Strong-coupling theory of superfluid transition temperatures for paramagnon models: Application to ^{3}He†

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The triplet superconducting transition temperature T_c for $l = 1$ paramagnon-induced pairing is computed within a Matsubara formulation of conventional strong-coupling theory, as a function of the interaction parameter $\bar{I}$. Using the scattering amplitudes of Fermi-liquid theory to fix $\bar{I}$ for ^{3}He at each pressure, we apply our results to ^{3}He. The computed values of T_c differ by less than a factor of 5 from those measured experimentally but the (slight) pressure dependence and the effective mass ratio $Z = m^*/m$ are incorrect. If Z is adjusted to be in better accord with experiment, we then obtain reasonable agreement with the measured magnitude and pressure dependence of T_c over the entire pressure range. General features of the paramagnon model are (i) $T_c(\bar{I})$ has a maximum value of $10^{-2} \sim 10^{-3}$ of the Fermi temperature T_F at $\bar{I} \simeq 0.995$; it seems doubtful that even under the most ideal conditions the paramagnon mechanism can be used to obtain high-temperature superconductors; (ii) $T_c(\bar{I})$ vanishes by $\bar{I} = 1$. (iii) Below $\bar{I} \sim 0.97$, the exponential form $T_c = \bar{\omega}_c e^{-b/\lambda}$ is obtained where b is close to unity, $\bar{\omega}_c$ is a constant of order $T_F/10$, and λ is the renormalized coupling constant λ^A/Z. The previously proposed analogous expression involving the spin-fluctuation frequency ω_{sf}, $T_c = \omega_{sf} e^{-1/\lambda}$, is inconsistent with our results over the entire range of $\bar{I}$. Strong-coupling corrections, deriving from the mass ratio Z, are extremely important in ^{3}He: They reduce the size of T_c by nearly two orders of magnitude and help make it relatively pressure insensitive. While the paramagnon model is undoubtedly an oversimplified description of ^{3}He, a strong-coupling calculation within this model represents a significant improvement over previous approaches in which theoretical values of T_c have differed from experiment by orders of magnitude or in which either ω_c or λ have been chosen to fit the T_c data.

I. INTRODUCTION

Recent studies of the nature of the pairing mechanism in superfluid ^{3}He have been along two main lines: phenomenological treatments[1-3] of quasiparticle-quasiparticle interactions (as in Landau Fermi-liquid theory) and microscopic theories of specific pairing mechanisms—in particular, paramagnon induced superfluidity.[4-6] The former approach, because it incorporates results of a number of different experimental measurements (of normal-state properties) probably gives a better estimate of the pairing coupling constant λ. However, because it is valid only for low-frequency phenomena it cannot be used to compute the superfluid transition temperature, which may be written $T_c = \omega_c e^{-1/\lambda}$, without introducing an arbitrary expression for the characteristic frequency ω_c. The latter approach, because it is based on an oversimplified description of the quasiparticle interactions, is not able to predict λ from first principles. However, because it is microscopic, it contains information about the dynamics of the interaction and can be used, together with conventional strong-coupling theory to compute the characteristic frequency ω_c.

It is the purpose of the present paper to set up a conventional strong-coupling theory for paramagnon induced pairing. In this way we obtain T_c as a function of $\bar{I}$ (the paramagnon interaction parameter) or, equivalently, we derive the characteristic frequency $\omega_c(\bar{I})$. Using the scattering amplitudes of Fermi-liquid theory to obtain $\bar{I}$, our results are then applied to ^{3}He. The values of T_c so obtained differ by less than a factor of 5 from those measured experimentally but the pressure dependence (although slight) is incorrect and the effective-mass ratio too large. Both of these shortcomings stem from the fact that the paramagnon model overestimates the frequency-dependent quantity $Z(\omega)$ [where $m^*/m \equiv Z(0)$]. If the mass ratio $Z = Z(0)$ is adjusted to be in better accord with experiment we then obtain semiquantitative agreement between theory and experiment for both the magnitude and pressure dependence of T_c over the entire pressure range.

The formalism used to compute $T_c(\bar{I})$ is the conventional strong-coupling (Eliashberg) theory.[7] The mass renormalization parameter $Z = m^*/m$ is derived from the Doniach-Engelsberg[8]-Penn[9] (DEP) expression. While vertex corrections are known[10] to be important for the paramagnon model, we justify the application of the usual Eliashberg theory (in which these corrections are ignored) to this problem by viewing $\bar{I}$ as a phenomenologically determined parameter. In this way, we hope to include some of the important effects which derive from vertex corrections. Using the Matsubara

representation of the gap equation, we reduce the solution of the Eliashberg equations to that of an eigenvalue problem[11] which is then solved numerically. We use the full Lindhard function in our computations; all the integrals are evaluated numerically. It was found that the usual approximations[4,8] to this function led to large overestimates of the "bare" coupling constant λ^{Δ} which resulted in order of magnitude errors in T_c.

The general features of the function $T_c(\bar{I})$ are as follows: (i) because of strong-coupling effects, which lead to a small renormalized coupling constant $\lambda \equiv \lambda^{\Delta}/Z$, the exponential expression $T_c = \bar{\omega}_c e^{-b/\lambda}$ (with $b \sim 1$ and $\bar{\omega}_c$ a constant of order one-tenth of the Fermi temperature T_F) is found to be valid for a large range of $\bar{I}$, $\bar{I} < 0.97$. We will often write T_c in the form $T_c = \omega_c e^{-1/\lambda}$, where $\omega_c = \bar{\omega}_c e^{(1-b)/\lambda}$ for $\bar{I} < 0.97$; (ii) at $\bar{I} \cong 0.995$, T_c reaches a maximum; and (iii) it falls to zero by $\bar{I} = 1.0$.

In addition to computing $T_c(\bar{I})$ we also compute the symmetric and antisymmetric Landau Fermi-liquid scattering amplitudes A^s and A^a as a function of $\bar{I}$. This calculation is relevant to our discussion of ^{3}He.

