# Josephson-junction threshold viewed as a critical point*

A. R. Bishop† and S. E. Trullinger‡
*Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853*
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We examine the expression for the mean thermal-noise voltage in the dc Josephson effect obtained by Ambegaokar and Halperin via a Brownian-motion analogy. We find that their expression can be reduced to closed form by two different methods which give the same result. This exact closed-form expression is used to derive analytic approximations in several limits and we find that the behavior of the mean voltage near the threshold current is characterized by "critical exponents" which bear a remarkable resemblance to those exhibited by an order parameter near the critical point of a continuous phase transition.

## I. INTRODUCTION

In recent years there has been considerable interest in comparisons between instabilities in dissipative nonlinear systems and thermodynamic phase-transition phenomena. The quantum-mechanical analysis of the threshold behavior of a single-mode laser has been studied most extensively[1–3] and the close analogy with a ferromagnetic phase transition appears to be well established. Similar analogies have been suggested for threshold instabilities in chemically reacting systems,[4] thermal instabilities in the Bénard problem,[5] the Gunn instability,[6] tunnel diode[7] and parametric oscillator[8] instabilities, the Wien bridge oscillator near threshold,[9] etc. General reviews of these topics are given in Ref. 10.

In this paper we investigate a close analogy between the behavior of the mean thermal-noise voltage $\nu$ in the dc Josephson effect and the behavior of an order parameter for a classical mean-field phase transition. The expression for $\nu$ obtained by Ambegaokar and Halperin[11] (AH) using a Brownian-motion analogy is reduced to closed form by two alternate methods which give the same result. This simple closed-form expression allows us to easily determine analytic approximations in several limits and by examining these limiting forms we extract "critical exponents" which govern the behavior of $\nu$ near the threshold current. We note that the exponents are reminiscent of a classical mean-field transition,[12] where $\nu$ plays the role of an order parameter, the applied current serves as the reservoir variable (temperature), and the temperature serves as an external field conjugate to the order parameter.

The outline of the paper is as follows. In Sec. II we briefly describe the AH theory which is based on a simple analogy to a Brownian-motion problem, we exhibit their expression for the mean thermal-noise voltage, and we replot their $I$-$V$ curves in a slightly different but suggestive fashion. In Sec. III we reduce the AH expression for $\nu$ to closed form and obtain several asymptotic expressions for $\nu$ and related quatities. In Sec. IV we examine the results of Sec. III to extract critical exponents and other information relating to the critical behavior of the system. In Sec. V we summarize our results and present comments and observations of a general nature.

## II. NOISE VOLTAGE AND THE BROWNIAN-MOTION ANALOGY (AH THEORY)

In a letter published a few years ago, Ambegaokar and Halperin[11] presented a calculation of the thermal-noise voltage arising from fluctuating-noise currents in the dc Josephson effect. They made use of a mechanical analogy to recast the problem in terms of the Brownian motion of a particle in a nonlinear potential (given below), and determined the noise voltage in the limit of small capacitance by solving a Smoluchowski equation for the steady-state distribution function of the "particle's" coordinate.

In terms of the mechanical analogy, the equations governing the motion of the "particle" are[11]

$$\frac{d\theta}{dt}=\frac{p}{M} \tag{2.1}$$

and

$$\frac{dp}{dt}=-\frac{dU}{d\theta}-\eta p+L(t) \quad . \tag{2.2}$$

The particle coordinate $\theta$ corresponds to the difference in the phases of the order parameter on opposite sides of the Josephson junction. The particle mass is given by $M=(\hbar/2e)^2C$, where $C$ is the capacitance of the junction and the particle momentum is given by $p=(\hbar C/2e)V$, where $V$ is the potential difference. The damping constant $\eta$ is given by $\eta=(RC)^{-1}$, where $R$ is the resistance of the junction; $L(t)$ is a thermal white-noise force arising from fluctuating-noise currents. The "potential energy" $U(\theta)$ is given by $U(\theta)=-\frac{1}{2}\gamma T(x\theta+\cos\theta)$ with $\gamma\equiv\hbar I_1(T)/eT$ and $x\equiv I/I_1(T)$, where $I_1(T)$ is the maximum Josephson current at temperature $T$ ($k_B=1$) in the absence of noise, and $I$ is the driving current from a constant current source.

