Nonequilibrium-enhanced supercurrents in short superconducting weak links*

M. Octavio,† W. J. Skocpol, and M. Tinkham
Department of Physics and Division of Applied Sciences, Harvard University, Cambridge, Massachusetts 02138
(Received 1 August 1977)

We have measured the low-voltage footlike features of the *I-V* characteristics of tin variable-thickness superconducting microbridges as a function of temperature. We find these to be consistent with a voltage-dependent enhancement of the supercurrent induced by the disequilibrium of the quasiparticles during the Josephson cycle. We discuss the physical origin of this effect and the more-specific microscopically based models of Golub and of Aslamazov and Larkin. We find the predictions of these models to be complementary, and in good agreement with many aspects of our experimental results.

I. INTRODUCTION

One of the most interesting ideas in recent studies of nonequilibrium superconductivity is the realization that perturbations which decrease the number or alter the energy distribution of quasiparticle excitations, may actually increase the magnitude of the energy gap and other superconducting properties. This idea has been used successfully in explaining the enhancement of the critical currents and energy gaps in superconducting films and junctions induced by microwaves and phonons.[1]

Recently, Golub[2] and Aslamazov and Larkin[3] (AL) have suggested that, because of the long relaxation time of the excitations, a similar enhancement will occur in short superconducting weak links with increasing voltage, leading to an increase in the average supercurrent flowing through the weak link. Gubankov *et al.*[4] have noted the qualitative similarity of their experimental *I-V* characteristics of tin variable-thickness microbridges to the shape obtained by Aslamazov and Larkin. In this paper, we discuss why the effect occurs and compare specific quantitative predictions of the theories of Golub and of Aslamazov and Larkin with our measurements of the properties of short tin variable-thickness microbridges. We find reasonable agreement in the regimes where the theories should be applicable.

II. EXPERIMENTAL PHENOMENA

The experimental phenomena to be explained have been observed in tin and indium microbridges made in various laboratories.[5] Our samples are tin variable-thickness microbridges, whose fabrication has been discussed elsewhere.[6] Briefly, the sharp straight edge of a diamond knife is pressed through a thick (1–3-μm) film perpendicular to the direction of a groove in the sapphire substrate. This leaves a small bridge in the groove which connects two much wider thicker banks. The thick banks provide increased cooling to minimize the buildup of Joule heat.[6] They also partially justify the usual theoretical assumption that the equilibrium properties of the banks can be imposed as boundary conditions at the ends of the narrow-bridge section.

In Fig. 1, we show the overall *I-V* characteristics of these microbridges. At high voltages there is a substantial amount of curvature due to heating.[7] At lower voltages one observes the characteristic features of subharmonic-gap structure.[8] Finally, at the lowest voltages (0–50 μV) there is a characteristic "foot," shown in detail in Fig. 2. Near T_c, the change to the resistive state above I_c is simply marked by a steep rise in voltage and a curve without inflection points. The curve shows an apparent excess supercurrent $\bar{I}_s$, defined as the zero-voltage intercept extrapolated from the higher-voltage parts of the curve. Near T_c, $\bar{I}_s$ is a fraction of the order of 0.6–0.8 times the critical current I_{c0}. At somewhat lower temperatures, the resistance in the vicinity of 10 μV appears somewhat depressed. At still lower temperatures, this feature develops continuously into an initial region (up to ~5 μV) of constant differential resistance, which is much smaller than the normal resistance R_N, followed by a rapid growth of the voltage with very little current change at a characteristic current I_{c1}. Above this region, there is an apparent excess supercurrent $\bar{I}_s$ which now appears to be related to I_{c1} rather than I_{c0}. Because of the well-cooled nature of our bridges, heating-related hysteresis[7] appears only at low temperatures, which allows us to follow the development of this "foot" over a wide range of temperatures. We find that the ratio I_{c1}/I_{c0} increases as the temperature is decreased, reaching a maximum of as much as two in some of our bridges at temperatures of order 3.2 K.

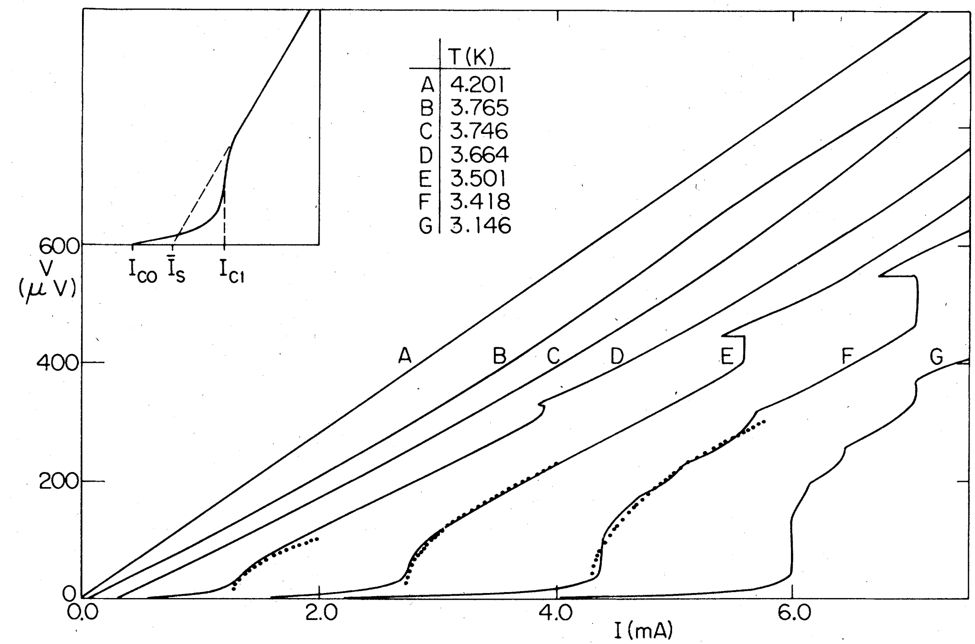

FIG. 1. I-V characteristics as a function of temperature for bridge No. 2. The inset defines the equilibrium critical current I_{c0}, the excess supercurrent $\bar{I}_s$, and the maximum enhanced supercurrent I_{c1}. The dotted curves show the shape above I_{c1} predicted by Aslamazov and Larkin.

At still lower temperatures this ratio decreases—an effect which we attribute to heating. The magnitudes and temperature dependences of the differential resistance at low voltages, of the ratio I_{c1}/I_{c0}, and of the apparent excess supercurrent $\bar{I}_s$ are the experimental facts to be accounted for.