To apply the paramagnon model to ^{3}He, we use Fermi-liquid theory to obtain the parameter $\bar{I}$ as a function of pressure. Since the pairing interaction and the Fermi-liquid scattering amplitudes (A^s and A^a) derive from the same vertex function, we base our choice of the parameter $\bar{I}$ on the experimentally measured A's.[12] We have done this in two ways which gave very similar results: (i) within paramagnon theory we find $\lambda \approx \frac{1}{9} A_1^s$. This relation holds for any model in which the effective interaction is of the form $-J\vec{s}\cdot\vec{s}$. Therefore, in this procedure we chose $\bar{I}$ so that A_1^s was equal to its experimental value at each pressure; (ii) within the s-p approximation, λ is equal[3] to $-\frac{1}{12}\Sigma_l (-1)^l (A_l^s + A_l^a)$. In this second approach we selected I so that λ was given by this experimental value at each pressure. It should be noted that in both approaches the magnitude and pressure dependence of λ is determined by the A's, which are of order unity and relatively pressure insensitive. The experimental values of the λ's determined in the two ways differ by less than 7% for pressures above 3 atm. We find λ is of order 10^{-1} and fairly independent of pressure. The associated values of $\bar{I}$ are in the exponential limit of $T_c(\bar{I})$, providing that (as noted above) the effective mass ratio is computed by scaling down slightly the DEP result to give better agreement with the experimental values of m^*/m.[12,13] We find $T_c = \omega_c e^{-1/\lambda}$ with $\omega_c \approx 0.1 T_F e^{0.3/\lambda}$, which is weakly dependent on pressure, over the narrow range of $\bar{I}$ from ≈ 0.92 to 0.97 which, we find, is appropriate to ^{3}He. These values are close to those obtained by fitting $\bar{I}$ to the magnetic susceptibility.[4]

Among the most detailed previous studies of the paramagnon contribution to T_c is that of Anderson and Brinkman.[4] These authors (along with Layzer and Fay[5]) were the first to propose that paramagnon exchange may be the source of the pairing interaction in ^{3}He. While they and their collaborators[4] demonstrated that paramagnons could explain the stability of the A state and the $A \rightarrow B$ transition in the superfluid, they had difficulty explaining the magnitude and pressure dependence of the transition temperature T_c. The exponential limit of our more general theory differs from theirs in several ways: (i) These authors guessed on the basis of spin fluctuation theory that $\omega_c = (1-\bar{I})T_F$, which we show to be incorrect for the paramagnon model. (ii) The authors of Ref. 4 used an approximate form for λ^{Δ} which we find, using numerical techniques, to be correct only for $(1-\bar{I}) \leq 10^{-7}$. This approximation to λ^{Δ} leads to orders of magnitude errors in T_c. (iii) Most importantly, the authors ignored strong coupling effects. Because of (ii) and (iii) their estimate of the paramagnon contribution gave a value for T_c which was two orders of magnitude too large (when the corrected paramagnon value for ω_c is used); it was therefore necessary to cancel most of the attractive interaction by introducing a phenomenological repulsive term.

It should be noted that strong-coupling corrections which arise because of the mass renormalization factor $Z = m^*/m$ are very significant. They have two important consequences in ^{3}He and their inclusion is essential in order to understand the phase diagram. First, because Z is large (it varies from 3.0 to ≈ 5.5 over the entire pressure range[12,13]), it leads to a reduction in the size of the effective pairing constant $\lambda (= \lambda^{\Delta}/Z)$ relative to its "bare" value by about half an order of magnitude. This will reduce the estimated values of T_c by about a factor of 50. Secondly, because Z is strongly pressure dependent it will compensate for much of the pressure dependence of λ^{Δ}, leading to a T_c which, as observed, varies by less than a factor of 3 over the entire pressure range. The importance of the effective-mass renormalization factor was recognized by Layzer and Fay.[5] However, these authors took an assumed form for ω_c and other parameters which can be calculated exactly. As a consequence of these approximations and their choice of a repulsive "pseudopotential" the authors obtained rather poor agreement between their calculated values of T_c and experiment.

The application of strong-coupling (Eliashberg) theory to the paramagnon model was first discussed by Berk and Schrieffer[14] who considered singlet pairing, for which the paramagnon interaction is

repulsive. Layzer and Fay[5] extended their work to the triplet case. In contrast to the authors of Refs. 5 and 14, who assumed a particular form for ω_c we treat the dynamics fully and can thus *derive* the cutoff frequency as well as the form for $T_c(\bar{I})$ away from the exponential region. Our solution of the Eliashberg equation is closely analogous to that of Allen and Dynes[11] who studied phonon induced pairing. In the same spirit as the authors of Ref. 14 we view $\bar{I}$ as renormalized by vertex corrections.

In contrast to the authors of Ref. 4 (who fixed ω_c and fitted λ), Patton and Zaringhalam[3] were able to fit the phase diagram by taking ω_c as a free parameter and determining λ from the s-p approximation of Fermi-liquid theory. They found a reasonably good fit to experiment for $\omega_c = 0.06 T_F$, which differs from our calculated value of the cutoff frequency by about a factor of 6. We will discuss the results of the s-p approximation to T_c, using our values for ω_c in Sec. IV.

An outline of the paper is as follows. In Sec. II we review the singlet and triplet vertex functions derived from paramagnon exchange. These quantities are then used to compute the triplet-pairing interaction and the scattering amplitudes of Landau Fermi-liquid theory. In Sec. III A we review the Eliashberg equations and discuss numerical results for $T_c(\bar{I})$ in Sec. III B. Section IV describes the applications of the model to ^{3}He. It is fairly self-contained, and the reader who is not interested in the details of the formal theory may want to skip directly to Sec. IV.

II. PARAMAGNON INDUCED INTERACTIONS BETWEEN FERMIONS

In this section we calculate the vertex function $\Gamma(p_1, p_2; p_1 + k, p_2 - k)$ for the paramagnon induced interactions between fermions in either the triplet or singlet state. This function is then related to the pairing interaction of triplet superconductivity (in the limit $p_1 = -p_2$) and the Fermi-liquid scattering amplitudes ($k \to 0$). Not surprisingly, these scattering amplitudes are found to be closely related to the pairing interaction; this fact will provide a simple method for applying the paramagnon model to ^{3}He in Sec. IV. While there has been considerable discussion of the pairing interaction in the literature,[4,5] the values of the Fermi-liquid parameters in the paramagnon model have not been computed at the same level of approximation as that used to treat the pairing. Since there is an intimate relation between λ and the Fermi-liquid scattering amplitudes, we compute them within the same formalism.

The interaction between fermions is the Hubbard contact interaction

$$\mathcal{H}^{\text{int}} = I \int d\vec{r}\, d\vec{r}'\, n_\uparrow(\vec{r}) n_\downarrow(\vec{r}') \delta(\vec{r} - \vec{r}'), \tag{2.1}$$

where I is the interaction parameter and $n_\uparrow(\vec{r})$ and $n_\downarrow(\vec{r})$ are the densities of spin-up and spin-down fermions at the point $\vec{r}$.