In the large damping limit, defined by

$$\Omega\equiv RC(2eI_1/\hbar C)^{1/2}\equiv\omega_J/\eta<<1\ ,$$

AH solve a Smoluchowski equation[13] for the steady-state coordinate distribution function and thereby obtain the dimensionless mean voltage $v\equiv\langle V\rangle/I_1R$:

$$v(x,\gamma)=\frac{4\pi}{\gamma}\left[(e^{\pi\gamma x}-1)^{-1}\left(\int_0^{2\pi}d\theta f(\theta)\right)\left(\int_0^{2\pi}\frac{d\theta'}{f(\theta')}\right)+\int_0^{2\pi}d\theta\int_\theta^{2\pi}d\theta'\frac{f(\theta)}{f(\theta')}\right]^{-1}\ , \tag{2.3}$$

where $f(\theta)$ is a Boltzmann factor given by

$$f(\theta)=\exp[-U(\theta)/T]$$
$$=\exp[\tfrac{1}{2}\gamma(x\theta+\cos\theta)]\ . \tag{2.4}$$

Plots of $v$ vs $x$ can be obtained by numerical evaluation of (2.3) or any of the alternate expressions given in Sec. III. In Fig. 1 we plot $v$ vs $x$ for several values of the temperature ($T\propto\gamma^{-1}$). Note that as the temperature approaches zero ($\gamma\to\infty$), a sharp threshold behavior appears and at $T=0$, no voltage develops until the current exceeds the maximum Josephson current ($x=1$). At finite temperature the thermal noise currents serve to round this threshold behavior. The nonlinear behavior of $v$ as a function of $x$ near $x=1$ can be seen more clearly in derivative plots ($dv/dx$ vs $x$) which we show in Fig. 2.

The behavior of $v$ exhibited in Fig. 1 suggests an analogy with a continuous phase transition where $v$ plays the role of an order parameter, $x$ serves as an external reservoir variable (e.g., temperature), and $T$ serves as an external field conjugate to the order parameter. With this identification we see that the order develops *above* the "critical temperature" ($x_c=1$) in zero external field ($T=0$). The analogy can perhaps be made more appealing if we instead regard $1/x$ as the reservoir variable so that the order develops below $x_c=1$. In Fig. 3 we plot $v/x$ vs $1/x$ for a few values of the "external field" ($\gamma^{-1}$). The resemblance to a mean-field ferromagnetic phase transition[12] is striking, and motivates the question of whether the "critical behavior" near $x_c=1$ is characterized by "critical exponents" which bear any relationship to those of the magnet problem. In Sec. III we pursue this question by investigating the mathematical behavior of $v$ in more detail.

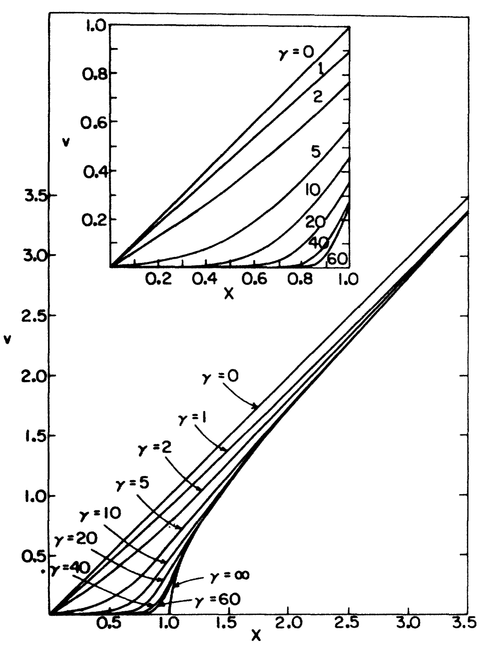


FIG. 1. Dimensionless mean voltage vs the dimensionless applied current for several values of the temperature ($T\propto\gamma^{-1}$) (cf. Ref. 15)

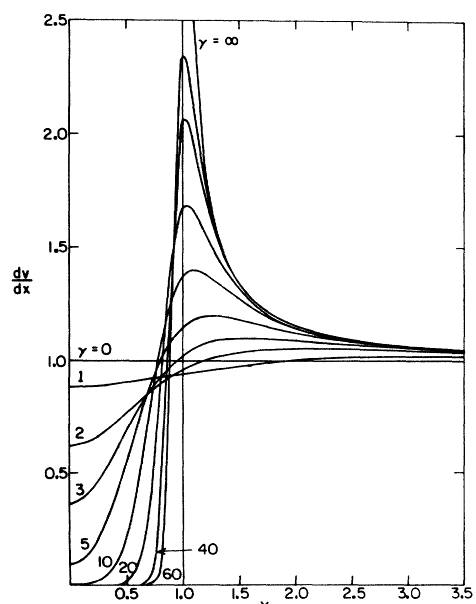


FIG. 2. Plots of $dv/dx$ vs $x$ showing a divergence at $x=1$ as the temperature is lowered to zero ($\gamma\to\infty$).