III. THEORETICAL INTERPRETATION

A particularly simple and useful model of the equilibrium behavior of weak links such as metallic point contacts and variable-thickness bridges has been proposed by Aslamazov and Larkin.[9] This model is based

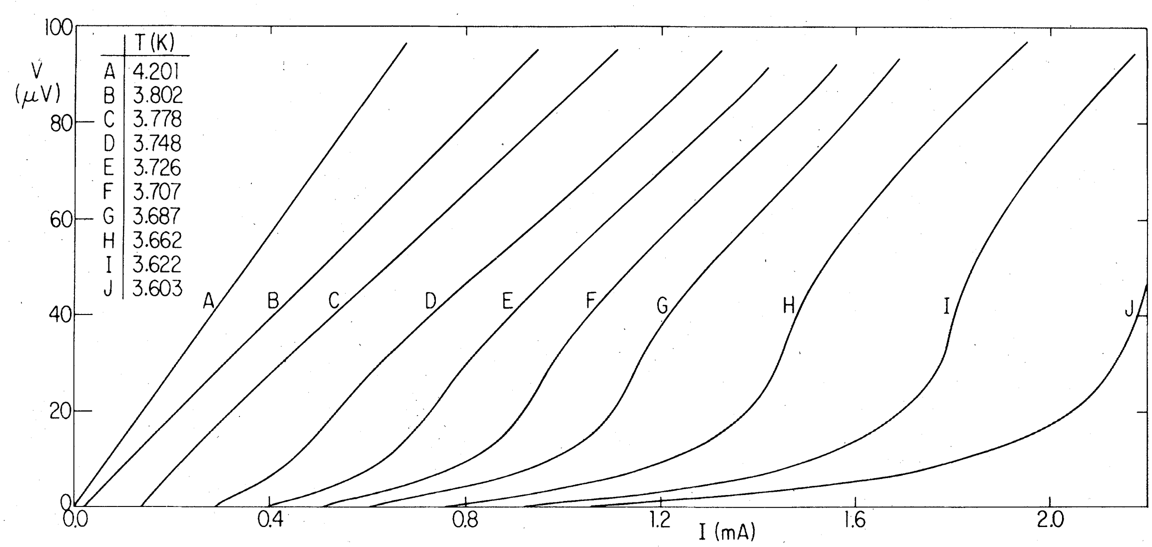

FIG. 2. Low-voltage part of the I-V characteristics of Fig. 1, showing the development of the foot as a function of temperature.

on the realization that if the length of the weak link is smaller than the temperature-dependent coherence length $\xi(T)$, the gradient term in the Ginzburg-Landau (GL) equations will dominate inside the weak link. In the simplest form of the model, the complex order parameter in the banks is assumed to have the values ψ_0 and $\psi_0 e^{i\phi_0}$, where ϕ_0 is the phase difference between the two sides, and to vary linearly across a one-dimensional bridge of length L connecting them

$$\psi = \psi_0(1 - x/L + e^{i\phi_0}x/L) \quad . \tag{1}$$

Thus the *magnitude* of the order parameter is fixed at $|\psi_0|$ at the ends of the weak link, and oscillates at the Josephson frequency between $|\psi_0|$ and 0 at the very center of the weak link. In this model, the supercurrent $I_s \propto \mathrm{Im}(\psi^* \nabla \psi)$ is a sinusoidal function of the phase difference ϕ_0, with a maximum value $I_{c0} = \pi\Delta^2/4eTR_N$ in agreement with the form for tunnel junctions near T_c. (Temperature is measured in energy units.)

The above model has served as the starting point for several previous considerations of the effect of the relaxation of the order parameter based on time-dependent Ginzburg-Landau (TDGL) theory and *ad hoc* extensions of it. Likharev and Yakobson[10] and Kramer and Baratoff[11] have numerically solved the TDGL equations for current-biased bridges of various lengths. Relaxation effects become important for times comparable to the Ginzburg-Landau time $\tau_{\mathrm{GL}} = \pi\hbar/8(T_c - T)$, which becomes significant primarily at high voltages. Likharev and Yakobson emphasized that end effects near the banks cause an "insufficient voltage," or apparent "excess supercurrent" of magnitude $0.75I_{c0}$ at high voltages, and found the appearance of an additional phase-dependent conductance which had little effect on the I-V curves. Jensen and Lindelof[5] subsequently noted that the phase-dependent conductance would have a much greater effect on the I-V characteristics if the relaxation time were taken to be two orders of magnitude longer; it then leads to features in the I-V characteristics which are qualitatively similar to the foot of interest here. However, the use of such long times was given no physical justification, and Kramer and Baratoff have noted that it leads to inconsistencies within the analytic approximations that Jensen and Lindelof use. Deaver *et al.*[12] have noted similar effects on the I-V characteristics resulting from time lags in a simple phenomenological nonequilibrium model.

Recent theories which go beyond the simple TDGL description take explicit account of the relationship between the nonequilibrium occupation of quasiparticle states and changes of the superconducting order parameter. We will first suggest qualitatively how such a relationship can lead to an enhanced supercurrent and the foot on the I-V characteristic, and then we will review the more-specific calculations by Golub and by Aslamazov and Larkin on which this picture is based.

A. Qualitative picture

To start, let us assume that the simple variation of the order parameter equation (1) also describes the space and time variation of the energy gap, whose magnitude determines the energy of each excitation through the relation $E_k = (\epsilon_k^2 + \Delta^2)^{1/2}$, where ϵ_k is the energy of an excitation in momentum state k in the absence of the pairing interaction. The occupation of states with a given energy will relax toward the equilibrium value given by the Fermi function $f_0(E) = [1 + \exp(\beta E)]^{-1}$ by two processes, inelastic scattering (primarily by phonons) to and from states of other energy, and spatial diffusion at constant energy to and from adjacent regions with different occupation. Diffusion over a distance L will occur over a characteristic time $\tau_D = L^2/D$, where D is the diffusion coefficient; for dimensions of the order of 1 μm, τ_D is comparable to $\tau_{\mathrm{GL}} = \xi^2/D$. Thus at energies which exceed the energy gap in the banks Δ_0, the occupation will tend to be fixed by diffusion at $f_0(E)$. Here we are neglecting the effects of Joule heating which would tend to increase the occupation of these states. Such effects are minimized by the variable-thickness geometry (which allows diffusion in three dimensions away from the bridge), and are insignificant at the low-voltage levels of interest here.

When the magnitude of the energy gap in the bridge is depressed below Δ_0, however, the only mechanism for relaxing the occupation of states with energies below Δ_0 involves inelastic phonon scattering, which is characterized by a relatively long time τ_E which for tin is of order 10^{-9}–10^{-10} sec. Thus we should expect different regimes of behavior depending on whether the Josephson period is long or short compared to τ_E.