Summing the ladder and bubble diagrams[4] and anti(symmetrizing) the result for the triplet (singlet) case, we get

$$\Gamma^{\text{triplet}}(p_1, p_2; p_1 + k, p_2 - k) = -\frac{I^2 \chi^0(k)}{1 - I^2[\chi^0(k)]^2} + \frac{I^2 \chi^0(k + p_1 - p_2)}{1 - I^2[\chi^0(k + p_1 - p_2)]^2} \tag{2.2}$$

and

$$\Gamma^{\text{singlet}}(p_1, p_2; p_1 + k, p_2 - k) = -2I + \frac{2I + I^2 \chi^0(k)}{1 - I^2[\chi^0(k)]^2} + \frac{2I + I^2 \chi^0(k + p_1 - p_2)}{1 - I^2[\chi^0(k + p_1 - p_2)]^2}, \tag{2.3}$$

where p_1, p_2, and k represent four momenta, and $\chi^0(k)$ is the dynamical spin susceptibility for a free fermion gas, which is given by the Lindhard function. The constant term $-2I$ appearing in Eq. (2.3) was included to avoid overcounting, since the lowest-order term in I appears in both the ladder and bubble series. There is some ambiguity about how to treat this term in the literature.[15] However, it is generally unimportant for the case of triplet pairing and for a strongly enhanced system.

From Eq. (2.2), the effective (dynamical) interaction for triplet superconducting pairs may be deduced[16]:

$$V^{\Delta}(p, p') = \tfrac{1}{2}[V^{\Delta}(p - p') - V^{\Delta}(p + p')], \tag{2.4a}$$

where

$$V^{\Delta}(p - p') = \frac{-I^2 \chi^0(p - p')}{1 - I^2[\chi^0(p - p')]^2}. \tag{2.4b}$$

In the static limit, this attractive interaction is the same as used by others.[4,5,17] As expected, in this limit, only odd-l Legendre polynomials contribute to $V^{\Delta}(\vec{p}, \vec{p}')$ for triplet pairs. For the case of singlet superconducting pairs, the effect of the paramagnon interaction is repulsive. The present results reduce to those obtained by Nakajima.[17] The results of Berk and Schrieffer[14] are contained in Eq. (2.3) when the bubble contributions and the less singular terms are also ignored. In the paramagnon model the effective mass is given by the DEP expression

$$1-\left(\frac{d\Sigma}{d\omega}\right)_{\omega=0,\,p=p_F} \equiv Z \equiv \frac{m^*}{m} = 1+\frac{1}{2k_F^2}\int_0^{2k_F} q\,dq\left(\frac{\bar{I}^3\bar{\chi}_0^2(\vec{q})}{1-\bar{I}\bar{\chi}_0(\vec{q})}+\frac{\bar{I}^2\bar{\chi}_0^2(\vec{q})}{1-\bar{I}^2\bar{\chi}_0^2(\vec{q})}\right). \tag{2.5}$$

Here $\bar{I}=IN(0)$ and $\bar{\chi}_0=\chi_0/N(0)$, where $N(0)$ is the bare density of states for a single spin. Thus the effective dynamical interaction which enters the expression for the mass renormalization is

$$V^z(p-p')=\frac{I^3\chi_0^2(p-p')}{1-I\chi_0(p-p')}+\frac{I^2\chi_0(p-p')}{1-I^2\chi_0^2(p-p')}, \tag{2.6}$$

which differs from the triplet pairing interaction as can be seen from Eq. (2.4). This follows because both ladder and bubbles contribute to V^z, whereas V^Δ can be computed by including only bubbles. For the case of phonon induced singlet pairing, these interactions are the same.

The quantities V^z and V^Δ are all that is needed to compute the transition temperatures. For later reference we compute the scattering amplitudes in the Landau Fermi-liquid limit ($k\to 0$), also using the vertex functions. Taking first the limit $\omega\to 0$ and then $\vec{k}\to 0$ corresponds to the scattering amplitudes (called A^s and A^a) rather than the interaction terms (called F^s and F^a).

We define, as in Fermi-liquid theory,[18] with $\vec{p}_1$ and $\vec{p}_2$ near the Fermi surface

$$A^a(\vec{p}_1,\vec{p}_2)=Z a_F^2\frac{N(0)}{2}[\Gamma^{\text{triplet}}(\vec{p}_1,\vec{p}_2;\vec{p}_1,\vec{p}_2)+\Gamma^{\text{singlet}}(\vec{p}_1,\vec{p}_2;\vec{p}_1,\vec{p}_2)], \tag{2.7}$$

$$A^s(\vec{p}_1,\vec{p}_2)=Z a_F^2\frac{N(0)}{2}[3\Gamma^{\text{triplet}}(\vec{p}_1,\vec{p}_2;\vec{p}_1,\vec{p}_2)-\Gamma^{\text{singlet}}(\vec{p}_1,\vec{p}_2;\vec{p}_1,\vec{p}_2)], \tag{2.8}$$

Here $a_F\cong m/m^*$ in the usual[14,18] approximation that derivatives of the self-energy with respect to momenta $(d\Sigma/dp)_{\omega=0,\,p=p_F}$, can be neglected compared to those with respect to frequency. Hence the coefficient of the term in [ ] is simply $N(0)/2Z$. This factor of $1/Z$ appears also in the pairing constant $\lambda^\Delta\to[ZN(0)](\lambda^\Delta/Z^2)=\lambda^\Delta/Z$. This will be seen in more detail below.

We find

$$A^a(\hat{p}_1,\hat{p}_2)=\frac{1}{Z}\left(\frac{-\bar{I}^2}{1-\bar{I}}-\frac{\bar{I}}{1-\bar{I}^2[\bar{\chi}_0(\vec{p}_1-\vec{p}_2)]^2}\right) \tag{2.9}$$

and

$$A^s(\hat{p}_1,\hat{p}_2)=\frac{1}{Z}\left(-\frac{\bar{I}^2}{1+\bar{I}}+2\frac{\bar{I}^2\bar{\chi}_0(\vec{p}_1-\vec{p}_2)}{1-\bar{I}^2[\bar{\chi}_0(\vec{p}_1-\vec{p}_2)]^2}+\frac{\bar{I}}{1-I^2[\bar{\chi}_0(\vec{p}_1-\vec{p}_2)]^2}\right). \tag{2.10}$$

The "forward-scattering sum rule" $A^s(\hat{p}_1,\hat{p}_1)+A^a(\hat{p}_1,\hat{p}_1)=0$ (Ref. 15) may be seen to follow from Eqs. (2.9) and (2.10). It follows from Eqs. (2.9) and (2.10) that the partial-wave components of the Fermi-liquid parameters A_0^s, A_0^a, and A_1^s are given by

$$A_0^a=\frac{1}{Z}\left\{-\frac{\bar{I}^2}{1-\bar{I}}-\frac{\bar{I}}{1-\bar{I}^2\bar{\chi}_0^2(q)}\right\}_{l=0}, \tag{2.11}$$

$$A_0^s=\frac{1}{Z}\left\{-\frac{\bar{I}^2}{1+\bar{I}}+2\frac{\bar{I}^2\bar{\chi}^0(q)}{1-\bar{I}^2\bar{\chi}_0^2(q)}+\frac{\bar{I}}{1-\bar{I}^2\bar{\chi}_0^2(q)}\right\}_{l=0}, \tag{2.12}$$

and

$$A_1^s=\frac{6}{Z}\left\{\frac{\bar{I}^2\bar{\chi}^0(q)}{1-\bar{I}^2\bar{\chi}_0^2(q)}+\frac{\frac{1}{2}\bar{I}}{1-\bar{I}^2\bar{\chi}_0^2(q)}\right\}_{l=1}, \tag{2.13}$$

where $\{\}_l$ refers to the projection of the interaction onto the lth Legendre polynomial. The normalizing factors $(2l+1)$ are explicitly included as prefactors in Eqs. (2.11)–(2.13).