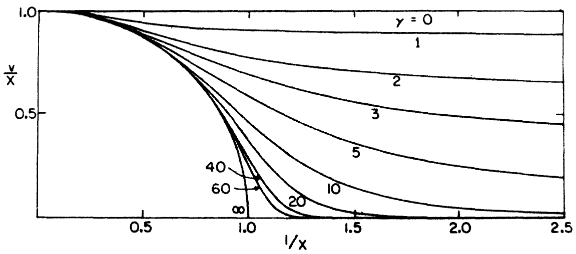


FIG. 3. Plots of $\nu/x$ vs $x^{-1}$ suggesting the analogy with magnetization curve for a ferromagnet. The temperature $T \propto \gamma^{-1}$ plays the role of an external applied field.

## III. BEHAVIOR OF THE MEAN VOLTAGE: A CLOSED-FORM EXPRESSION AND LIMITING ASYMPTOTIC FORMULAS

In this section we note that the AH expression for $\nu$ [Eq. (2.3)] can be reduced to closed form by two different methods which give the same result. This closed-form expression is then used to obtain several analytic approximations which aid the investigation of the phase-transition analogy.

The two alternative routes for reducing Eq. (2.3) to closed form consist of straightforward manipulations of intermediate results obtained by other authors. The first of these starts from a simple expression for $\nu$ obtained by Henkels[14] using a method due to Stratonovich[15]:

$$\nu(x,\gamma) = \frac{\sinh\frac{1}{2}\pi\gamma x}{\gamma}\left[\int_0^{\pi/2} \cosh(\gamma x y) \times I_0(\gamma\cos y)\, dy\right]^{-1}, \quad (3.1)$$

where $I_0$ is the modified Bessel function. Upon making the substitution $\cosh(\gamma x y) = \cos(i\gamma x y)$, the integral appearing in Eq. (3.1) can be found in tables,[16] with the result that

$$\nu(x,\gamma) = \frac{2}{\pi\gamma}\frac{\sinh\frac{1}{2}\pi\gamma x}{|I_{i(\gamma/2)x}(\frac{1}{2}\gamma)|^2}, \quad (3.2)$$

where $I_{i(\gamma/2)x}(\frac{1}{2}\gamma)$ is the modified Bessel function of imaginary order.

The second method of arriving at Eq. (3.2) begins with a series form for $\nu$ found by Falco, Parker, and Trullinger.[17] Starting directly from Eq. (2.3), these authors used the series representation[18]

$$e^{\pm(\gamma/2)\cos\theta} = \sum_{k=-\infty}^{+\infty} I_k\left(\pm\frac{\gamma}{2}\right)\cos k\theta, \quad (3.3)$$

to obtain

$$\nu(x,\gamma) = \frac{1}{\gamma^2 x}\left[\sum_{k=-\infty}^{+\infty} (-1)^k \frac{I_k^2(\frac{1}{2}\gamma)}{\gamma^2x^2+4k^2}\right]^{-1}. \quad (3.4)$$

The summation in Eq. (3.4) can be performed in a straightforward manner using the method of residues and the relation[18] $I_k = I_{-k}$ to obtain the result given by Eq. (3.2). It is worth noting that Eq. (3.1) can be obtained directly from Eq. (3.4) by a method[19] which employs Poisson's summation formula.[20]

The compact form exhibited for $\nu$ in Eq. (3.2) in terms of the modified Bessel function allows us to easily determine analytic approximations (in terms of simpler functions) in various limits, using known asymptotic properties of Bessel functions.[18,21] For small $\gamma$ with $x$ arbitrary, we use an ascending series formula[18] for $I_{\pm i(\gamma/2)x}(\frac{1}{2}\gamma)$ to find

$$\nu(x,\gamma) \cong x\left[1 - \frac{\frac{1}{8}\gamma^2}{(1+\frac{1}{4}\gamma^2x^2)} + \frac{(\frac{1}{256}\gamma^4)(8-\frac{3}{4}\gamma^2x^2+\frac{1}{16}\gamma^4x^4)}{(1+\frac{1}{4}\gamma^2x^2)^2(4+\frac{1}{4}\gamma^2x^2)} + \cdots\right] \quad (\gamma \text{ small, } x \text{ arbitrary}) \quad (3.5)$$

