Spin anisotropy and crossover in the Potts model

D. J. Wallace
Department of Physics, The University, Southampton S09 5NH, England

A. P. Young*
Institut Laue-Langevin, 156X Centre de Tri, 38042 Grenoble Cédex, France
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A group-theoretical analysis is presented of the perturbations in the $(n+1)$-state Potts model, which arise when the equivalence between the states is broken. In the continuum version of this model, described by n real fields, all possible quadratic perturbations transform according to two irreducible representations of the underlying symmetry group, so there are two associated crossover exponents. One of these exponents is equal to the order-parameter exponent β for any n. In the limit $n \to 0$ the other exponent ϕ describes crossover between percolation and random Ising behavior and is also related to an exponent describing the conductivity of a random resistor network close to the percolation threshold. We show that $\phi = 1$, in this limit, to all orders in perturbation theory. Consequently, there are no new exponents associated with crossover to anisotropic behavior for $n \to 0$.

I. INTRODUCTION

The Potts[1] model is a generalization of the Ising model in which there are $n+1$ spin states on each lattice site and the energy between a neighboring pair takes one value if the spin states of the pair are the same and another value if they are different. As $n \to 0$ this model describes[2] the percolation problem, which may be viewed[3] as the $T=0$ limit of a dilute Ising model where the interaction between neighbors has value J or 0 at random. Except for $n=1$, which is the Ising model, the Potts model has different exponents from the more commonly studied spin systems even in mean-field theory which in this case is known to be valid[4] for $d>6$, whereas the mean-field theory usually holds for $d>4$.

The dilute Ising model at finite temperature may be described by an m-component spin model[5] with cubic symmetry in the limit $m \to 0$ and so has exponents different from the percolation problem. Temperature is therefore a relevant perturbation to the $T=0$ dilute Ising problem which causes a crossover from percolation to random Ising behavior. Recently, Stephen and Grest[6] have shown that the exponent characterizing this crossover is equal to a crossover exponent ϕ for anisotropy induced in the $n \to 0$ Potts model when the equivalence between the $n+1$ states is broken. They show that $\phi = 1$ up to order ϵ^2 where $\epsilon = 6-d$. It is therefore of interest to discuss in some detail the anisotropic Potts model and in this note we classify, by group-theoretical methods, the types of perturbation that may be induced. As a byproduct we show that $\phi = 1$ to all order in perturbation theory.

Additional interest in the anisotropic Potts model has arisen from a recent renormalization-group treatment[7] of the random resistor network. The bulk conductivity Σ vanishes, as the concentration of conducting links p approaches the percolation threshold. The corresponding exponent μ is then shown in Ref. 7 to be related to a crossover exponent describing anisotropy in the $n \to 0$ Potts model.

II. GROUP-THEORETICAL ANALYSIS

The partition function of the Potts model may be written[8]

$$Z = \sum_{\{e\}} \exp\left(-\frac{1}{2} \sum_{l,l'} K_{ll'} e_i(l) e_i(l')\right) , \tag{1}$$

where the statistical sum runs over the $n+1$ n-dimensional vectors e_i^α ($\alpha = 1,\ldots,n+1$, $i=1,\ldots,n$) on each site l, which satisfy the relations[8]

$$e_i^\alpha e_i^\beta = (n+1)\delta^{\alpha\beta} - 1 , \tag{2a}$$

$$e_i^\alpha e_j^\alpha = (n+1)\delta_{ij} , \tag{2b}$$

$$\sum_{\alpha=1}^{n+1} e_i^\alpha = 0 . \tag{2c}$$

Standard techniques enable one to rewrite (1) as a continuum model with n real fields ϕ, and "Hamiltonian" given by[8]

$$H=\int d^dx\left\{\frac{1}{2}[r_0\phi^2+(\nabla\phi)^2]+\frac{1}{3!}g_0d_{ijk}\phi_i\phi_j\phi_k+\frac{1}{4!}(u_0d_{ijkl}+v_0s_{ijkl})\phi_i\phi_j\phi_k\phi_l\right\}\ , \tag{3}$$

where

$$d_{ijk}=\sum_{\alpha=1}^{n+1}e_i^\alpha e_j^\alpha e_k^\alpha\ , \tag{4a}$$

$$d_{ijkl}=\sum_{\alpha=1}^{n+1}e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha\ , \tag{4b}$$

$$s_{ijkl}=\tfrac{1}{3}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\ , \tag{4c}$$

and terms of higher order in fields and derivatives are neglected.