At very low voltages where the Josephson period τ_J is much longer than τ_E, the relaxation will occur quickly on the scale of τ_J and the corrections to the equilibrium model will be small. As the Josephson period decreases with increasing voltage, the rate of change of the order parameter will increase and the disequilibrium will become more important. Let us assume that the gap follows the equilibrium behavior and then estimate the correction to it due to the disequilibrium. When the magnitude of the gap is decreasing (which occurs while the bridge is carrying current in the forward direction) the energies of the quasiparticles $E_k = (\epsilon_k^2 + \Delta^2)^{1/2}$ will be reduced. In the absence of inelastic scattering and diffusion, the occupation fraction of any state will remain unchanged and therefore will be smaller than the equilibrium occupation at the reduced energy. Inelastic scattering and diffusion will tend to restore equilibrium, but to the extent that there is a lag in this process, there will still be fewer excitations than in the equilibrium model, as if the bridge were cooler. Thus the magnitude of the order parameter will be somewhat larger, and will lead to a larger forward supercurrent during this half of the cycle. On the return part of the cycle, the gap is increasing, the occupation numbers are larger than thermal, the magnitude of the order parameter is decreased, and the supercurrent (which is negative in this part of the cycle) is therefore less negative. Thus during both parts of the cycle an extra positive supercurrent occurs, compared to the equilibrium model, and its magnitude should be proportional to the rate of change of the gap, and hence the voltage. Since a current-biased junction at low voltages already carries a current $\sim I_{c0}$ in the equilibrium model (because most of the time is spent near the maximum of the forward supercurrent part of the cycle), this additional supercurrent proportional to the voltage will result in a characteristic which starts from I_{c0} and rises with a slope smaller than the normal resistance of the link, up to voltages of order $h/2e\tau_E$ (several μV for tin). This describes the observed foot.

As the voltage begins to exceed $h/2e\tau_E$, the quasiparticle occupations cease to follow closely the equilibrium values and there should be a deficiency of quasiparticles in those states below Δ_0 throughout the entire cycle. This suggests that the enhancement of the magnitude of the order parameter will reach a limiting value (corresponding to no relaxation at all), and the supercurrent will be enhanced in both the forward and reverse part of the cycle. Thus there should be a transition to a regime with the usual behavior of the weak link (imperfectly understood though that may be) but with an enhanced critical current parameter I_{c1}. This is the region on each I-V curve which appears to have a critical current I_{c1} and a value of $\bar{I}_s$ which is a fraction of I_{c1}.

The key to a more-quantitative treatment of the situation described above is the derivation of generalized Ginzburg-Landau equations for the order parameter which contain terms depending explicitly on the quasiparticle distribution function, together with Boltzmann equations for the quasiparticle distribution function. Both Golub and Aslamazov and Larkin adopt this approach, but they make rather different physical and mathematical approximations. As a result, Golub's theory is more appropriate to describe the time variation within the Josephson cycle at low voltages, while Aslamazov and Larkin's is more appropriate to describe the time-averaged limiting behavior near I_{c1}.

B. Golub's theory ($\tau_J >> \tau_E$)

Golub describes the derivation of a modified form of the Ginzburg-Landau equation appropriate to the short weak-link geometry, and a perturbation scheme for solving it at low voltages. The equation has the form

$$\left[u_T\left(\frac{\partial}{\partial t}+i\mu\right)+u_L\frac{1}{|\psi|}\frac{\partial|\psi|}{\partial t}\right]\psi = \xi^2\frac{\partial^2\psi}{\partial x^2}+(1-|\psi|^2)\psi \ , \quad (2)$$

where the order parameter $\psi=\Delta e^{i\phi}/\Delta_0$ is normalized to its magnitude in the banks, and the times are normalized to the current relaxation time $\tau_0=\tau_{GL}/u_0$, where $u_0=\pi^4/14\zeta(3)=5.79$. The usual TDGL equation would correspond to the values $u_T=u_0$ and $u_L=0$. Instead, Golub shows that u_T is very small and the new term is the most important one. As we shall see, the smallness of u_T is related to the fact that the quasiparticle diffusion length rather than the coherence length governs the variation of quasiparticle electrochemical potentials, and the more important u_L term arises from the slow relaxation of disequilibrium generated by the mechanism we discussed above.

Golub's equation is derived within the particular microscopic picture described by Schmid and Schön[13] (which Golub relates to the work of Larkin and Ovchinnikov[14]). In this picture, the distribution function $f(E)$ of quasielectron states as a function of energy E (with the convention that states inside the Fermi surface have negative energies) is given by the equilibrium Fermi function $f_0(E)$, with odd and even correction terms associated with the longitudinal and transverse modes, respectively. The longitudinal correction f_L adds equal numbers of electronlike and holelike excitations, respectively, above and below the Fermi surface, and is related to changes in the magnitude of the order parameter. The transverse correction f_T is associated with changes of the gauge-invariant electrochemical-potential difference $\mu-\mu_p=\Phi=\frac{1}{2}(2e\varphi+\partial\phi/\partial t)$, where ϕ is the phase of the order parameter and φ is the scalar potential; it adds quasielectrons both above and below the Fermi surface corresponding to a net charge.

In computation the distribution function $f(E)$ is weighted by appropriate functions $N_1(E)$, $N_2(E)$, and $R_2(E)$ which are rounded off by the lifetime-shortening effects of inelastic electron-phonon scattering characterized by the time τ_E. Figure 3 shows the shape of these functions for three different values of $\tau_E\Delta$. $N_1(E)$ is the normalized density of states, rounded off by phonon broadening which leads to the existence of a small but nonzero density of states between $-\Delta$ and Δ. $R_2(E)$ looks similar to the density of states, but it is an odd function of E and falls as Δ/E far from the Fermi surface. For large $\tau_E\Delta$, $R_2(E)\sim\Delta/(E^2-\Delta^2)^{1/2}$, which is the usual weighting factor for the quasiparticle occupations in the BCS self-consistent gap equation; moreover, this form is precisely $N_1(E)\,\partial E/\partial\Delta$, where $E=(\epsilon^2+\Delta^2)^{1/2}$. The physical significance of $N_2(E)$, which is important