Note that the first term in A_1^s is proportional to the triplet superconducting pairing interaction [Eq. (2.4)]. This fact will be exploited in discussing the phase diagram of superfluid ^{3}He in Sec. IV. It also follows that, because Γ is approximately of the form $-J\vec{s}\cdot\vec{s}$, $A_l^a\cong-\frac{1}{3}A_l^s$ for $l>0$.

The present calculation of the Landau Fermi-liquid parameters in the paramagnon model can be compared with the previous approach to this problem of Babu and Brown.[19] These authors computed the interaction parameters (F's) using the same diagrams (although unsymmetrized) as we used here to compute the scattering amplitudes. Thus their corresponding A's involve complicated "cross-channel" couplings—with bubbles and ladders appearing in the same vertex function, in contrast to the present case. While these authors include higher order diagrams in Eqs. (2.2) and (2.3), our own approach is preferred because (i) we seek to treat the pairing interaction (without including these cross-channel diagrams) within the same level of approximation as our computation of the A's and (ii) problems of consistency occur when these more complicated cross channel diagrams are included.

III. CONVENTIONAL STRONG-COUPLING THEORY FOR PARAMAGNONS

A. Theory

In this section the theoretical framework for applying Eliashberg theory to the pairing interaction of Eq. (2.4) will be discussed. In Sec. III B numerical results for the values of T_c as a function of $\bar{I}$ are presented. While vertex corrections in the paramagnon diagrams are not negligible (as they are in the phonon case) we effectively include some of these higher-order terms by choosing the parameter $\bar{I}$ to fit the measured scattering amplitudes (which are available for ^{3}He). In this sense we justify applying Eliashberg theory to the paramagnon problem.

The linearized Eliashberg equations for the normal and anomalous components of the self-energy

are given by[7]

$$(1-Z(p))i\omega_n = +\frac{1}{\beta}\sum_{\omega_{n'}}\int d\vec{\mathrm{p}}\,\frac{V^z(p-p')Z(p')i\omega_{n'}}{[Z(p')i\omega_{n'}]^2-\epsilon'^2_{\vec{\mathrm{p}}}} \quad (3.1)$$

and

$$\varphi(p) = -\frac{1}{\beta}\sum_{\omega_n'}\int d\vec{\mathrm{p}}'\,\frac{V^{\Delta}(p-p')\varphi(p')}{[Z(p')i\omega_{n'}]^2-\epsilon'^2_{\vec{\mathrm{p}}}}\,. \quad (3.2)$$

Here $\epsilon(p')$ is the energy and $\{\omega_n\}$ are the Matsubara frequencies, $\omega_n=(2n+1)\pi T$. Defining $\Delta\equiv\varphi/|\omega_n|$, these two equations can be combined[11] to yield

$$\frac{\beta|\omega_n|}{\pi}\,\Delta(p) = \sum_{\vec{p}'n'}\delta(\epsilon_{\vec{p}'})\,[V^z(p-p')\,\mathrm{sgn}n'\Delta(p) - V^{\Delta}(p-p')\Delta(p')]. \quad (3.3)$$

Because we are considering triplet pairing, Eq. (3.3) may be expanded in a series of Legendre polynomials involving odd-l contributions only. We define the coupling constants

$$\lambda^{\Delta}_l(\omega_n) = N(0)[V^{\Delta}(\vec{\mathrm{p}}-\vec{\mathrm{p}}',\omega_n)]_l \quad (3.4)$$

and

$$\lambda^z(\omega_n) = N(0)[V^z(\vec{\mathrm{p}}-\vec{\mathrm{p}}',\omega_n)]_0\,. \quad (3.5)$$

Note that it is the bare density of states $N(0)$ which enters into the coupling constants. This factor arises from a change of variables from $\vec{\mathrm{p}}$ to $\epsilon'_{\vec{p}}$ in Eq. (3.3). Thus

$$\frac{\beta}{\pi}\,|\omega_n|\Delta^l(\omega_n) = \sum_{n'}\left[\lambda^{\Delta}_l(\omega_n-\omega_{n'})-\delta_{nn'}\sum_m\lambda^z(\omega_n-\omega_m)\,\mathrm{sgn}m\right]\Delta^l(\omega_{n'}), \quad (3.6)$$

where the parameters $\Delta^l(\omega_n)$ are the coefficients of the lth (normalized) spherical harmonic in the expansion of $\Delta(\vec{\mathrm{p}},\omega_n)$.[20]

We now assume that $l=1$, for definiteness and because it is most appropriate for ^{3}He.

Using Eqs. (2.4b) and (2.6) it follows that

$$\lambda^{\Delta}(\omega_n-\omega_m) = \frac{\bar{I}^2}{2k_F^2}\int_0^{2k_F} dq\,q \times\frac{(1-q^2/2k_F^2)\bar{\chi}_0(q,\omega_n-\omega_m)}{1-\bar{I}^2\bar{\chi}_0^2(q,\omega_n-\omega_m)} \quad (3.7)$$

and

$$\lambda^z(\omega_n-\omega_m) = \frac{\bar{I}^2}{2k_F^2}\int_0^{2k_F} dq\,q\left(\frac{\bar{I}\bar{\chi}_0^2(q,\omega_n-\omega_m)}{1-\bar{I}\bar{\chi}_0(q,\omega_n-\omega_m)} + \frac{\bar{\chi}_0(q,\omega_n-\omega_m)}{1-\bar{I}^2\bar{\chi}_0^2(q,\omega_n)}\right), \quad (3.8)$$

where we have suppressed the subscript $l=1$ on $\lambda^{\Delta}(\omega_n-\omega_n)$. The parameter $\lambda^{\Delta}(0)\equiv\lambda^{\Delta}$ is the same as the coupling constant used in the calculation of Anderson and Brinkman[4]; $\lambda^z\equiv\lambda^z(0)=m^*/m-1$ as can be seen from (2.5). Here k_F is the Fermi momentum and $\bar{\chi}_0(q,\omega_m)$ is the analytic continuation of the Lindhard function onto the imaginary axis. This quantity is given by[21]

$$\bar{\chi}_0(q,\omega_n) = \mathrm{Re}\left[\frac{1}{2}+\frac{1}{2\bar{q}}\left(1-\frac{(\frac{1}{2}\bar{q}^2-i\bar{\omega}_n)^2}{\bar{q}^2}\right)\times[\ln(\tfrac{1}{2}\bar{q}^2-i\bar{\omega}_n+\bar{q}) - \ln(\tfrac{1}{2}\bar{q}^2-i\bar{\omega}_n-\bar{q})]\right], \quad (3.9)$$

where $\bar{q}=k/k_F$, $\bar{\omega}_n=\omega_n/2E_F$, and ln stands for the principal value of the complex logarithm. In deriving Eq. (3.9), the Fermi functions were replaced by their zero-temperature values, which approximation is justified, for we find $T_c\ll T_F$ for all $\bar{I}$.