The underlying symmetry of the Potts model arises from the equivalence of the $n+1$ states which means that the indices $\alpha,\beta,\gamma,\ldots$ can be permuted amongst each other without changing the physics. The corresponding group is therefore the symmetric (or permutation) group of $n+1$ objects S_{n+1}.[9] It will now be demonstrated that the ϕ_i transform like the representation $(n,1)$ of this group.

Consider the quantities M^α defined by

$$M^\alpha=\phi_ie_i^\alpha\ , \tag{5}$$

and the transformed quantities M'^α obtained by a permutation of the states, so

$$M'^\alpha=M^\beta\tilde{D}^{\beta\alpha}\ , \tag{6}$$

where $\tilde{D}^{\beta\alpha}$ is a permutation matrix with one element unity in each row and column and the remaining elements, zero. Through Eq. (5) this transformation on M induces a transformation on ϕ:

$$\phi_i\rightarrow\phi_i=\phi_jD_{ji}\ , \tag{7}$$

where

$$D_{ij}=e_i^\alpha\tilde{D}^{\alpha\beta}e_j^\beta/(n+1)\ . \tag{8}$$

The character of the representation χ is $\mathrm{Tr}D$ so

$$\chi=\frac{e_i^\alpha\tilde{D}^{\alpha\beta}e_i^\beta}{n+1}=\mathrm{Tr}\tilde{D}-(n+1)^{-1}\sum_{\alpha,\beta}\tilde{D}^{\alpha\beta}\ .$$

$\mathrm{Tr}D$ is the number of states which are not permuted (number of one cycles) which is denoted by α, and $\sum_{\alpha\beta}\tilde{D}^{\alpha\beta}=n+1$, so

$$\chi=\alpha-1\ , \tag{9}$$

which is just the character of the representation $(n,1)$.[9]

Now the direct product of two irreducible representations contains the identity representation once if the representations are the same and not at all if they are different. It immediately follows that there is only one quadratic invariant of the ϕ's, which is clearly ϕ^2. To investigate the number of third- and fourth-order invariants we need to know how the direct product $(n,1)\times(n,1)$ decomposes into irreducible representations. The result is[9]

$$(n,1)\times(n,1)=(n+1)+(n,1)+(n-1,2)+(n-1,1^2)\ , \tag{10}$$

where $(n+1)$ is the identity representation ϕ^2, and by inspection we deduce that $(n,1)$ corresponds to $d_{ijk}\phi_j\phi_k$. The representation $(n-1,1^2)$ is $\frac{1}{2}n(n+1)$-fold degenerate and so must correspond to the antisymmetric combination, while $(n-1,2)$ has degeneracy $\frac{1}{2}(n+1)(n-2)$. From (10) it follows that $(n,1)^3$ contains the identity representation once so there is just one cubic invariant which must be d_{ijk} defined in (4a). Since $(n,1)\times(n,1)$ contains four irreducible representations, $(n,1)^4$ contains the identity representation four times and there are four quartic invariants

$$d_{ijkl},\quad \delta_{ij}\delta_{kl},\quad \delta_{ik}\delta_{jl},\quad \delta_{il}\delta_{jk}\ . \tag{11}$$

Of course only the symmetric tensors d_{ijkl} and s_{ijkl} can appear in Hamiltonian (3). [For $n=2$, $(n-1,2)$ is not a standard tableau; there are ony three representations in the decomposition (10) and d_{ijkl} is equivalent to s_{ijkl}.]

Equation (10) also tells us that all perturbations to the quadratic term in (3) which destroy the isotropy of that term, must transform according to the two irreducible representations $(n,1)$ and $(n-1,2)$ of S_{n+1}, so there are two associated crossover exponents. One of these will turn out to be the percolation-random Ising crossover exponent of the Stephen-Grest theory so it seems appropriate to give a few remarks on their calculation at this stage.

At zero temperature the effective Hamiltonian for the dilute Ising model is[6,10]

$$\overline{H}=-\frac{1}{2}\sum_{l,l'}\overline{K}_{ll'}\left[1-\prod_{\tau=1}^{m}(1+\sigma_l^\tau\sigma_{l'}^\tau)\right] \tag{12}$$

in the limit $m\rightarrow0$, where $\sigma_l^\tau=\pm1$. The product in (12) is zero unless $\sigma_l^\tau=\sigma_{l'}^\tau$, for all τ and so (12) corresponds to a 2^m state Potts model where a "state" denotes a particular configuration of the σ^τ on a site.

