Instability threshold of a one-dimensional Bloch wall

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Possible nucleation modes for a one-dimensional Bloch wall in a field antiparallel to the magnetization at the wall center are analyzed and the corresponding threshold instability fields are calculated. For the case of a mode uniform in the plane of the wall the exact result $H_b(0) = 4\pi M/3$ is obtained, but it is then shown that modes exhibiting buckling in this plane will have a lower threshold. These modes are characterized by the constraint that the wall azimuthal angle remains at its equilibrium value until an instability in the polar angle is reached. Rigorous upper- and lower-bound calculations show that the buckling-mode threshold instability field will be in the range $0.034 \leq H_b(q)/4\pi M \leq 0.149$. An alternate nucleation mode, characterized by zero magnetostatic self-energy, is also analyzed. For this corrugating mode we find a rigorous upper bound to the threshold instability field of $H_c(q \rightarrow 0)/H_k = 0.543$. The implications of these results are discussed.

I. INTRODUCTION

A basic problem in micromagnetics[1-4] is the calculation of the threshold or nucleation field at which a given magnetization configuration becomes unstable. This particular problem is, in many cases, amenable to solution as the general nonlinear and constrained equilibrium equations can be linearized and the stability of a given equilibrium configuration with respect to small deviations can then be considered.

The starting configuration which has been most studied is that of uniform magnetization. This state is attained by applying a sufficiently large negative and uniform external field $-H$ to a specimen of suitable geometry. The field is then quasistatically reduced and, if necessary, reversed until a point is reached at which the uniform configuration is no longer stable. To determine this instability point, the stability with respect to particular deviation modes, such as rotation in unison, buckling, and curling is studied by solving the resulting eigenvalue problem for each mode. If there is more than one solution, then only that with the (algebraically) smallest value of H is meaningful. A full discussion of the above points and solutions of the nucleation problem for various geometries can be found in the literature.[1-4]

We shall here consider a somewhat different type of instability threshold problem, one in which the starting configuration is *not* uniform.[5] As our initial state we shall instead take a one-dimensional Bloch wall. The equilibrium one-dimensional 180° Bloch-wall configuration was first calculated by Landau and Lifshitz[6] in a classic paper. It is one of the very few examples of a rigorous solution of the nonlinear equations of micromagnetics. We shall show that, as the magnitude of a magnetic field applied antiparallel to the magnetization at the wall center is increased, there always exists a one-dimensional wall configuration. For suitable boundary conditions,[7] this wall remains stable until the external field is increased to the instability threshold. We stress that the orientation of the applied field is critical, even a small component perpendicular to the direction indicated could result in wall motion or a change in the wall configuration.

The motivation of the present work is twofold: First, there is the practical interest in finding the instability threshold for the specific case considered. Second, and possibly more important, there is the methodological interest in the exposition of a stability analysis for a spatially nonuniform configuration.

In Sec. II A, we derive the equilibrium configuration of the one-dimensional Bloch wall in an external field. Following this, in Sec. II B, we calculate the threshold field under the constraint that the instability mode is itself one-dimensional. In Sec. II C we relax this constraint and consider more general instability modes. While an analytic solution of the general variational equations could not be found, these equations are used to obtain rigorous upper bounds on the instability threshold field for given buckling or corrugating-mode wave vectors. We treat these two types of instability modes in Secs. II D and II E. For the buckling mode, we also give a lower bound calculation of the instability threshold field, obtained by employing a method proposed originally by Brown[1,8] for approximating the system's magnetostatic self-

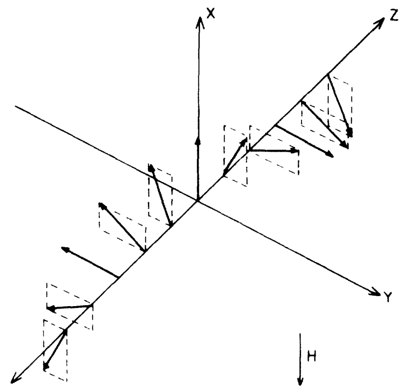

FIG. 1. Bloch wall in an external field.

energy.