Equation (3.6) may be converted to an eigenvalue equation by introducing the pair-breaking parameter[11] ρ:

$$\rho\Delta(\omega_n) = \sum_{m=-N-1}^{N}\left(\lambda^{\Delta}(\omega_n-\omega_m)-\delta_{mn}\,\frac{\beta}{\pi}\,|\tilde{\omega}_n|\right)\Delta(\omega_m), \quad (3.10a)$$

where

$$\tilde{\omega}_n = \omega_n+\frac{\pi}{\beta}\left(\lambda^z(0)+2\sum_{l=1}^{n}\lambda^z(\omega_l)\right) \quad (3.10b)$$

and N is arbitrarily large.

The physical-gap equation is that limit of Eq. (3.10a) in which $\rho=0$. Thus the solution of the gap equation is reduced to that of solving the eigenvalue equation

$$\sum_{n=0}^{N}(K_{mn}-\rho\delta_{mn})\Delta(\omega_n)=0, \quad (3.11a)$$

where

$$K_{mn} = \lambda^{\Delta}(\omega_m-\omega_n)+\lambda^{\Delta}(\omega_m+\omega_n) - \delta_{mn}\left(2m+1+\lambda^z(0)+2\sum_{l=1}^{m}\lambda^z(\omega_l)\right). \quad (3.11b)$$

Here we have used the fact that λ^z and Δ are even in ω_n.

The highest temperature at which the largest eigenvalue $\rho^{\max}=0$ is the transition temperature T_c, as we shall see below.

B. Numerical results

There are several simple results which follow directly from the analytic form of K_{nm}: (i) At infinitely high temperatures, all the eigenvalues are negative. This corresponds physically to the fact that the normal state is always more stable

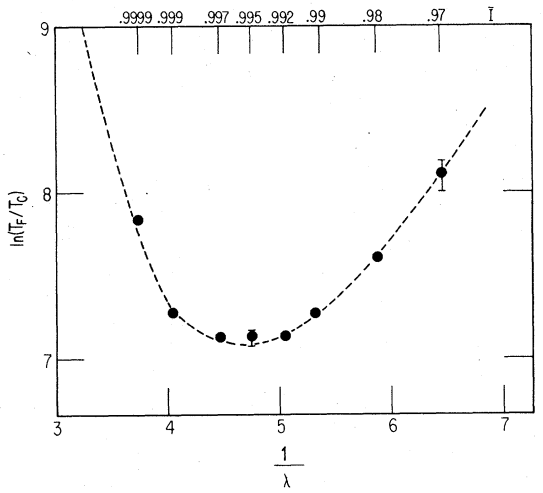

FIG. 1. $\ln T_c/T_F$ vs the inverse coupling constant $1/\lambda$; corresponding values of $\bar{I}$ are shown in the upper scale. Error bars indicate numerical uncertainty.

as $T \to \infty$. (ii) As $\bar{I} \to 1$, K_{nm} is the diagonal matrix $-\infty 1$. This follows because in this limit $\lambda^z(0) \simeq 3\lambda^\Delta(0) = \frac{9}{2}\left|\ln(1-\bar{I})\right|$ so that $[\lambda^\Delta(0) - \lambda^z(0)] \to -\infty$. Thus by $\bar{I}=1$, there is no superfluid phase transition. The physical consequence of this is that there is no competition between the magnetic instability which occurs at $\bar{I}=1$ and the triplet superconducting one: the superconducting instability "shuts off" before the magnetic one occurs. *Thus, in contrast to the phonon case,*[11] *as the paramagnon becomes arbitrarily soft it suppresses the superconducting instability*. The difference between the two cases arises in part because the parameters λ^z and λ^Δ are unequal (equal) for paramagnon (phonon) induced pairing. From (ii) we can deduce that $T_c(\bar{I})$ must have a maximum at some $\bar{I}$. This in in contrast to the situation for phonons[11] in which $T_c(\lambda)$ increases monotonically with λ.

To solve for T_c, we truncate the matrix at some finite value of N and obtain its largest eigenvalue $\rho^{\max}(N)$ as a function of T. Because K is Hermitian we thus find a lower bound to the transition temperature: it is that temperature at which $\rho^{\max}(N)$ changes sign (with decreasing T) from a negative to positive value. The size of the matrix N is then increased and the same procedure followed until convergence is obtained. For most of the cases considered here, it was necessary to take $N=256$. To avoid errors, all the integrals $\lambda^z(\omega_n)$ and $\lambda^\Delta(\omega_n)$ were computed numerically with high precision.

The computed values of $\ln(T_F/T_c)$ as a function of $\lambda^{-1} = [1+\lambda^z(0)]/\lambda^\Delta(0)$ are shown in Fig. 1. The upper scale corresponds to the value of $\bar{I}$ for each λ. The points on the curve are also tabulated in Table I. Error bars indicate the uncertainty in our result in several regions of the dotted curve due to incomplete convergence by $N=256$. As can be seen, T_c reaches a maximum value of about $8\times10^{-4}T_F$ at $\bar{I} \cong 0.995$. For values of $\bar{I} \leq 0.97$, the curve approaches a straight line corresponding to an exponential form $T_c = \bar{\omega}_c e^{-b/\lambda}$, with b near unity. We were not able to obtain enough points in the straight-line region (due to the small size of T_c which led to convergence difficulties at small $\bar{I}$) to determine accurately the cutoff parameter ω_c; it is of order $(\frac{1}{10}T_F)$. We were also unable to fit our curve to the form (which has been suggested)[4] $T_c/T_F = (1-\bar{I})e^{-1/\lambda}$, for any values of $\bar{I}$. That this form is inadequate can be understood as follows. As $\bar{I}$ approaches 1 the paramagnon frequencies become arbitrarily soft so that the "static approximation" breaks down: the T_c-vs-$\bar{I}$ curve will no longer have the exponential form. As $\bar{I}$ decreases away from 1, the small-q approximation to the Lindhard function $\chi^0(q,\omega)$ is invalid and the characteristic frequencies are not related to $(1-\bar{I})T_F$.