For large values of $\gamma$ we make use of the relation[18]

$$I_{i(\gamma/2)x}(\tfrac{1}{2}\gamma) = e^{\pi\gamma x/4} J_{i(\gamma/2)x}(\tfrac{1}{2}\gamma), \quad (3.6)$$

in order to use various asymptotic forms of $J_\nu(\nu z)$ when $|\nu| \gg 1$. For $\gamma$ large and $x \gg 1$ we set $\nu = i(\gamma/2)x$ and $z = 1/x \ll 1$ and use Meissel's extension of Carlini's formula[22] to obtain

$$\nu(x,\gamma) \cong (x^2-1)^{1/2}\left[\exp\frac{4x^2+1}{2\gamma^2(x^2-1)^3}\right] \quad (\gamma \text{ large, } (x^2-1) \gg \gamma^{-2/3}). \quad (3.7)$$

In the limit $\gamma \to \infty$, Eq. (3.7) is valid for all $x > 1$, i.e., $\nu \to (x^2-1)^{1/2}$. For $\gamma$ large and $x \ll 1$ we use Meissel's second expansion[23] to obtain

$$v(x,\gamma) \cong \frac{(1-x^2)^{1/2}\exp[-\frac{1}{2}\pi\gamma x + (x^2+\frac{1}{4})/\gamma^2(1-x^2)^3]\sinh\frac{1}{2}\pi\gamma x}{1+2\sinh^2\{\frac{1}{2}\gamma[(1-x^2)^{1/2}-\frac{1}{2}\pi x + x\sin^{-1}x] + (2x^2+3)/12\gamma(1-x^2)^{3/2}\}}$$

$$(\gamma \text{ large}, \ (1-x^2) >> \gamma^{-2/3}) \ . \quad (3.8)$$

In the limit of very large $\gamma$, Eq. (3.8) gives the limiting form obtained by AH[11]:

$$v(x,\gamma) \cong 2(1-x^2)^{1/2}\exp\{-\gamma[(1-x^2)^{1/2} + x\sin^{-1}x]\}\sinh\tfrac{1}{2}\pi\gamma x \underset{\gamma\to\infty}{\longrightarrow} 0 \quad (x<1) \ . \quad (3.9)$$

The approximate forms given by Eqs. (3.7) and (3.8) are valid as long $x$ is not too close to unity. The value of $x$ may be taken closer to unity as $\gamma$ increases, but not arbitrarily close for a fixed finite value of $\gamma$. Fortunately, approximate expressions do exist[24] for the region $|1-x| \lesssim \gamma^{-2/3}$, i.e., when $x$ is slightly less than or greater than unity. We obtain the following approximate form for $v$ in this region ($\epsilon \equiv 1-x$):

$$v(\epsilon,\gamma) \cong \frac{9\pi}{\gamma}\left|\sum_{m=0}^{\infty} B_m\left(i\frac{\gamma}{2}\epsilon\right)\sin[\tfrac{1}{3}(m+1)\pi] \times \Gamma(\tfrac{1}{3}m+\tfrac{1}{3})\left(i\frac{\gamma}{12}\right)^{-(1/3)(m+1)}\right|^{-2}$$

$$(\gamma \text{ large}, \ |\epsilon| \lesssim \gamma^{-2/3}) \quad (3.10)$$

where the functions $B_m(\epsilon z)$ are $m$th order polynomials. The first few of these are[24]

$$\begin{aligned} &B_0(\epsilon z) = 1 \ , \quad B_1(\epsilon z) = \epsilon z \ , \\ &B_2(\epsilon z) = \tfrac{1}{2}\epsilon^2 z^2 - \tfrac{1}{20} \ , \\ &B_3(\epsilon z) = \tfrac{1}{6}\epsilon^3 z^3 - \tfrac{1}{15}\epsilon z \ , \\ &B_4(\epsilon z) = \tfrac{1}{24}\epsilon^4 z^4 - \tfrac{1}{24}\epsilon^2 z^2 + \tfrac{1}{280} \ , \\ &B_5(\epsilon z) = \tfrac{1}{120}\epsilon^5 z^5 - \tfrac{1}{60}\epsilon^3 z^3 + \tfrac{43}{8400}\epsilon z \ . \end{aligned} \quad (3.11)$$