In Sec. III, we review and discuss the results of our calculation.

II. CALCULATION OF THE INSTABILITY THRESHOLD

A. Equilibrium configuration

Consider an unbounded uniaxial ferromagnet with magnetization M, easy axis along the y direction, and in a magnetic field $\vec{H} = -H\vec{u}_1$ applied in the negative x direction (see Fig. 1). In polar coordinates, the magnetization components are

$$M_x = M\cos\phi\,\sin\theta\,,\quad M_y = M\sin\phi\,\sin\theta\,,\quad M_z = M\cos\theta\,. \tag{1}$$

The exchange, anisotropy, applied-field, and magnetostatic self-energy or demagnetizing energy densities are given by the usual expressions[1-4]

$$e_x = A[(\nabla\theta)^2 + \sin^2\theta(\nabla\phi)^2]\,, \tag{2a}$$

$$e_a = -K\sin^2\phi\,\sin^2\theta\,, \tag{2b}$$

$$e_f = MH\cos\phi\,\sin\theta\,, \tag{2c}$$

$$e_d = -\tfrac{1}{2}\vec{M}\cdot\vec{H}_d\,, \tag{2d}$$

where A and K are the exchange and anisotropy constants, respectively, and H_d is the demagnetizing field.

As is well known,[5] the one-dimensional Bloch-wall configuration is characterized by $\theta = \frac{1}{2}\pi$, $\vec{M} = \vec{M}(z)$, and $\vec{H}_d = 0$. We therefore restrict our attention to the energy contributions of Eqs. (2a)–(2c).

Setting $\theta = \frac{1}{2}\pi$ in Eq. (2), we find the equilibrium equation for $\phi = \phi(z)$ by requiring that the first variation of the total energy be zero. This yields

$$\phi_{zz} + \delta_0^{-2}\sin\phi(\cos\phi + h) = 0\,, \tag{3}$$

where the subscript denotes differentiation with respect to z, and

$$h = H/H_k = MH/2K,\ \delta_0^2 = A/K\,. \tag{4}$$

We are interested in nonuniform solutions of Eq. (3) satisfying the boundary conditions

$$\phi(\pm\infty) = \pm\phi_0\,, \tag{5}$$

where the angle ϕ_0 is given by

$$\cos\phi_0 = +1 \quad \text{for } -1 > h\,, \tag{6a}$$

$$\cos\phi_0 = -h \quad \text{for } -1 < h < +1\,, \tag{6b}$$

$$\cos\phi_0 = -1 \quad \text{for } h > +1\,. \tag{6c}$$

Equation (3) has solutions satisfying Eq. (5) which describe Bloch walls with $\phi(0) = 0$ or π, centered at an arbitary value z_0 of z. We select for further study a wall centered at $z_0 = 0$ with $\phi(0) = 0$. This solution is given by

$$\phi = 0 \quad \text{for } -1 > h\,, \tag{7a}$$

$$\tan(\tfrac{1}{2}\phi) = \tan(\tfrac{1}{2}\phi_0)\tanh(z\sin\phi_0/2\delta_0) \quad \text{for } -1 < h < +1\,, \tag{7b}$$

$$\tan\frac{\phi}{2} = \left(\frac{h}{h-1}\right)^{1/2}\sinh\frac{(h-1)^{1/2}z}{\delta_0} \quad \text{for } h > +1. \tag{7c}$$

Equation (7b) for $0 < h < +1$ is the equilibrium equation of a Bloch-type wall in which the overall angle of magnetization rotation is between 180° and 360° while Eq. (7c) describes a 360° Bloch wall. In particular, for $h \gg 1$, the latter reduces to

$$\tan(\tfrac{1}{2}\phi) = \sinh(z/\delta) \tag{8a}$$

or

$$1 + \cos\phi = 2/\cosh^2(z/\delta)\,, \tag{8b}$$

where we have introduced a field-dependent width δ by

$$\delta^2 = \delta_0^2/h = 2A/MH\,. \tag{8c}$$

We now wish to study the stability of the domain-wall structure described by Eqs. (7b), (7c) or, for $h \gg 1$, by Eq. (8).