Among the most interesting physical consequences of the model calculation is the fact that the maximum value of T_c is of order $10^{-3}T_F$. The implication of this result for triplet (metal) superconductors is that *even under the most ideal circumstances the paramagnon mechanism is unlikely to lead to high-temperature superconductivity*. For most systems the highest attainable T_c will be comparable to that achieved by the ordinary phonon mechanism. That these transition temperatures are so low is a consequence of the large mass renormalization factor Z.

TABLE I. Theoretical values of T_c/T_F, effective-mass ratio, and scattering amplitudes for different values of $\bar{I}$, obtained using the paramagnon model.

$\bar{I}$	T_c/T_F	m^*/m	A_1^s	A_0^s	A_0^a
0.9999	$4\pm0.5\times10^{-4}$	34.36	2.41	1.01	−291.3
0.999	7×10^{-4}	24.03	2.20	1.01	−50.22
0.995	8×10^{-4}	16.91	1.89	1.02	−12.26
0.990	7×10^{-4}	13.92	1.75	1.02	−8.03
0.980	5×10^{-4}	11.02	1.52	1.02	−4.76
0.970	3×10^{-4}	9.39	1.39	1.03	−3.75

IV. APPLICATION TO SUPERFLUID ^{3}He

In this section we discuss the predictions of the paramagnon model for the pressure P dependence of T_c in ^{3}He and the Landau Fermi-liquid scattering amplitudes. The formalism of Secs. II and III established a means of computing $T_c(\bar{I})$. To compare with experimental data on $T_c(P)$ it is necessary to obtain $\bar{I}$ as a function of pressure.

While the paramagnon model is undoubtedly oversimplified, we apply it here to ^{3}He because (i) it is a microscopic theory which enables us to *derive* (in the exponential limit) the cutoff frequencies ω_c which help to set the scale for the size of T_c. Fermi-liquid theories cannot be used to obtain these characteristic frequencies. (ii) It has met with great success in explaining the temperature dependence of the susceptibility[22] and has been somewhat successful in explaining the magnitude of the mass enhancement[8] in normal ^{3}He. (iii) A spin-dependent fermion-fermion interaction clearly plays an important role in stabilizing the A phase relative to the B phase in the high pressure region. Paramagnon theory has been utilized to explain this stabilization.[4]

It is expected that the oversimplified contact interaction model [Eq. (2.1)] leads to sensible results for the scattering amplitudes only for $l>0$. It probably fails for $l=0$, since these scattering processes sample directly the $\delta(\vec{r}-\vec{r}')$ term in the interaction. Thus for the computation of the scattering amplitudes A_l^a and A_l^s for $l>0$ and the transition temperatures for $l=1$ pairing, the paramagnon model will yield a reasonable first-order approximation.

There is clearly no unique way for obtaining the values of the pressure-dependent parameter $\bar{I}(P)$ which are appropriate to ^{3}He. Three Fermi-liquid parameters F_0^a, F_0^s, and F_1^s (or alternatively A_0^a, A_0^s, and A_1^s) are known experimentally,[12] while the model contains only one adjustable parameter. It has previously been suggested that $\bar{I}(P)$ be chosen by fitting the theoretically computed normal-state static susceptibility to experiment.[4] This procedure is thus based on the Fermi-liquid parameter F_0^a. We argue here that it is more appropriate to fit λ directly than to fit χ. In this section we propose two methods for selecting the parameter $\bar{I}$, both of which give very similar results. The first is based on the scattering amplitude A_1^s which is found to be proportional to λ; we call this the "A_1^s method." The second is based on the s-p approximation[3] which yields an expression for λ in terms of the three measured scattering amplitudes. We also point out that when the DEP model for m^*/m is modified slightly to give better agreement with experiment, then the value of $\bar{I}(P)$ we obtain in these two ways are very close to what would be deduced from the static susceptibility in paramagnon theory. It is quite remarkable that all reasonable procedures for obtaining $\bar{I}(P)$ yield results for T_c in ^{3}He which are in close agreement.

We now obtain $\bar{I}$ at each pressure P in two different ways based on the scattering amplitudes of Fermi-liquid theory. The integrals in Eq. (2.13) for A_1^s were performed; it was found that for $\bar{I}>0.90$ the second term in Eq. (2.13) is equal to one-half the first to within 4%. Therefore,

$$A_1^s \cong \frac{9}{Z}\left\{\frac{\bar{I}^2\bar{\chi}^0(q)}{1-\bar{I}^2\bar{\chi}_0^2(q)}\right\}_{l=1} . \tag{4.1}$$

Noting also that

$$\lambda = \frac{\lambda^\Delta}{Z} = \frac{1}{Z}\left\{\frac{\bar{I}^2\bar{\chi}^0(q)}{1-\bar{I}^2\bar{\chi}_0^2(q)}\right\}_{l=1} , \tag{4.2}$$

it follows that $\lambda \cong \frac{1}{9} A_1^s$. Thus a reasonable choice for parameter $\bar{I}(P)$ is to choose that $\bar{I}$ which fits A_1^s to the experimentally measured value at each pressure. It is important to note that this result is generally true in any model in which the effective interaction is of the form $-J\vec{s}\cdot\vec{s}$.

The second approach which will also be considered here is the s-p approximation used by Patton and Zaringhalam.[3] In this approximation,[23]

$$\lambda = \frac{-1}{12}\sum_l (-1)^l (A_l^s + A_l^a), \tag{4.3}$$

so that $\bar{I}(P)$ is chosen by fitting $\lambda(\bar{I})$ to the right-hand side of Eq. (4.3) at each P. Since only A_0^s, A_0^a, and A_1^s are known experimentally, the sum over l in Eq. (4.3) is truncated after $l=1$ and A_1^a is chosen so as to satisfy the forward-scattering sum rule. We begin by using the DEP model for computing m^*/m.

To obtain $T_c(P)$ in the A_1^s method we use the theoretically computed values of A_1^s which are tabulated in Table I. We fit the parameter A_1^s to the experimentally measured values (examples of which are shown in Table II) and obtain values of $\bar{I}$ that vary from 0.997 to more than 0.9999 from $P=0$ to $P=34.36$ bar. All the experimental numbers shown in Table II and those used throughout this paper are taken from Ref. 12 unless indicated otherwise. The resulting pressure dependent T_c is plotted in the dotted line in Fig. 2(a). This line terminates because we were unable to compute T_c for $\bar{I}>0.9999$. The solid line plots the experimental results.[12] While there is very good agreement between the *magnitude* of the theoretical results and experiment, the pressure dependence is

TABLE II. Experimental values of T_c/T_F, m^*/m, and scattering amplitudes for ^{3}He at different pressures. All values are from Ref. 12, except when indicated otherwise.