If we keep only the first few dominant terms appearing in the summation in Eq. (3.10), we have

$$v(\epsilon,\gamma) \cong \frac{12\pi}{\gamma}\left\{[\Gamma(\tfrac{1}{3})]^2\left(\frac{\gamma}{12}\right)^{-2/3} + \sqrt{3}\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3})\left(\frac{\gamma\epsilon}{2}\right)\left(\frac{\gamma}{12}\right)^{-1} + [\Gamma(\tfrac{2}{3})]^2\left(\frac{\gamma\epsilon}{2}\right)^2\left(\frac{\gamma}{12}\right)^{-4/3} - \tfrac{1}{3}\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3})\left[\frac{1}{8}\left(\frac{\gamma\epsilon}{2}\right)^4 + \frac{1}{40}\left(\frac{\gamma\epsilon}{2}\right)^2 - \frac{1}{280}\right]\left(\frac{\gamma}{12}\right)^{-2} + O(\gamma^{-8/3})\right\}^{-1}$$

$$(\gamma \text{ large}, \ |\epsilon| << \gamma^{-2/3}) \ . \quad (3.12)$$

In the limit as $x \to 1$ ($|\epsilon| \to 0$), this becomes

$$v(x=1,\gamma) \cong \frac{\pi}{[\Gamma(\frac{1}{3})]^2}\left(\frac{\gamma}{12}\right)^{-1/3} \times \left[1 - \frac{1}{840}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\left(\frac{\gamma}{12}\right)^{-4/3} + \cdots\right]$$

$$(\gamma \text{ large}) \ . \quad (3.13)$$

From Eq. (3.12), we also find

$$\frac{dv}{dx}\Big|_{x=1} \cong 6\sqrt{3}\pi\frac{\Gamma(\frac{2}{3})}{[\Gamma(\frac{1}{3})]^3}\left(\frac{\gamma}{12}\right)^{1/3} \times \left[1 - \frac{1}{420}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\left(\frac{\gamma}{12}\right)^{-4/3} + \cdots\right]$$

$$(\gamma \text{ large}) \quad (3.14)$$

It should be emphasized that Eq. (3.14) is obtained by taking $x \to 1$ while keeping $\gamma$ finite. If, on the other hand, we take $\gamma \to \infty$ first, we find from Eq. (3.7) that

$$\lim_{x\to 1^+} \frac{dv}{dx}\bigg|_{\gamma=\infty} = (x^2-1)^{-1/2} , \quad (3.15)$$

and from Eq. (3.9) we find

$$\lim_{x\to 1^-} \frac{dv}{dx}\bigg|_{\gamma=\infty} = 0 . \quad (3.16)$$

We also obtain the following results from Eqs. (3.7) and (3.9):

$$\frac{dv}{d(1/\gamma)} \cong (x^2-1)^{-5/2}(4x^2+1)\gamma^{-1} \underset{\gamma\to\infty}{\longrightarrow} 0$$

$$[\gamma \text{ large}, \ (x^2-1) >> \gamma^{-2/3}] , \quad (3.17)$$

$$\frac{dv}{d(1/\gamma)} \cong (1-x^2)^{1/2}\exp\{-\gamma[(1-x^2)^{1/2}+x\sin^{-1}x]\}$$

$$\times \{2\gamma^2[(1-x^2)^{1/2}+x\sin^{-1}x]$$

$$\times \sinh\tfrac{1}{2}\pi\gamma x - \pi\gamma^2 x\cosh\tfrac{1}{2}\pi\gamma x\} \underset{\gamma\to\infty}{\longrightarrow} 0$$

$$[\gamma \text{ large}, \ (1-x^2) << \gamma^{-2/3}], \quad (3.18)$$

$$\lim_{x\to 1^+} \frac{d^2v}{d(1/\gamma)^2}\bigg|_{\gamma=\infty} = 5(x^2-1)^{-5/2} \quad (3.19)$$

and

$$\lim_{x\to 1^-} \frac{d^2v}{d(1/\gamma)^2}\bigg|_{\gamma=\infty} = 0 . \quad (3.20)$$

Finally, from Eq. (3.9) we find

$$v(x,\gamma) \cong \pi\gamma x\left[1+\left(\frac{\pi^2\gamma^2}{24}-\frac{1}{2}\right)x^2+O(x^4)\right]$$

$$\times \exp[-\gamma(1+\tfrac{1}{2}x^2+\cdots)]$$

$$(\gamma \text{ large}, \ x << \gamma^{-1}) . \quad (3.21)$$

In the limit as $x \to 0$, Eq. (3.2) yields

$$\lim_{x\to 0} \frac{v}{x} = \left[I_0\left(\frac{\gamma}{2}\right)\right]^{-2} \quad (\text{all } \gamma) , \quad (3.22)$$

in agreement with the result obtained by AH. Equation (3.22) also agrees with the $x \to 0$ limit of Eq. (3.21) when $\gamma$ is large.