In fact, expanding the product in (12) and comparing with (1) one sees that the values of the 2^m-1 operators

$$\sigma_1,\sigma_2,\ldots,\sigma_m,\sigma_1\sigma_2,\ldots,\sigma_1\sigma_2,\ldots,\sigma_m$$

in the 2^m states form a representation of the vectors e_i^α with all components ±1. One can readily verify that the relations in equation (2) are satisfied. If we take $m=2$, for example, the three operators are σ_1, σ_2, and $\sigma_1\sigma_2$ whose values in the four states $(\uparrow\uparrow)$, $(\uparrow\downarrow)$, $(\downarrow\uparrow)$, $(\downarrow\downarrow)$ give the vectors (1,1,1),

$(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$, which are indeed the corners of a three-dimensional tetrahedron. In general we see that the hypertetrahedron of 2^m vertices in 2^m-1 dimensions can be neatly embedded in the cube $(\pm 1, \pm 1, \ldots, \pm 1)$ and a particular representation for this is

$$\sigma_1, \sigma_2, \ldots, \sigma_m, \sigma_1\sigma_2, \ldots, \sigma_1\sigma_2, \ldots, \sigma_m$$

with $\sigma_i = \pm 1$ $(i=1,\ldots m)$. In this representation, with $n=2^m-1$, $d_{iijk}=(n+1)\delta_{jk}$ from (2b) and d_{ijk} is zero if any two of the indices are equal.

For T small but nonzero Stephen and Grest[6] show that the pair interaction of the Potts model $Ke_i(l)e_i(l')$ is replaced by $\sum_i K_i e_i(l)e_i(l')$, where the K_i are no longer all equal. The equivalence between the Potts states is broken and in the above representation the quadratic part of the continuum model is $\sum_i r_i \phi_i^2$. This necessarily corresponds to a perturbation involving the representation $(n-1,2)$ because the other possibility $d_{ijk}\phi_j\phi_k$ has only off diagonal $(j \neq k)$ terms in this representation of the e_i^α. Actually removing the equivalence between the states in the lattice model also changes the cubic and higher terms in the continuum version, in a way consistent with the lower symmetry, but power counting arguments show that the quadratic perturbation is the most relevant, at least for dimensions greater than two.

In their treatment of the random resistor network Dasgupta *et al.*[7] relate this problem to an anisotropic s^m state Potts model in the limits $m\to 0$, $s\to 0$, where the limit $m\to 0$ is to be taken first. Again the quadratic part of the continuum model is diagonal and corresponds to a perturbation of the $(n-1,2)$ representation.

III. CROSSOVER EXPONENTS

Having completed the group-theoretical part of this analysis we now show that the crossover exponent for quadratic perturbations transforming like $(n-1,2)$ is unity to all orders in perturbation theory for $n\to 0$.

Define[11,12] a vertex function $\Gamma^{kl}_{\phi_i\phi_j}$ with two external legs ϕ_k and ϕ_l, and an insertion of an operator $\phi_i\phi_j$. The structure of the Feynman graphs is shown in Fig. 1. The Γ's must be invariant quantities and therefore be expressible in terms of the four invariant tensors in Eq. (11). Since the insertion $\phi_i\phi_j$ is symmetric in $i\to j$, we can write

$$\Gamma^{kl}_{\phi_i\phi_j} = A d_{ijkl} + B(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + C\delta_{ij}\delta_{kl} \ , \quad (13)$$

where A, B, and C are invariant functions of the external momenta, etc.

It is now straightforward to show that in the limit $n\to 0$ one obtains the *same* vertex function for inser-

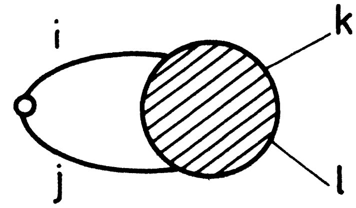

FIG. 1. Schematic representation of a graph contributing to $\Gamma^{kl}_{\phi_i\phi_j}$.

tions corresponding to the representations $(n+1)$ $(\equiv \phi^2 = \sum_{i=1}^n \phi_i^2)$ and $(n-1,2)$. For a ϕ_1^2 insertion gives the vertex function

$$\Gamma^{kl}_{\phi_1^2} = [(n+1)A + C]\delta_{kl} + 2B\delta_{k1}\delta_{l1} \ , \quad (14)$$

using definition (4b) of d_{ijkl} and the explicit representation for the e_i^α's in terms of the σ's. Hence the eigenoperators are ϕ^2, with

$$\Gamma^{kl}_{\phi^2} = [n(n+1)A + nC + 2B]\delta_{kl} \quad (15a)$$

and $\{\phi_1^2\} \equiv \phi_1^2 - \phi^2/n$, with

$$\Gamma_{\{\phi_1^2\}} = 2B[\delta_{k1}\delta_{l1} - (1/n)\delta_{kl}] \ . \quad (15b)$$