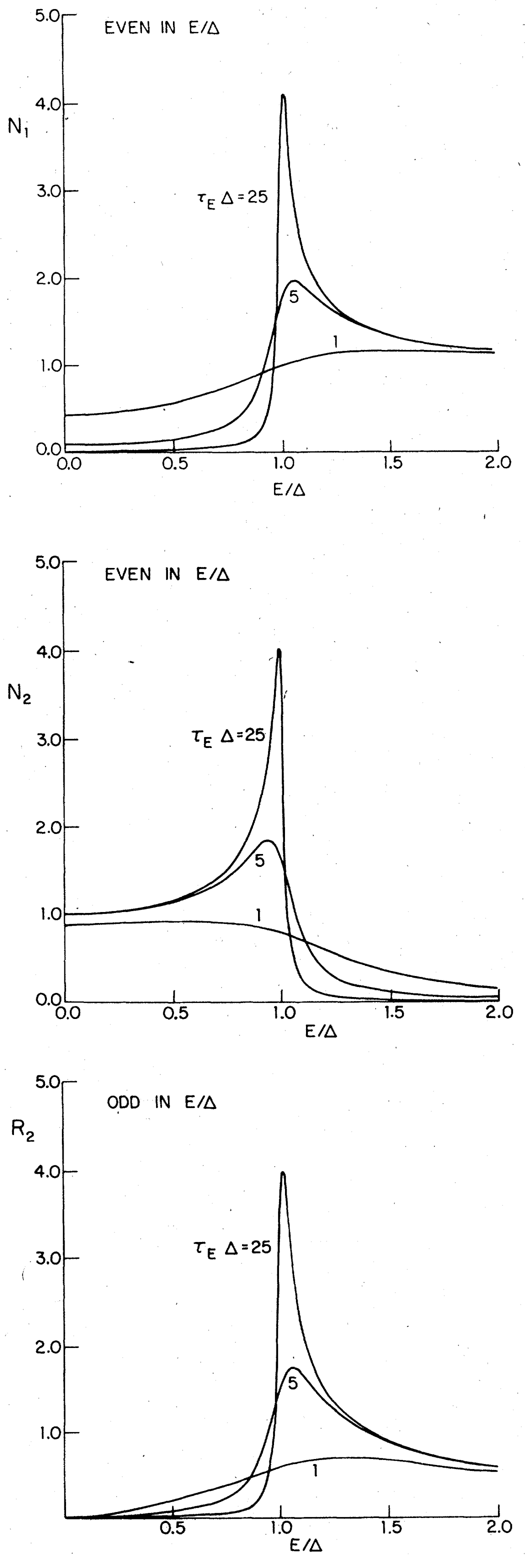

FIG. 3. Functions N_1, N_2, and R_2 as a function of E/Δ for $\tau_E\Delta = 1$, 5, and 25.

only in the region of "unphysical" states between $-\Delta$ and Δ, is less clear, but it appears to pick out states for which the transverse imbalance equilibrates rapidly with the superconducting condensate.

The steady-state Boltzmann equation for f_T balances inelastic electron-phonon scattering, spatial diffusion, and a transverse-mode quasielectron sink rate $2\Delta N_2 f_T$. In situations where the spatial variation of f_T is slow, the equation has the approximate solution

$$f_T = \left|\frac{\partial f_0}{\partial E}\right| \frac{N_1(E)}{N_1(E) + 2\tau_E \Delta N_2(E)} \Phi \ , \qquad (3)$$

which is like the distribution function for a normal metal, except for the missing energy range between $-\Delta$ and Δ where the sink rate is large. This solution also approximately satisfies the condition of electroneutrality

$$\Phi = \int_{-\infty}^{\infty} dE\, N_1(E) f_T \ . \qquad (4)$$

The generalized GL equation which depends on f_T is

$$\frac{\pi D}{8 T_c}\left[\Delta^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial x}\frac{\partial \Delta^2}{\partial x}\right] = \Delta \int_{-\infty}^{\infty} dE\, N_2 f_T \ , \qquad (5)$$

which for f_T given by (2) can be shown to reduce approximately to

$$\mathrm{div}\vec{J}_S = (1/\lambda_Q^2)(\sigma/e)\Phi \ , \qquad (6)$$

where

$$\lambda_Q = (D\tau_Q)^{1/2} = (D\tau_E T_C/\Delta)^{1/2} >> \xi(T)$$

is the branch-mixing quasiparticle diffusion length[15] and σ is the normal-state conductivity. With $\vec{J}_n = -(\sigma/e)\vec{\nabla}\Phi$ and the continuity equation $\mathrm{div}(\vec{J}_s + \vec{J}_n) = 0$, this establishes λ_Q as the length scale for variations of Φ.[16]

The steady-state Boltzmann equation for f_L balances electron-phonon scattering, diffusion, and the generation of disequilibrium at a rate

$$R_2 \left|\frac{\partial f_0}{\partial E}\right| \frac{\partial \Delta}{\partial t} \ ,$$

which for large $\tau_E\Delta$ reduces to

$$N_1 \left|\frac{\partial f_0}{\partial E}\right| \frac{\partial E}{\partial \Delta}\frac{\partial \Delta}{\partial t} \ .$$

This justifies our interpretation of the origin of the disequilibrium as arising from the changes of the quasiparticle energies as the gap changes. Golub assumes that this disequilibrium relaxes by spatial diffusion alone (although the effect of phonon scattering does enter through the broadened density of states), which leads to the solution

$$f_L = \frac{L^2}{12D}\left|\frac{\partial f_0}{\partial E}\right|\frac{R_2}{N_1^2 - R_2^2}\frac{\partial \Delta}{\partial t} , \tag{7}$$

where L is the length of the weak link. Thus f_L, shown in Fig. 4, is a function localized between $-\Delta$ and Δ where the coefficient $(N_1^2 - R_2^2)$ of the diffusion term in the Boltzmann equation is small. The insets of Fig. 4 show, in exaggerated form, the shape of the total quasielectron occupation $f(E) = f_0 + f_L$ (with $f_T = 0$). Increased quasielectron occupation at negative energies corresponds to fewer holelike excitations below the Fermi surface. Thus we see that a decreasing gap decreases the number of electronlike and holelike excitations, as suggested qualitatively above, and an increasing gap increases the number of excitations, compared to the equilibrium model. The resulting corrections to the gap are obtained directly from the generalized Ginzburg-Landau equation (neglecting unimportant static terms)

$$\frac{\pi D}{8T_c}\left[\frac{\partial^2}{\partial x^2} - \left(\frac{\partial \phi}{\partial x}\right)^2\right]\Delta = \frac{\pi}{8T_c}\frac{\partial \Delta}{\partial t} + \int_{-\infty}^{\infty} dE\, R_2 f_L . \tag{8}$$

The two real equations (5) and (8) can be combined into a single complex equation of the form (2), where the u_T and u_L terms arise from the integrals of f_T and f_L, using the forms (3) and (7). At the level of approximation which leads to Eq. (6), $u_T \equiv \beta u_0$ is simply τ_{GL}/τ_Q and is small because the quasiparticle diffusion length $\lambda_Q = (D\tau_Q)^{1/2}$ is much longer than the coherence length $\xi = (D\tau_{GL})^{1/2}$ so that the quasiparticle current and hence the supercurrent must be nearly

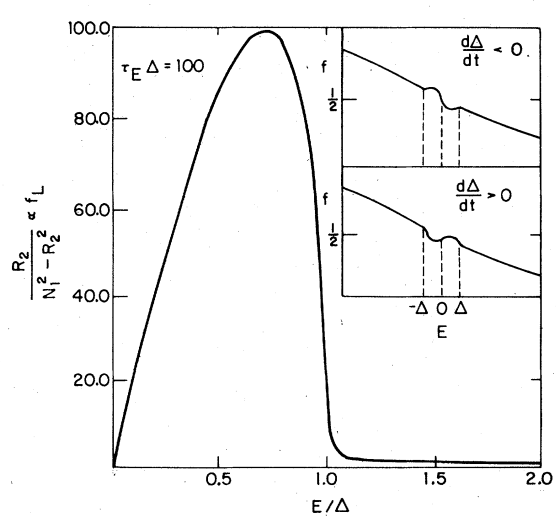

FIG. 4. Shape of the longitudinal correction f_L to the equilibrium distribution function f_0 as a function of E/Δ. The inset shows the overall distribution function with the nonequilibrium correction greatly exaggerated for the cases $d\Delta/dt > 0$ and $d\Delta/dt < 0$.