## IV. CRITICAL-POINT ANALOGY

The form of the threshold illustrated in Fig. 1 (and Fig. 3) is reminiscent of a continuous phase transition in equilibrium systems,[12] with temperature ($T \sim \gamma^{-1}$) playing the role of the symmetry-breaking field conjugate to an order parameter (the mean voltage $v$) and with the applied current $x$ playing the role of the temperature or external reservoir variable. The threshold is sharp at $T=0$ ($\gamma=\infty$) but smeared at all finite $T$, in much the same way as a magnetic field smears the transition in a ferromagnet. The power-law behavior found in Sec. III for $v(x,\gamma)$ near the point $(x,\gamma)=(1,\infty)$ suggests that this point be viewed as a "critical point"[12] in an equilibrium phase transition analogy. In this section we explore this analogy in detail, with particular regard to the identification of "critical exponents" which characterize the power-law behavior near the critical point. In the following, we regard $v(x,\gamma)$ as the "order parameter" although at the level of a discussion of critical exponents $x^{-1}v(x,\gamma)$ may serve just as well and it may also be somewhat more appealing (see Fig. 3).

In Table I we have collected the key results from Sec. III for the behavior of the mean voltage. For clarity we have restored the dimensions for the various quantities ($V,I,T$) and for comparison we have also listed appropriate quantities for the ferromagnet. In Table II we list the critical exponents for the Josephson junction and the mean-field exponents[12] for the ferromagnet. We note that all of the exponents are the same for the two systems, except for the exponent $\gamma$ (not to be confused with the conjugate field $\gamma^{-1}$). The source of the zero value of $\gamma$ for the Josephson junction lies in the fact that $V(I,-T)=V(I,T)$, i.e., the order parameter is an *even function* of the conjugate field; thus the first-order susceptibility must vanish at zero field. It is interesting to note that (as in other mean-field systems ) the Coopersmith inequality[12]

$$\phi+2\psi-1/\delta \geqslant 1 \quad (4.1)$$

is satisfied as an equality.

These observations lead naturally to the question of whether the predictions of the static scaling hypothesis[12] for the ferromagnet are also satisfied by the Josephson junction. In terms of dimensionless quantities, the equation of state [Eq. (3.2)] should scale as

$$v(\epsilon,\gamma)=|\epsilon|^{\beta}v\left(\frac{\epsilon}{|\epsilon|},\gamma|\epsilon|^{\beta\delta}\right) \quad (\epsilon\equiv 1-x) , \quad (4.2)$$

close to the critical point ($\epsilon=0$, $\gamma=\infty$), with $\beta=\frac{1}{2}$ and $\delta=3$ for a mean-field system. It is customary to define scaling functions $F_{\pm}(z)$ so that

$$v(+1,\gamma|\epsilon|^{3/2})=F_+(\gamma|\epsilon|^{3/2}) \quad (4.3a)$$

and

TABLE I. Threshold behavior of the Josephson junction compared to the critical behavior of a ferromagnet.

| Josephson junction | Ferromagnet | |
|---|---|---|
| $\langle V\rangle$ (or $\langle V\rangle/I$) | $\langle M\rangle$ | Order parameter |
| $I$ (or $I^{-1}$) | $T$ | Reservoir variable |
| $T$ | $H$ | Field conjugate to order parameter |
| $(I=I_1,\ T=0)$ | $(T=T_c,\ H=0)$ | Critical point |
| $\langle V\rangle_{T=0}\sim(I-I_1)^{1/2},\ I>I_1$; $=0,\ I<I_1$ | $\langle M\rangle_{H=0}\sim(T_c-T)^{\beta},\ T<T_c$; $=0,\ T>T_c$ | Coexistence curve |
| $\langle V\rangle_{I=I_1}\sim T^{1/3}$ | $\langle M\rangle_{T=T_c}\sim H^{1/\delta}$ | Critical isotherm |
| $\left(\frac{\partial\langle V\rangle}{\partial T}\right)_{T=0}=0$ | $\left(\frac{\partial\langle M\rangle}{\partial H}\right)_{H=0}\sim\lvert T-T_c\rvert^{-\gamma}$ | Zero-field susceptibility |
| $\left(\frac{\partial^2\langle V\rangle}{\partial T^2}\right)_{T=0}\sim(I-I_1)^{-5/2}$ | $\left(\frac{\partial^2\langle M\rangle}{\partial H^2}\right)_{H=0}\sim(T_c-T)^{\beta(1-2\delta)}$ | Second-order susceptibility |
| $\left(\frac{\partial\langle V\rangle}{\partial I}\right)_{I=I_1}\sim T^{-1/3}$ | $\left(\frac{\partial\langle M\rangle}{\partial T}\right)_{T=T_c}\sim -H^{\psi-1}$ | |
| $\left(\frac{\partial\langle V\rangle}{\partial I}\right)^2_{I=I_1}\left(\frac{\partial\langle V\rangle}{\partial T}\right)^{-1}_{I=I_1}\sim T^0$ | $\left(\frac{\partial\langle M\rangle}{\partial T}\right)^2_{T=T_c}\left(\frac{\partial\langle M\rangle}{\partial H}\right)^{-1}_{T=T_c}\sim H^{-\phi}$ | Specific heat at constant field |