B. Uniform mode

To begin, we shall test the stability of the Bloch wall under the constraint that the instability mode describing the departure from equilibrium is a uniform one, i.e., that it is a function of z and not of x or y. Under such a constraint, the demagnetizing field is simply $\vec{H}_d = -4\pi M_z\vec{u}_3$ and the associated energy density is given by

$$e_d = 2\pi M^2\cos^2\theta\,. \tag{9}$$

As we shall see, the threshold field for this mode will turn out to be of the order of $4\pi M$. Since for materials of interest $4\pi M \gg H_k$, i.e., $h \gg 1$, the anisotropy-energy contribution to the total energy is negligible and we shall ignore this term. The total energy per unit wall area, obtained by summing the contributions of Eqs. (2a), (2c), and (9), is given by

$$E = \int_{-\infty}^{+\infty} e\,dz$$

$$= \int_{-\infty}^{+\infty} \{A[\theta_z^2 + \sin^2\theta(\phi_z^2) + (2/\delta^2)\cos\phi\sin\theta] + 2\pi M^2\cos^2\theta\}\,dz\,. \qquad (10)$$

We expand this expression to second order in the variations $\alpha(z)$, $\beta(z)$ of the angles θ, ϕ from their equilibrium values, thus

$$\theta_1 = \tfrac{1}{2}\pi - \alpha(z)\,, \qquad (11a)$$

$$\phi_1 = \phi(z) + \beta(z)\,. \qquad (11b)$$

As might be expected, there is no coupling between the α and β variations. The β mode corresponds to a translation of the wall along z, with respect to which the wall configuration is neutral at any field, and need not be considered further in the uniform case (the nonuniform case will be discussed in Sec. II E). The α mode consists of a rotation of the wall magnetization out of the wall plane. Letting E_0 be the energy per unit wall area of the equilibrium configuration, we obtain

$$\Delta E = E - E_0$$

$$= \int_{-\infty}^{+\infty} dz\,\left\{A\alpha_z^2 - \left[A\phi_z^2 + \left(\frac{A}{\delta^2}\right)\cos\phi - 2\pi M^2\right]\alpha^2\right\}$$

$$= 0\,. \qquad (12a)$$

From Eq. (8b) we find

$$(\delta^2\phi_z^2 + 1)\cos\phi = [6/\cosh^2(z/\delta)] - 1\,. \qquad (13)$$

Introducing the reduced coordinate $\zeta = z/\delta$, we can write Eq. (12a) in the form

$$\Delta E = \frac{A}{\delta}\int_{-\infty}^{+\infty} d\zeta\left[(\alpha')^2 - \left(\frac{6}{\cosh^2\zeta} - 1 - \frac{4\pi M^2}{H}\right)\alpha^2\right], \qquad (12b)$$

where a prime denotes differentiation with respect to ζ. In order for the equilibrium state $(\theta = \frac{1}{2}\pi, \phi)$ to be stable, ΔE must be positive for arbitrary variations $\alpha(\zeta)$. The stability limit $H_b(0)$ is the smallest value of H for which there exists an instability mode such that $\Delta E = 0$. We thus obtain the variational problem

$$\frac{4\pi M}{H_b(0)} = \max\left\{\int_{-\infty}^{+\infty} d\zeta\left[-(\alpha')^2 + \left(\frac{6}{\cosh^2\zeta} - 1\right)\alpha^2\right] \Big/ \int_{-\infty}^{+\infty} d\zeta\,\alpha^2\right\}. \qquad (14)$$