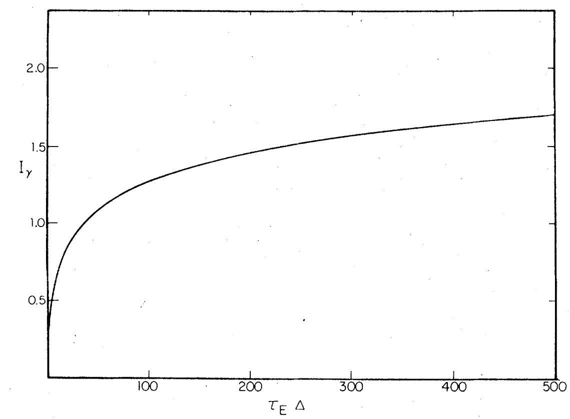

FIG. 5. Variation of I_γ with $\tau_E \Delta$.

uniform on the length scale of the bridge. The other term u_L is given by $u_L = (1 + \gamma - \beta)u_0$, where $\gamma = I_\gamma (T_c/\Delta)(L^2/D)\tau_{GL}^{-1}$ and

$$I_\gamma = \frac{4}{3\pi^2}\int_{-\infty}^{\infty} d\left(\frac{E}{\Delta}\right) R_2^2 (N_1^2 - R_2^2)^{-1} . \tag{9}$$

The variation of I_γ with $\tau_E \Delta$ is shown in Fig. 5. For large $\tau_E \Delta$, I_γ is relatively independent of τ_E, and the diffusion time L^2/D plays a major role.

Even for $u_T = 0$ it is difficult to obtain an exact solution to (2), but Golub has obtained a perturbation solution which is valid at low voltages. He assumes that the solution to (2) is of the form $\psi = \psi_0 + \psi_1$ where ψ_0 is given by Eq. (1) and ψ_1 is a small nonequilibrium correction. Then one only needs to solve

$$\xi^2 \frac{d^2 \psi_1}{dx^2} = -u_L \frac{x}{L}\frac{\psi_0}{|\psi_0|^2}\frac{d\phi_0}{dt}\sin\phi_0 , \tag{10}$$

with boundary conditions $\psi_1(0) = \psi_1(L) = 0$. In Fig. 6(a) we show the real versus the imaginary part of the order parameter at equal time intervals for $\psi_0(u_T = u_L = 0)$ assuming that $\psi_0(L) = e^{i\phi_0}$, $\psi_0(0) = 1$, and $d\phi_0/dt = 1$ (voltage-bias). Each line corresponds to a projection of the spatial variation of the order parameter onto the complex plane with the ends of each line corresponding to the value at the ends of the bridge. In this case the supercurrent $I_s = I_{c0}\sin\phi_0$ is forward during the first half of the cycle and reversed during the second half. In Fig. 6(b), we show a similar plot but now $u_T = 0$ and $u_L = 5$ so that $\psi = \psi_0 + \psi_1$. Now, because of the disequilibrium, during the first part of the cycle the order parameter (which is decreasing in magnitude) is larger in the middle of the brige than the $u_L = 0$ (equilibrium) value. At the middle of the cycle there is little difference from the equilibrium case because the magnitude of ψ is stationary at $\phi_0 = \pi$. But in the second half of the cycle the order parameter in the middle of the bridge (which is now increasing in value) again lags and is now smaller than the equilibrium value. Thus, as discussed qualitatively earlier, the magnitude of the supercurrent is increased during the forward part of the cycle and decreased in the reverse part of the cycle leading to a net *enhancement* of the supercurrent through the bridge. Golub's analytic expression for the supercurrent is only approximate in that at a given time the actual supercurrent is not spatially uniform along the bridge as required by Eq. (8) so that Golub's solution (like that of Likharev and Yakobson for $u_T = u_0, u_L = 0$) is based on the spatial average of the supercurrent. In general, the deviations of J_s from spatial uniformity are small compared to the enhancement.

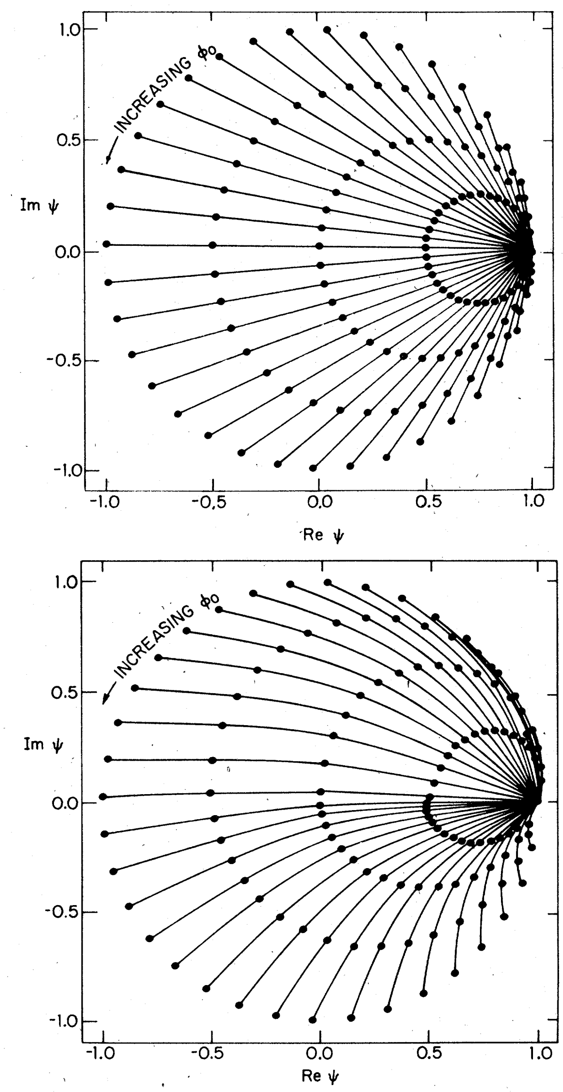

FIG. 6. Plot of the complex order parameter at equal time intervals; each line corresponds to the values of the order parameter along the length of the bridge. (a) Equilibrium case, $u_T = u_L = 0$; (b) nonequilibrium case for $L = \xi$, $u_T = 0$, $u_L\, d\phi_0/dt = 5$.

If one uses Golub's expression for the supercurrent, one then obtains an effective conductance at low voltages due to the increasing supercurrent as the voltage is increased which is given by

$$\frac{1}{R_{\text{eff}}} = \frac{1}{R_N}\left[1 + \frac{u_T L^2/\xi^2}{15} + \frac{u_L L^2/\xi^2}{24}\right] , \tag{11}$$

where R_N is the normal resistance of the link. Although this was derived under the voltage-bias assumption, we assume that in the current-biased case it will describe the differential resistance of the foot above I_{c0}. As will be shown later, for the materials and temperatures of interest here $\gamma >> \beta$ so that at low voltages the effective resistance of the linear foot is given by

$$\frac{R_{\text{eff}}}{R_N} = \frac{1}{1 + \frac{1}{24}(1+\gamma) u_0 L^2/\xi^2} . \tag{12}$$

Because of the initial assumption of steady state Boltzmann equations for the quasiparticles at each point of the Josephson cycle, and because of the assumption of small corrections necessary for the perturbation method of solution, Golub's results should only be valid at small voltages.

