections should be negligible. If, however, some effect which measures the violation of the accidental weak-coupling symmetry $K_L^{(0)} - 3K_T^{(0)} = 0$ can be found, this will provide information on the leading frequency dependence of the triplet pairing interaction. Thus, although the nontrivial strong-coupling corrections to the bending energies and to the magnetic energies are relatively small, they nevertheless can yield valuable new insight into the nature of the microscopic pairing interaction.

ACKNOWLEDGMENTS

This work was supported in part by the Nordic Institute for Theoretical Atomic Physics and by the NSF through Grant No. H040748000.

APPENDIX

The purpose of this appendix is to describe simple techniques for evaluating the momentum integrals in the $l=1$ strong-coupling free energy. By following the steps leading to Eqs. (3.20) and (3.21) of I, one can reduce the strong-coupling momentum integrals to integrals over the arguments $\cos(\frac{1}{2}\theta)\phi$ of the scattering amplitudes; for $l=1$ pairing the integrands contain functions of the form

$$\int \frac{d\Omega_{\hat{n}}}{4\pi} \int_0^{2\pi} \frac{d\psi}{2\pi} (\hat{k}_1 \cdot \hat{e}_\alpha)(\hat{k}_2 \cdot \hat{e}_\beta) \times (\hat{k}_3 \cdot \hat{e}_\gamma)(\hat{k}_4 \cdot \hat{e}_\sigma) . \tag{A1}$$

Here $\{\hat{e}_\mu\}$ is a set of orthogonal unit vectors, and $\hat{k}_1, \ldots, \hat{k}_4$ are unit vectors with fixed orientations relative to one another, which form a rigid body whose absolute orientation is given by $\hat{n}$ and ψ. Hence the integral (A1) is over all orientations of the rigid tetrad of vectors $\hat{k}_1, \ldots, \hat{k}_4$. On symmetry grounds, (A1) must have the form

$$\delta_{\alpha\beta}\delta_{\gamma\sigma} A(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4) + \delta_{\alpha\gamma}\delta_{\beta\sigma} A(\hat{k}_1,\hat{k}_3;\hat{k}_2,\hat{k}_4) + \delta_{\alpha\sigma}\delta_{\beta\gamma} A(\hat{k}_1,\hat{k}_4;\hat{k}_2,\hat{k}_3) , \tag{A2}$$

with

$$A(\hat{k}_i,\hat{k}_j;\hat{k}_l,\hat{k}_m) = a_1(\hat{k}_i \cdot \hat{k}_j)(\hat{k}_l \cdot \hat{k}_m) + a_2[(\hat{k}_i \cdot \hat{k}_l)(\hat{k}_j \cdot \hat{k}_m) + (\hat{k}_i \cdot \hat{k}_m)(\hat{k}_j \cdot \hat{k}_l)] . \tag{A3}$$

To find the constants a_1 and a_2 we evaluate (A1) in the special cases $\alpha=\beta \neq \gamma=\sigma$, $\hat{k}_1=\hat{k}_2=\hat{k}_3=\hat{k}_4$ and $\alpha=\beta\neq\gamma=\sigma$, $\hat{k}_1=\hat{k}_2$, $\hat{k}_3=\hat{k}_4$, $\hat{k}_1\cdot\hat{k}_3=0$. In this way we obtain $a_1=\frac{2}{15}$ and $a_2=-\frac{1}{30}$.

We also note that in the *s*-*p* approximation with A_1^a fixed by the forward scattering sum rule, the expression for the triplet scattering amplitude $T_t(\theta',\phi')$ simplifies considerably. Using Eq. (3.22) of I, we find

$$T_t(\theta',\phi') = [(A_0^s + A_0^a) + (A_1^s + A_1^a)\cos\theta']\cos\phi' = (A_1^s + A_0^a)(3\cos^2(\tfrac{1}{2}\theta) - 1 + \sin^2(\tfrac{1}{2}\theta)\cos\phi) \tag{A5}$$

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