The factors δ_{kl} and $[\delta_{k1}\delta_{l1} - (1/n)\delta_{kl}]$ in Eqs. (15a) and (15b) give the same tensor structure as the operator insertions ϕ^2 and $\{\phi_1^2\}$, respectively. Dividing out this tensor part both vertex functions are equal to $2B$ in the limit $n\to 0$ so the operator ϕ_1^2 behaves like ϕ^2. Therefore, in terms of reduced temperature[13] t the crossover exponent of $\{\phi_1^2\}$ and hence of all operators in the $(n-1,2)$ representation is 1 to all orders in perturbation theory.[14] This result arises entirely from the Potts symmetry and is not tied to an expansion in $\epsilon(=6-d)$ which would correspond to considering only the cubic interaction in Eq. (3). Even with the inclusion of quartic and higher terms, which might possibly be necessary at sufficiently low dimensionality, the result that $\phi = 1$ still holds within perturbation theory. On the basis of numerical calculations Dasgupta *et al.*[7] argue that $\phi \neq 1$ for $d<4$. If real this discrepancy cannot be explained on the grounds that quartic interactions may be needed in the continuum model for this dimensionality range.

This derivation of the result depends on the choice of the particular representation of the e's valid only for $n=2^m-1$. In fact the result can be proved for arbitrary n. The ϕ^2 insertion is always as in (15a) and since from (13),

$$d_{mij}\Gamma^{kl}_{\phi_i\phi_j} = [(n^2-1)A + 2B]d_{mkl} \ , \quad (16)$$

there is always an n-fold degenerate eigenoperator $d_{mij}\phi_i\phi_j$ $(m=1,2,\ldots n)$ with vertex function

$A(n^2-1)+2B$. However the sum of all the vertex functions equals the sum of the diagonal elements

$$\sum_{ij} \Gamma^{ij}_{\phi_i \phi_j} = n[n(n+1)A + C + (n+1)B] \;\; .$$

Hence by simple subtraction the remaining $\frac{1}{2}(n+1)(n-2)$ vertex functions must have the value $2B$ as in Eq. (15b).

Finally, we discuss crossover induced by the eigenoperators $d_{mij}\phi_i\phi_j$ following an argument of Symanzik.[15] The graphs which contribute are as in Fig. 1 except that the insertion now looks exactly like another three point vertex d_{mij}. Hence order by order in perturbation theory we have

$$\frac{1}{2} g_0 \Gamma^{kl}_{d_{mij}\phi_i\phi_j} = d_{mkl}\Gamma^{(3)} \;\; , \qquad (17)$$

where the right-hand side is the usual three-point function, the two vertex functions being evaluated at the same external momenta. Thus the scaling behavior of the operator $d_{mij}\phi_i\phi_j$ is determined by the scaling behavior of ϕ alone. Specifically, if we take the vertex functions at zero momentum one has[16] $\Gamma^{(N)} \sim t^{d\nu - N\beta}$, with a single ϕ^2 insertion the exponent becomes $d\nu - N\beta - 1$ and one can show that inserting an anisotropic quadratic operator characterized by crossover exponent $\bar{\phi}$ the vertex function varies as $t^{d\nu - N\beta - \bar{\phi}}$. Consequently

$$\frac{1}{2} g_0 \Gamma^{kl}_{d_{mij}\phi_i\phi_j} \Big/ \Gamma^{(2)} \sim t^{-\bar{\phi}} \;\; .$$

However the left-hand side is just $\Gamma^{(3)}/\Gamma^{(2)} \sim t^{-\beta}$ so the crossover exponent is[17] β.

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*Present address: Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, N. Y. 14853

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[13]This is of course the reduced temperature characterizing the deviation of the Potts model from criticality and is *not* the temperature of the dilute Ising model, discussed in the Introduction, which is being represented by a Potts Hamiltonian.

[14]If the limit $n \to 0$ is taken *before* the diagonalization of the insertions of the ϕ_i^2 and ϕ^2 operators then one obtains a Jordan canonical form

$$\begin{bmatrix} 2B & A+C \\ 0 & 2B \end{bmatrix}$$

which cannot be diagonalized. The coupled renormalization-group equations for $\Gamma^{kl}_{\phi_1^2}$ and $\Gamma^{kl}_{\phi^2}$ then exhibit logarithmic violations of scaling. It is not clear to us on physical grounds which is the correct procedure.

[15]See the Erratum to A. J. Macfarlane and G. Woo, Nucl. Phys. B 77, 91 (1974); namely, 86, 548(E) (1975).

[16]See, e.g., E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, London, 1976), Vol. 6, p. 125.

[17]These simple scaling arguments are of course verified by the renormalization group.