C. Aslamazov and Larkin's theory ($\tau_J << \tau_E$)

At higher voltages, the approach taken by Aslamazov and Larkin is more appropriate. They write the nonequilibrium Ginzburg-Landau equation in the form

$$\frac{\pi D}{8T}\left[\frac{\partial^2}{\partial x^2} - (\nabla\phi)^2\right]\Delta - \frac{T-T_c}{T_c}\Delta - \frac{7\zeta(3)}{8\pi^2 T_c^2}\Delta^3 + \Delta\int_\Delta^\infty \frac{f'(E)\,dE}{(E^2-\Delta^2)^{1/2}} = 0 , \tag{13}$$

where $f' = f - f_0$ is the difference between the nonequilibrium distribution function and the Fermi function. This assumes that the occupation of electronlike and holelike excitations is equal, i.e., that the transverse mode is irrelevant to the problem. Note that with the neglect of certain terms, this equation is completely equivalent to Eq. (8). In particular $\Delta(E^2-\Delta^2)^{-1/2}$ is just the limiting form of $R_2(E)$ for $\tau_E\Delta \to \infty$.

The real difference between the two approaches lies in the calculation of the nonequilibrium distribution function of the quasiparticles. Golub retains an explicit time dependence of the order parameter and integrates over energy, so that the magnitude (and sign) of the correction to the equilibrium distribution function of the quasiparticles depend explicitly on the time evolution of the gap. Aslamazov and Larkin, on the other hand, calculate the distribution function using an approach in which the quasiparticle distribution in the weak link is averaged over space and time very early in the calculation, and only the energy dependence is explicitly retained. This leads to rather different physical and mathematical approximations, and is justified only when the Josephson period is much smaller than the relaxation times. At low voltages where the linear foot is observed the inelastic scattering time τ_E (of the order of 10^{-9}–10^{-10} sec for tin) is comparable to the Josephson period. Thus, the low-voltage predictions of Aslamazov and Larkin should not be applicable. At higher voltages they calculate a time-averaged distribution function for the excitations of the form

$$f(E)=\tfrac{1}{2}\{1-(\Delta_0/2T)[1-(1-E/\Delta_0)^{5/4}]\} \tag{14}$$

for $E<\Delta_0$. At energies above Δ_0, they assume that the distribution is thermal, because of the rapid diffusion. This function corresponds to a smaller than equilibrium occupation number for those excitations with energies less than Δ_0, just as was the case for Golub's theory during the most-important (forward-current) part of the cycle. Using (14) for f, they obtain that the last term in Eq. (13) which they denote $\phi(\Delta)$ is given by

$$\phi(\Delta)=\frac{\sqrt{2}}{3}\frac{\Delta_0^2}{T_c}\left(1-\frac{\Delta}{\Delta_0}\right)^{3/2} . \tag{15}$$

When this and the gradient terms are assumed to be the most important ones, the order parameter variations are characterized by a length scale

$$\eta=\xi(T)[(T_c-T)/T_c]^{1/4} .$$

If the length of the bridge L is $>>\eta$ they obtain that the supercurrent is enhanced and its maximum is given by

$$I_{c1}/I_{c0}=KL/\eta , \tag{16}$$

where K is a numerical coefficient not calculated but of order one. This result, unfortunately, is derived under the rather stringent assumptions that $\xi>>L>>\eta$, which can be satisfied only extremely near to T_c, since ξ and η differ only by a factor of three even at $0.99T_c$; nevertheless, it may be useful enough for comparison with the data. For currents larger than I_{c1}, Aslamazov and Larkin calculate the shape of the I-V curve based on certain assumptions about the current-biased nature of the weak link near the forward current portion of the Josephson cycle. The result that they obtain is that the I-V relation should be given by

$$V=I_{c0}R\,(I/I_{c1}-1)^{1/2} . \tag{17}$$

Since this depends on the pulselike nature of the response of the current-biased junction, it is presumably valid only for currents which do not greatly exceed I_{c1}.

IV. DISCUSSION OF EXPERIMENTAL RESULTS

Taken together, the Golub and the Aslamazov and Larkin models provide a semiquantitatively satisfactory explanation for the I-V curves that we observe.

In Fig. 7, we compare Golub's prediction equation (11) of the normalized resistance of the foot region as a function of temperature for several values of the bridge-length parameter, with experimental data from six different bridges of similar geometrical lengths but varying resistances, as shown in Table I. In generating the theoretical curves, we have assumed an average value of Δ throughout the cycle of $0.8\Delta_0$, a value $\tau_E=8\times10^{-10}$ sec corresponding to that determined experimetally by Skocpol *et al.*,[17] and a coherence length $\xi(0)=0.13$ μm appropriate to our moderatly clean films (mean free path ~0.1 μm). The calculations are relatively insensitive to the value of τ_E because for temperatures less than $0.99T_c$, $\tau_E\Delta$ is at least 100, so that I_γ does not depend strongly on it. For bridge lengths of 0.5–0.8 μm and temperatures of 0.9–0.98T_c, the parameter γ in Eq. (12) is typically 10–20, while $u_0=5.79$ so the corrections to R_N are substantial.

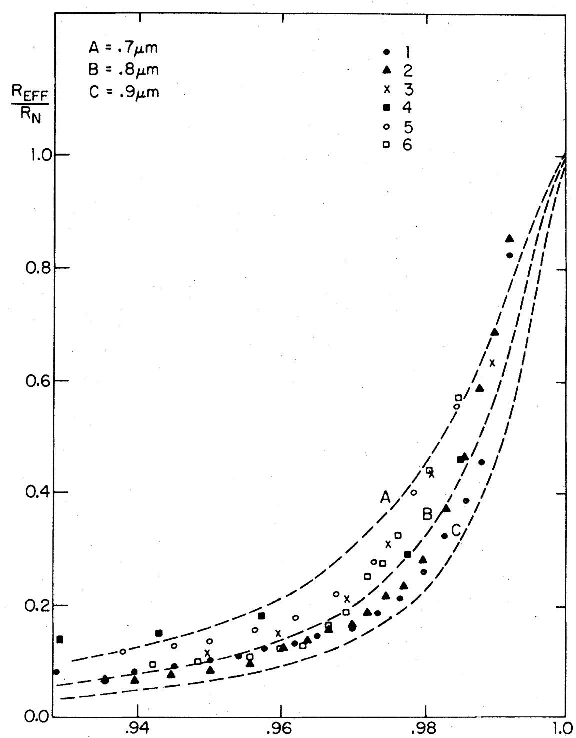

FIG. 7. Normalized differential resistance of the foot vs temperature. The data points are for the six different microbridges listed in Table I, and the dashed curves are Golub's predictions for several values of the bridge-length parameter L.