Variation with respect to α leads to the linear variational equation

$$\alpha'' + [(6/\cosh^2\zeta) - \lambda^2]\alpha = 0\,, \qquad (15)$$

with the eigenvalue parameter

$$\lambda^2 = 1 + 4\pi M/H\,. \qquad (16)$$

The solutions of Eq. (15) are

$$\alpha_1 = [\tanh^2\zeta + \tfrac{1}{3}(\lambda^2 - 1)]\cosh\lambda\zeta - \lambda\tanh\zeta\sinh\lambda\zeta\,,$$
$$\alpha_2 = [\tanh^2\zeta + \tfrac{1}{3}(\lambda^2 - 1)]\sinh\lambda\zeta - \lambda\tanh\zeta\cosh\lambda\zeta\,. \qquad (17)$$

We require that the physically realizable instability mode have a finite amplitude for all ζ and, in particular, as $\zeta \to \infty$. The solutions α_1, α_2 satisfy this condition if and only if

$$\lambda^2 - 3\lambda + 2 = 0; \quad \text{i.e., if } \lambda^2 = 1, 4\,. \qquad (18)$$

The solution $\lambda^2 = 1$ corresponds to $H = \infty$. However, the alternate solution, $\lambda^2 = 4$, yields

$$H_b(0) = \tfrac{4}{3}\pi M\,. \qquad (19)$$

Thus, we find that the magnitude of the applied field at which the uniform-mode instability threshold is reached is $\frac{4}{3}\pi M$. The eigenfunction corresponding to this mode is given by the even function α_1 and is

$$\alpha_0 = \alpha_1\ (\lambda = 2) = \frac{1}{\cosh^2\zeta} = \frac{1}{[\cosh^2 z(2\pi M^2/3A)^{1/2}]}\,. \qquad (20)$$

C. Nonuniform modes

The instability threshold field $H_b(0)$ was obtained in Sec. II B by "constraining" the instability mode to be independent of x and y. We shall now show that relaxing this condition will lead to a lower threshold. To do this, we must first derive the equations describing the threshold field for more general instability modes. We thus generalize Eq. (11) to

$$\theta_1 = \tfrac{1}{2}\pi - \alpha(x, y, z)\,, \qquad (21a)$$

$$\phi_1 = \phi(z) + \beta(x, y, z)\,. \qquad (21b)$$

To second order in α and β, the total energy per unit wall area is given by

$$E=\frac{1}{\tau}\int dV\{A\phi_z^2+MH\cos\phi-K\sin^2\phi+2A\phi_z\beta_z-(MH\sin\phi+2K\sin\phi\cos\phi)\beta+A(\nabla\alpha)^2$$
$$-(A\phi_z^2+\tfrac{1}{2}MH\cos\phi-K\sin^2\phi)\alpha^2+A(\nabla\beta)^2-[\tfrac{1}{2}MH\cos\phi+K(\cos^2\phi-\sin^2\phi)]\beta^2\}+E_d\,. \qquad (22)$$

Here, $d\tau$ is an element of area in the x-y plane, and E_d is the contribution of the demagnetizing energy to the total energy per unit area. Note that Eq. (22) includes the anisotropy energy associated with the domain wall.

Now,

$$\int dz[A\phi_z^2+MH\cos\phi-K\sin^2\theta]=E_0\,, \qquad (23a)$$

the energy per unit area of the equilibrium configuration. Also,

$$\int dV[2A\phi_z\beta_z-(MH\sin\phi+2K\sin\phi\cos\phi)\beta]$$
$$=\int dV(2A\phi_z\beta)_z$$
$$=0\,. \qquad (23b)$$

We again introduce reduced coordinates

$$\xi=\frac{x}{\delta},\quad \eta=\frac{y}{\delta},\quad \zeta=\frac{z}{\delta},\quad \sigma=\frac{\tau}{\delta^2},\quad \Omega=\frac{V}{\delta^3}\,. \qquad (24)$$

Using Eqs. (7), (23), and (24), Eq. (22) becomes

$$\Delta E=E-E_0=(A/\delta)[-\gamma+(4\pi M/H)\gamma_d]\,, \qquad (25)$$

where, for $h>1$,

$$\gamma=\frac{1}{\sigma}\int d\Omega\left\{-(\nabla\alpha)^2-(\nabla\beta)^2+\alpha^2\left[(2+3\cos\phi)-\left(\frac{2}{h}\right)\sin^2\phi\right]+\beta^2\left[\cos\phi+\left(\frac{1}{h}\right)(\cos^2\phi-\sin^2\phi)\right]\right\}, \qquad (26)$$

$$\gamma_d=E_d/2\pi M^2\delta\,, \qquad (27a)$$

and the gradient operator is now with respect to the reduced coordinates.