$$\nu(-1,\gamma\lvert\epsilon\rvert^{3/2})=F_-(\gamma\lvert\epsilon\rvert^{3/2})\ . \tag{4.3b}$$

Then $\nu$ should have the form

$$\nu=\lvert\epsilon\rvert^{1/2}F_\pm(\gamma\lvert\epsilon\rvert^{3/2})\ . \tag{4.4}$$

We find that, as long as $\gamma\lvert\epsilon\rvert^{3/2}\gg 1$, $\nu$ does indeed have the form (4.4). From Eq. (3.7) we find

$$F_-(z)\cong\sqrt{2}\exp[5/(16z^2)+\cdots]\quad(z\gg 1)\ , \tag{4.5}$$

and from Eq. (3.8), we find

$$F_+(z)\cong\sqrt{2}\exp\left[-\sqrt{2}z-\frac{5\sqrt{2}}{24z}+\frac{5}{32z^2}+\cdots\right]\quad(z\gg 1)\ . \tag{4.6}$$

TABLE II. Comparison of critical exponents obtainable from the equation of state for the Josephson junction and the mean-field ferromagnet.

| Critical exponent | Josephson junction | Mean-field ferromagnet |
|---|---|---|
| $\beta$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\delta$ | 3 | 3 |
| $\gamma$ | 0 | 1 |
| $\psi$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| $\phi$ | 0 | 0 |

We wish to emphasize that Eqs. (4.4)–(4.6) are valid only when $|\epsilon|$ is small *and* $\gamma$ is large enough ($T$ small enough) so that $\gamma|\epsilon|^{3/2} >> 1$. Thus, if the conjugate field ($\gamma^{-1}$) is held fixed, the scaling laws (4.4) are valid as long as the reservoir variable ($x$) *does not approach too close* to the critical point ($x=1$). When $\gamma|\epsilon|^{3/2} << 1$ the equation of state has the approximate form given by Eq. (3.10) [or (3.12)] and does *not* appear to possess any simple scaling property, although it does give classical values for the exponents $\delta$, $\psi$, and $\phi$ (Table II). Even in the region ($\gamma|\epsilon|^{3/2} >> 1$), where the scaling laws are obeyed, the scaling functions (4.5) and (4.6) are exponentials rather than the power laws of more familiar examples (e.g., the ferromagnet[12]), thus reflecting the more complex nature of the junction threshold.

## V. DISCUSSION

In Sec. IV we noted a very close analogy, at the level of critical exponents, between the threshold behavior of a nonequilibrium system (Josephson junction) and the critical behavior found near an equilibrium mean-field phase transition. Such an analogy is novel but certainly not unique; there has been considerable interest in recent years in comparisons between instabilities in several dissipative nonlinear systems and thermodynamic phase-transition phenomena. In several of the examples cited in Sec. I, it has been possible to demonstrate an explicit analogy between the nonequilibrium system and a mean-field phase transition by constructing an "order-parameter" expansion of phenomenological Landau form.[10,12] This approximation usually follows implicitly from a self-consistent-field treatment of the nonequilibrium system (e.g., the quasi-plane-wave approximation for the laser[10]). Sufficiently near threshold such approximations break down and deviations from mean-field behavior are found,[3,9] analogous to critical fluctuation corrections.[12] In the case of the Wien-bridge oscillator[9] there is sufficient external control on the Landau-expansion coefficients (i.e., choice of nonlinear elements) that classical first-order, second-order, and multicritical-point phase transitions can be simulated, and critical deviations from classical behavior can be enhanced by an external white-noise generator. We suspect that the "mean-field" exponents listed in Table II result from the restriction to the large-damping limit in the theory of Ambegaokar and Halperin.[11] This view is consistent with critical fluctuations being suppressed in a Landau prescription.[12] Similar assumptions are implicit in laser descriptions,[10] and mean-field exponents for the Wien-bridge oscillator only arise in the zero-noice limit.[9] Dramatic effects of finite (rather than infinite) damping have been found in the Brownian-motion system (dc Josephson effect) by Monte Carlo studies[25] and some approximate analytic procedures.[11,26] We can expect to find deviations from mean-field exponents if these latter results are extended sufficiently close to the threshold (low-$T$ region). We recall from Sec. IV that the equation of state exhibits simple scaling behavior near the critical current only if the temperature is sufficiently low and it will be most interesting to examine the nonscaling region $\gamma|\epsilon|^{3/2} << 1$ in the presence of finite damping.