TABLE I. Bridge parameters.

Bridge No.	R_N (Ω)	$\frac{dI_{c0}}{dT}$ (mA/K)
1	0.06	8.98
2	0.14	5.05
3	0.20	5.8
4	0.33	1.69
5	1.14	0.92
6	0.09	7.69

The agreement between theory and experiment is fairly good if we use bridge lengths of 0.7–0.85 μm. Although scanning electron microscope pictures of our samples show that the distance between the edges of the banks is usually 0.2 μm smaller than that, the appropriate length for comparison with the theory is not well defined and may well be longer than that distance. Near T_c, the experimental effective resistances are somewhat higher than predicted by the theory. This may occur because the temperature dependence of the theory is not quite right, or it may be a result of the differing T_c of the bridge compared to that of the banks, so that additional contributions of the proximity effect affect the results. For T_c we have used the temperature at which the linear temperature dependence of the critical current extrapolates to zero, which may differ by ~10 mK from the temperature at which the critical current actually appears to go to zero and by ~10 mK from the temperature at which the additional resistance of the entire banks is observed.

One feature of the prediction which is difficult to verify but appears somewhat suspect is the strong L^4 length dependence (arising from the L^2/D in γ together with the additional L^2/ξ^2 arising from the calculation of the enhanced order parameter). Our fabrication method is unable to make bridges with substantially shorter geometrical lengths, and attempts to make longer bridges by moving the knife sideways have been inconclusive because the first cut appears to always leave a noticeably deeper cut than the scraped elongation. Such bridges have normalized differential resistance values comparable to those of unelongated bridges. We have also varied the mean free path of the films, and found that the normalized differential resistance of moderately dirty films also does not differ decisively from the range observed in our moderately clean films. Since the theoretical assumption of a well-defined length with rigid banks is obviously an idealization, the extent to which quantitative agreement with the normalized differential resistance and its length dependence should be expected is not clear.

Golub's approach does much better than the expression given by Aslamazov and Larkin for the slope of

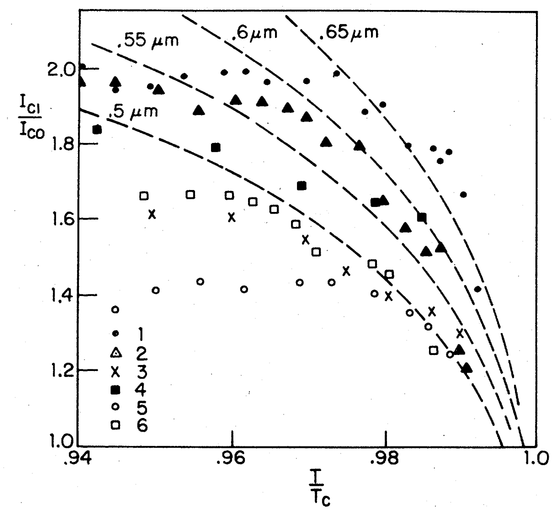

FIG. 8. Ratio of the enhanced critical current I_{c1} to the equilibrium critical current I_{c0} as a function of temperature for the same bridges as in Fig. 7. The dashed curves correspond to the Aslamazov and Larkin prediction, for different values of KL, where $K \sim 1$.

the foot, calculated assuming $\tau_E >> \tau_J$. The latter is several orders of magnitude smaller than the data. Since τ_J is comparable to τ_E throughout the foot region of our bridges, that expression should not be expected to apply, although the extent of disagreement is somewhat surprising and may indicate some difficulty in that particular calculation by AL.

At voltages above this initial region, the Aslamazov and Larkin solution is more appropriate. According to their model the supercurrent is enhanced up to a current I_{c1} which is of order $(L/\eta)I_{c0}$. In Fig. 8, we plot the ratio I_{c1}/I_{c0} as a function of temperature for the same six samples used in Fig. 7. Near T_c, the ratio grows very rapidly in a manner consistent with the temperature dependence $I_{c1}/I_{c0} \propto (1-t)^{1/4}$. At lower temperatures the ratio seems to saturate and remain constant rather than continue growing as predicted by theory. At much lower temperatures (not shown), where heating effects which lead to hysteresis may reduce the apparent value of I_{c1}, the ratio decreases. The dashed lines in the figure correspond to Eq. (15). The appropritate values for KL vary from 0.5 to 0.7 μm. For $K=0.8$ the range and systematic variation of the effective lengths from sample to sample is in agreement with the values determined in Fig. 7. The saturation observed at low temperatures for this ratio is not contained in the theory. As previously mentioned, the theory was derived under the conditions $\eta << L << \xi(T)$; both of these conditions are only marginally satisfied in our bridges in the range shown. At most, the length of our bridges is two times the nonequilibrium length η, and at low temperatures where the saturation is observed, the coherence length becomes shorter than the length of the bridge. Nevertheless, at least the order of magnitude of the enhancement effect is correctly given by the AL theory.

For a limited range of currents above I_{c1}, AL's prediction equation (17) for the shape of the I-V characteristic can be fit to the experimental data reasonably well, although the test is far from stringent. Figure 1(a) includes three theoretical curves of this form fit with the same value of R and adjusting I_{c1} and I_{c0} to an appropriate value for each curve.

For still higher voltages the theoretical situation is not at all clear. This is the region of apparent excess supercurrent $\bar{I}_s$ which is a feature of both the phase-slip model of weak links[17,18] and the TDGL theory.[10,11] Both predict that $\bar{I}_s$ should be a definite fraction of I_c. Presumably in the nonequilibrium situation described here the relevant critical current is I_{c1}, the effective critical current for the actual operating conditions. In Fig. 9, we plot I_{c0}, I_{c1}, and $\bar{I}_s$ for a typical bridge. Near T_c, $\bar{I}_s$ is about $0.6 I_{c0}$, while at lower temperatures, it is about $0.7 I_{c1}$. The qualitative changeover to a dependence on I_{c1} is clear cut, and within this interpretation tends to support I_{c1} as the effective critical current under operating conditions. Note that I_{c0}, not I_{c1}, has the linear temperature dependence extrapolating to T_c expected for the zero-dissipation critical current.