P (bar)	T_c/T_F	m^*/m	A_1^s	A_0^a	A_0^s
0	6.1×10^{-4} [a]	3.01	2.04	−1.91	0.91
18.0	19.0×10^{-4}	4.93	2.39	−2.79	0.98
34.36	25.5×10^{-4}	6.22(5.5) [b]	2.52	−2.80	0.99

[a] T. Alvesalo, D. Yu, H. K. Collan, O. V. Launasmaa, and P. Wennerstrom, Phys. Rev. Lett. 30, 962 (1973).
[b] After Ref. 13.

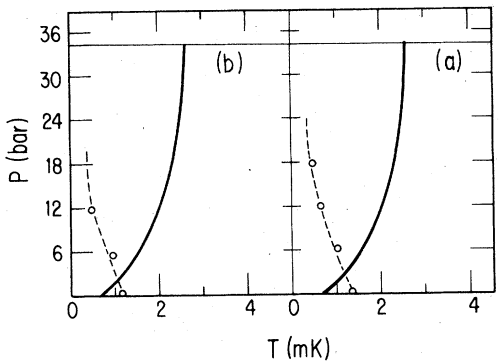

FIG. 2. Comparison between experiment (solid curves, after Ref. 12), and theory (dashed curves) as a function of pressure using (a) a fit to the A_1^s scattering amplitude and (b) the s-p approximation.

wrong. This can be traced to the fact that at these large values of $\bar{I}$, we are in the region of the curve of Fig. 1 in which T_c is decreasing with increasing $\bar{I}$. Thus in this model ^{3}He would not be described by an exponential expression for T_c. Similar results obtain for the s-p approximation method in which $\bar{I}$ varied from ≈ 0.990 to ≈ 0.9999. These are shown in Fig. 2(b). It should be noted that in both cases the incorrect pressure dependence of T_c is related to another unsatisfactory property: the effective mass ratios m^*/m are an order of magnitude larger than the experimental values, as can be seen by comparing Tables I and II.

In summary, we see that this model yields good agreement between theory and experiment for the magnitudes of the transition temperatures, but that the pressure dependence is incorrect. In addition the values of $\bar{I}$ are unexpectedly large and the associated values of m^*/m are in disagreement with experiment. Although they may appear to be unrelated, all of these discrepancies derive from a single factor: the DEP model yields values for $Z = m^*/m$ which are too large.

We can obtain a more physically reasonable paramagnon model for ^{3}He, in which A_1^s and the mass ratios m^*/m are simultaneously close to the experimental values, by reducing the magnitude of the function $Z(\omega)$ so that the values of $Z(0) = m^*/m$ are brought into better agreement with experiment.[24] It should be noted that we are *not* fitting T_c here but only normal Fermi-liquid properties. As will be emphasized here any reasonable procedure which accomplishes this will not appreciably affect the magnitude of the computed transition temperatures. Formally, we can view this reduction of $Z(\omega)$ as coming from the inclusion of vertex corrections, which affect λ^z and λ^Δ in different ways. These corrections have been shown[10] to decrease the effective-mass ratio. Phenomenologically, the readjustment of $Z(\omega)$ is justified for the following reasons. (i) Our results for the *magnitude* of T_c are extremely insensitive to theoretical models used for Z provided Z is of order unity (rather than of order 10 as in the model just discussed). This follows because $\bar{I}$ is chosen so that the renormalized coupling constant λ^Δ/Z is given by the *experimental* value in terms of the A's. *Changing the effective mass ratio has only the effect of rescaling the parameter* $\bar{I}$. An order-of-magnitude reduction in the effective-mass ratio will lead to values of $\bar{I}$ which are in the exponential limit of the T_c curve. (ii) Fermi-liquid theories show that m^*/m and A_1^s are related, so that a more-appropriate model for ^{3}He is one in which both parameters are close to the experimental values.

We now redefine Z: $Z(\omega) = 1 + s\lambda^z(\omega)$, where s is a scaling factor. Thus the DEP model corresponds to the case $s = 1$. As can be seen from Table II, the experimentally measured mass ratio varies from 3.0 to about 5.5. For a scale factor $s = 0.5$ we obtained reasonably good agreement with the measured values of m^*/m over the entire pressure range. These results are shown in Table III. The numbers in this table are obtained using the A_1^s method, while the numbers in parentheses are obtained from the s-p approximation. A smaller scale factor of $s = 0.4$ was also tried in which m^*/m varied from ≈ 3.0 to about 4.0. For both values for s the transition temperatures were within 20% of one another and the A's as a function of pressure were essentially unchanged. It should be emphasized that we do not view s as a fitting parameter. We have made no attempt to find the "best" s or the best functional form for $Z(\omega)$. We use this model to illustrate how in the paramagnon model, the normal Fermi-liquid properties can brought into reasonable agreement with experiment.

The results for $T_c(\bar{I})$ in the rescaled effective mass model (with $s = 0.5$) are shown in Fig. 3 which plots $\ln(T_c/T_F)$ vs $1/\lambda \equiv (1+\lambda^z)/\lambda^\Delta$. Note that the general features of the curve are the same as those obtained when the unmodified DEP model for $Z(\omega)$ is used (see Fig. 1). For $\bar{I} \leq 0.97$ the curve could be fit to the form

$$T_c = (0.1T_F)e^{-0.7/\lambda}, \tag{4.4a}$$

or equivalently

$$T_c = \omega_c e^{-1/\lambda}, \quad \text{where } \omega_c = 0.1T_F e^{0.3/\lambda}, \tag{4.4b}$$

so that the cutoff frequency ω_c is weakly dependent on pressure for the narrow range of $\bar{I}$ appropriate to ^{3}He. For this reason, Patton and Zaringhalam[3] were able to fit the phase diagram of ^{3}He using an equation of the form (4.4b) with ω_c equal to a cons-

TABLE III. Theoretical values of $\bar{I}$, T_c/T_F, m^*/m, and the scattering amplitudes for ^{3}He at each pressure, obtained using the rescaled paramagnon model. The numbers not in parentheses are obtained by fitting the A_1^s scattering amplitude while those in parentheses are obtained from the s-p approximation.