Comparison with a simple Landau order-parameter expansion is quite direct in, e.g., the laser case because of a simpler nonlinearity in the order parameter (field amplitude of the lasing atoms) equation of motion; typically, the steady-state solution to the Fokker-Planck equation takes a relatively simple form in which low-order powers of the order parameter appear. In the Brownian-motion problem studied here, the steady-state mean velocity (voltage across the junction) has a nontrivial form [Eq. (3.2)] and we have not yet found any simple mapping onto an equivalent equilibrium system or phenomenological Landau form. Basically, the only information we have is the "equation of state" [Eq. (3.2)], but not the relevant "thermodynamic potential." Although we have not yet found an explicit equilibrium phase transition representation, we anticipate that the mapping (if one exists) should be to a real order parameter, zero-dimensional quantum system, or alternatively a time-dependent Ginzburg-Landau problem in zero dimensions.[9] Equivalently, we might transform the dynamic system to a one-dimensional static one via[27] $it \rightarrow x$ (position). The lack of explicit spatial dependence in the AH theory suggests that there should be no strict phase transition in the analogy we seek, even at $T=0$. A mean-field treatment gives[12] dimension-independent exponents and a finite transition point even in one dimension. The finite damping simulations of Ref. 25 indeed suggest that the $T=0$ threshold current is reduced by decreasing the damping constant so that it at least shows a tendency to approach zero as $\eta \rightarrow 0$.

Recently, it has become possible to solve[28] the problem of an infinite array of *coupled* nonlinear pendula (or Josephson junctions, etc.) undergoing driven Brownian motion in the overdamped limit. We expect that this problem may map onto a *two-dimensional* static phase transition problem. Nonlinear spatial fluctuations play a dominant role in this case[28] and the effects of finite damping should be especially interesting. It is found[28] that the large damping threshold exponents at $T=0$ (where there are no spatial fluctuations) are unchanged from the single-pendulum (junction) case, as we expect for a mean-field theory.

In this paper we have demonstrated,[29] at the level of exponent comparisons, an interesting analogy between the threshold characteristics of a nonequilibrium, nonlinear driven Brownian-motion problem (Josephson junction), and the critical-point properties of an equilibrium classical phase transition. We consider this problem to be a new member of a growing class[10] of nonequilibrium systems exhibiting such "critical" instabilities. In a future publication, we hope to consider (i) the effects of finite damping (e.g., $\eta^{-1}$ expansions) on the location of the threshold, on the classical exponents, and on the size of the nonclassical critical region; (ii) a transformation to a zero-dimensional time-dependent Ginzburg-Landau formalism; (iii) corresponding analysis of the coupled junction system in the light of recent analytical advances[28]. and (iv) extension to space-time correlation functions[10,30] (critical slowing down, etc.).

*Note added in proof.* *Finite*-order expansions in $\eta^{-1}$ do not appear to change the mean-field exponents reported here [T. Schneider *et al.* (unpublished)]. However, an interesting additional "critical" variety is possible for sufficiently small damping: hysteresis occurs at zero temperature [D. E. McCumber, J. Appl. Phys. 39, 3113 (1968); W. C. Stewart, Appl. Phys. Lett. 12, 277 (1968)], and one branch is unstable (cf. metastable states) at finite temperature [P. A. Lee, J. Appl. Phys. 42, 325 (1970].

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†Present address: Dept. of Physics, Queen Mary College, Mile End Rd., London E1 4NS, England.
‡Present address: Dept. of Physics, University of Southern California, Los Angeles, Calif. 90007.

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