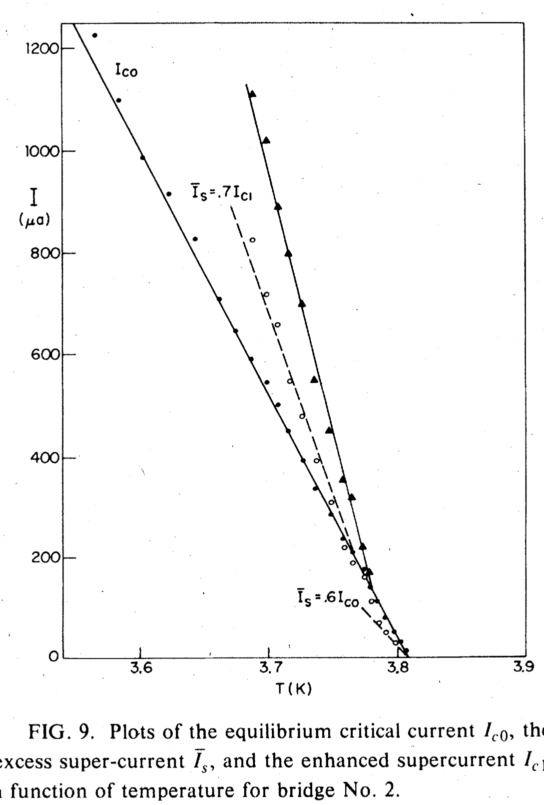

FIG. 9. Plots of the equilibrium critical current I_{c0}, the excess super-current $\bar{I}_s$, and the enhanced supercurrent I_{c1} as a function of temperature for bridge No. 2.

Some comments should be made about the material dependence of the effects discussed. The characteristic feature of interest has been observed only in microbridges made of tin and indium. Since these materials are very similar the theory should describe both equally well. On the other hand the I-V characteristics of short aluminum microbridges do not show similar effects.[5] We have calculated the effective resistance of the low voltage region for aluminum microbridges of lengths similar to ours and find that the change from the normal resistance would be much smaller than that predicted for tin. This is due to the very long coherence length of aluminum which will tend to oppose changes in the order parameter different from the equilibrium case. In order to observe the supercurrent enhancement in aluminum, one would require microbridges with L/ξ ratios comparable to those of our tin microbridges. However, because of the longer inelastic scattering time of aluminum[19] the enhancement would appear at very low voltages where it might not be practical to observe it. In the case of lead, which has a shorter scattering time, the transverse time might become comparable to the longitudinal one, but it is not presently feasible to fabricate bridges satisfying the condition $L < \xi(T)$ because of the shorter coherence length in lead.

Klapwijk and Mooij[5] have suggested that the foot might be a result of flux flow. We believe that our bridges are, in most cases, too narrow to contain a single vortex. Yet, the enhancement is observed in all of our bridges. In order to test the flux-flow concept, we have fabricated lead microbridges where one might expect flux flow to be more important.[20] Our lead microbridges show a linear temperature dependence of the critical current near T_c as well as strong Josephson effects up to voltages comparable to those in tin.[6] However, the I-V characteristics are quite different and do not have any footlike features. This would seem to rule out flux flow as the origin of this feature.

V. CONCLUSIONS

In this paper we have presented systematic evidence which supports the idea that the supercurrent through tin variable-thickness microbridges at nonzero voltages can be enhanced by the effect of disequilibrium of the quasiparticles inside the weak link. We have presented a qualitative discussion of the physical origin of the enhancement, based on the more specific calculation by Golub and by Aslamazov and Larkin. These theories have a fundamentally similar point of view, but they make rather different physical and mathematical approximations and provide complementary contributions to the description of the complete I-V curves.

By comparing the various quantitative predictions of these theories with our experimental data, we have found many points of agreement in the regimes of applicability of the theories. While the validity of some of the physical and mathematical approximations cannot be settled simply by comparison with experimental data, the general agreement indicates that these theories are a promising approach to understanding nonequilibrium systems.

ACKNOWLEDGMENT

The authors would like to express their gratitude to Dr. L. N. Smith for stimulating discussions which aided in clarifying the physical content of the theories and for numerous useful suggestions.

*Supported in part by ONR and NSF.

†Supported in part by Instituto Venezolano de Investigaciones Cientificas.

[1]G. M. Eliashberg, JETP Lett. 11, 114 (1970); T. J. Tredwell and E. H. Jacobsen, Phys. Rev. Lett. 35, 244 (1975), and Phys. Rev. 13, 2931 (1976); T. M. Klapwijk, J. N. van der Bergh, and J. E. Mooij, J. Low Temp. Phys. 26, 385 (1977); T. Kommers and J. Clarke, Phys. Rev. Lett. 38, 1091 (1977).

[2]A. A. Golub, Zh. Eksp. Teor. Fiz. 71, 341 (1976) [Sov. Phys.-JETP 44, 178 (1976)].

[3]L. G. Aslamazov and A. I. Larkin, Zh. Eksp. Teor. Fiz. 70, 1340 (1976) [Sov. Phys.-JETP 43, 698 (1976)].

[4]V. N. Gubankov, V. P. Koshelets, G. A. Ovsyannikov, IEEE Trans. Magn. Mag-13, 228 (1977).

[5]G. M. Daalmans, T. M. Klapwijk, and J. E. Mooij, IEEE Trans. Magn. Mag-13, 719 (1977); H. Højaard Jensen and P. E. Lindelof, J. Low Temp. Phys. 23, 469 (1976), and Ref. 4.

[6]M. Octavio, W. J. Skocpol, and M. Tinkham, IEEE Trans. Magn. Mag-13, 739 (1977).

[7]W. J. Skocpol, M. R. Beasley, and M. Tinkham, J. Appl. Phys. 45, 4054 (1974).

[8]L. E. Hasselberg, M. T. Levinsen, and M. R. Samuelsen, J. Low Temp. Phys. 21, 567 (1975), and references therein.

[9]L. G. Aslamazov, and A. I. Larkin, JETP Lett. 9, 87 (1969).

[10]K. K. Likharev, and L. A. Yakobson, Zh. Eksp. Teor. Fiz. 68, 1150 (1975) [Sov. Phys.-JETP 41, 570 (1975)].

[11]L. Kramer and A. Baratoff, Phys. Rev. Lett. 38, 518 (1977).

[12]B. S. Deaver, B. G. Boone, and R. Rifkin, Phys. Lett. A 57, 186 (1976).

[13]A. Schmid and G. Schön, J. Low Temp. Phys. 20, 207 (1975).

[14]A. I. Larkin and Yu N. Ovchinnokov, Zh. Eksp. Teor. Fiz. 68, 1915 (1975) [Sov. Phys.-JETP 41, 960 (1975)].

[15]M. Tinkham and J. Clarke, Phys. Rev. Lett. 28, 1366 (1972); M. Tinkham, Phys. Rev. B 6, 1747 (1972).

[16]M. L. Yu and J. E. Mercereau, Phys. Rev. B 12, 4909 (1975); S. N. Artemenko and A. F. Volkov. Zh. Eksp. Teor. Fiz. 70, 1051 (1976) [Sov. Phys.-JETP 43, 548 (1976)].

[17]W. J. Skocpol, M. R. Beasley, and M. Tinkham, J. Low Temp. Phys. 16, 145 (1974).

[18]H. A. Notarys and J. E. Mercereau, Physica (Utr.) 55, 424 (1971).

[19]T. M. Klapwijk and J. E. Mooij, Phys. Lett. A 57, 97 (1976).

[20]K. K. Likharev, Zh. Eksp. Teor. Fiz. 61, 1700 (1971) [Sov. Phys.-JETP 34, 906 (1972)].