P	$\bar{I}$	T_c/T_F	m^*/m	A_1^s	A_0^a	A_0^s
0	0.948	4.9×10^{-3}	4.2	2.06	−4.64	1.77
	(0.919)	(2.8×10^{-3})	(3.4)	(1.71)	(−3.80)	(1.72)
6	0.957	6.4×10^{-3}	4.65	2.21	−5.40	1.80
	(0.959)	(6.6×10^{-3})	(4.67)	(2.24)	(−5.53)	(1.80)
12	0.963	7.4×10^{-3}	4.87	2.32	−5.91	1.81
	(0.970)	(8.4×10^{-3})	(5.29)	(2.47)	(−6.65)	(1.82)
18	0.966	7.9×10^{-3}	4.92	2.38	−6.23	1.82
	(0.976)	(9.6×10^{-3})	(5.92)	(2.63)	(−7.80)	(1.84)
24	0.969	8.3×10^{-3}	5.24	2.46	−6.60	1.82
	(0.979)	(9.9×10^{-3})	(5.95)	(2.71)	(−8.38)	(1.84)
30	0.971	8.6×10^{-3}	5.38	2.49	−6.83	1.83
	(0.979)	(9.9×10^{-3})	(5.95)	(2.71)	(−8.38)	(1.84)
34.36	0.972	8.7×10^{-3}	5.52	2.52	−7.01	1.83
	(0.978)	(9.9×10^{-3})	(5.94)	(2.69)	(−8.18)	(1.84)

tant. We were unable to fit our curve to a function of the form $T_c=(1-\bar{I})T_F e^{-1/\lambda}$, in which the cutoff frequency is the spin-fluctuation frequency. The reasons for this were discussed earlier. For $\bar{I}\cong 0.995$, $T_c(\bar{I})$ again has a maximum of magnitude $\approx 10^{-2}T_F$. From Table III it should be noted that the values of $\bar{I}$ which are appropriate to ^{3}He range from about 0.95 to 0.97 in the first method and from 0.92 to 0.98 in the second. These values of $\bar{I}$ are close to those used previously.[4] However, in these earlier estimates of T_c the mass renormalization factor Z was neglected and λ^Δ was computed from the expression $\lambda^\Delta=\frac{3}{2}|\ln(1-\bar{I})|$. The second approximation, which results from a small-q expansion of the Lindhard function, is only valid for $(1-\bar{I})\lesssim 10^{-7}$. For $\bar{I}=0.95$, we find using numerical techniques that $\lambda^\Delta=1.0$; in Ref. 4 it was deduced that $\lambda^\Delta=4.5$. For this $\bar{I}$, $m^*/m=4.3$. Consequently, $\lambda^\Delta/Z=0.23$ which should be compared with the previous estimate 4.5. These order of magnitude differences in λ are greatly magnified in T_c.

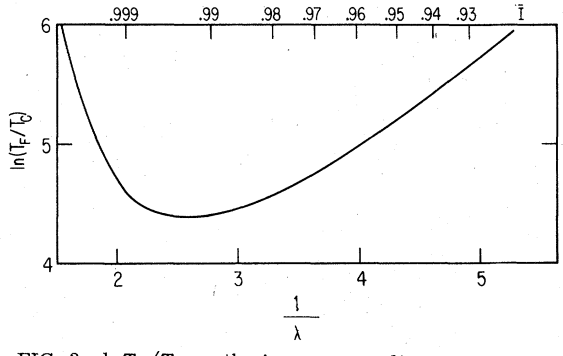

FIG. 3. $\ln T_c/T_F$ vs the inverse coupling constant. Corresponding values of $\bar{I}$ are indicated in the upper scale. The curve is obtained by adjusting the theoretical values of m^*/m to yield better agreement with experiment than the model in Fig. 1.

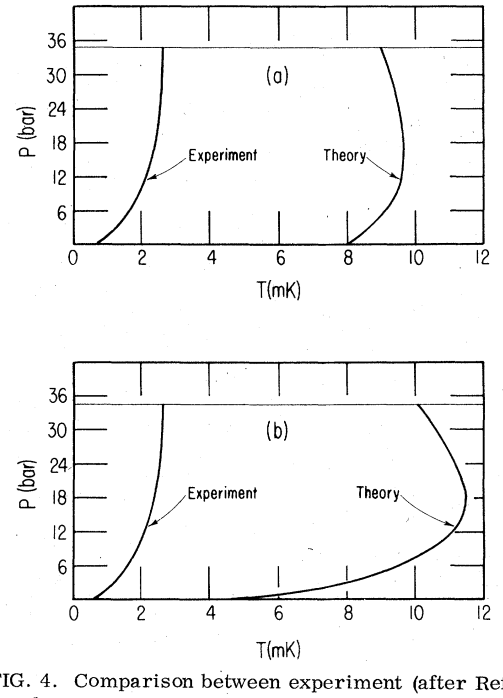

FIG. 4. Comparison between experiment (after Ref. 12) and theory for T_c as a function of pressure using (a) a fit to the A_1^s scattering amplitude and (b) the s-p approximation. The theoretical points were obtained by adjusting m^*/m to yield better agreement with the experimental masses than the model in Fig. 2.

As might be expected, the theoretical values for A_0^a and A_0^s differ from experiment (by about a factor of 2), since the $l=0$ scattering amplitudes are not properly given by a contact potential interaction. However, agreement between theory and experiment is not poor, even for these amplitudes. We find that A_1^a is about $-0.8 \approx -A_1^s/3$ at melting pressure. Other estimates[25] of this amplitude give $A_1^a \approx -0.73$ at this pressure.

Finally, in Figs. 4(a) and 4(b) are shown the pressure dependence of T_c in the A_1^s and s-p approximation method, respectively. As can be seen both approaches yield semiquantitative agreement with experiment both in the magnitude of T_c and its pressure dependence. This agreement is quite good in view of the fact that small[26] errors in λ are greatly magnified in T_c. That the theoretical curves are slightly re-entrant is due to the fact that T_F decreases with increasing pressure.[12] The fact that the theoretical values are somewhat higher than the experimental ones suggests that there is probably[26] a very small direct repulsive interaction between parallel-spin fermions in ^{3}He. A modified model which contains this effect will be discussed in more detail in a future paper.

In summary, we have shown how to solve the Eliashberg equations for T_c exactly, within the paramagnon model, and we have discussed the implications of these solutions. In applying these results to ^{3}He, we find that when conventional strong-coupling effects are properly incorporated into a modified paramagnon model, it is possible to obtain reasonably good agreement between theoretical and experimentally measured values of T_c. This work suggests that if the parameters in a simple model for ^{3}He (like the paramagnon picture) are chosen to fit the normal Fermi-liquid properties, it is indeed possible to understand the magnitude and pressure dependence of the normal → superfluid transition temperature.

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$$\lambda_{L=1} \sim \frac{1}{3} \frac{N(0)}{Z} [\Gamma(p_1,p_1+k;-p_1,-p_1-k)] ,$$

whereas the A's depend on

$$\frac{2N(0)}{Z} [\Gamma(p_1,p_2;p_1,p_2) \pm \Gamma(p_1,p_2;p_2,p_1)] ,$$

where the [+ (−)] sign refers to singlet (triplet) scattering processes. The difference between the two expressions for λ arises because of the scattering amplitudes $\Gamma(p_1,p_2;p_1,p_2)$ and $\Gamma(p_1,p_1+k,-p_1,-p_1-k)$ are not related in the same way in the paramagnon and s-p approximations.
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