The reduced demagnetizing energy γ_d is given in terms of the magnetostatic potential ψ by

$$\gamma_d=\frac{1}{\sigma}\int d\Omega\left(-\beta\sin\phi\frac{\partial\psi}{\partial\xi}+\beta\cos\phi\frac{\partial\psi}{\partial\eta}+\alpha\frac{\partial\psi}{\partial\zeta}\right)\,, \qquad (27b)$$

$$\nabla^2\psi=-\sin\phi\frac{\partial\beta}{\partial\xi}+\cos\phi\frac{\partial\beta}{\partial\eta}+\frac{\partial\alpha}{\partial\zeta}\,, \qquad (27c)$$

with the appropriate boundary conditions. Thus γ_d does not depend explicitly on the applied field. The stability limit H_t is again the smallest value of H for which $\Delta E=0$ for some instability mode $[\alpha(\zeta),\beta(\zeta)]$. Thus the true instability field is determined by the variational problem

$$4\pi M/H_t=\max(\gamma/\gamma_d)\,. \qquad (28a)$$

It might appear that the explicit dependence of γ on $h=H/H_k$ can lead to difficulties. However, the maximization can be carried out at constant h. When the maximum has been found for a given h, this h is then identified with H/H_k, yielding the implicit equation

$$4\pi M/H_t=F(H_t/H_k) \qquad (28b)$$

for the determination of the instability threshold field.

While Eq. (28) has not been solved analytically for arbitrary variations α, β, it can be used to find rigorous upper bounds on the instability threshold field for given instability modes. In particular, we shall consider two types of variations, which we refer to as buckling and corrugating modes.

D. Buckling-mode instability

Buckling-type nucleation is obtained by suppressing the β variation (i.e., constraining β to be zero) and considering only modes described by α. It is thus, in a sense, an extension of the uniform mode considered earlier in Sec. II B, and consists of a rotation of wall magnetization out of the wall plane in an alternating pattern described by a wave vector $\vec{q}$. We assume that the threshold field will again be of order $4\pi M$, i.e., that $h\gg 1$, and therefore neglect the anisotropy energy contribution. Then γ becomes independent of h and the instability field is given directly by Eq. (28a).

In order to study the buckling modes in the x-y plane, we Fourier transform the x and y coordinates by writing

$$\alpha=\sum_{\vec{q}}\alpha_{\vec{q}}(\zeta)e^{i\vec{q}\cdot\vec{\rho}};\quad \alpha_{-\vec{q}}=\alpha_{\vec{q}}^*\,, \qquad (29a)$$

$$\psi=\sum_{\vec{q}}\psi_{\vec{q}}(\zeta)e^{i\vec{q}\cdot\vec{\rho}};\quad \psi_{-\vec{q}}=\psi_{\vec{q}}^*\,, \qquad (29b)$$

where

$$\vec{\rho}=(\xi,\eta,0),\quad \vec{q}=(q_\xi,q_\eta,0)\,. \qquad (29c)$$

Setting $\beta=0$ in Eqs. (26) and (27) and using Eq. (29), we obtain for $\psi_{\vec{q}}$

$$\psi_{\vec{q}}''=q^2\psi_{\vec{q}}=\alpha_{\vec{q}}'\,, \qquad (30)$$

Using Green's functions the solution of Eq. (30) satisfying the boundary conditions $\psi_{\vec{q}}(\zeta)\to 0$ as $\zeta\to\pm\infty$ is found to be

$$\psi_{\vec{q}}(\zeta)=\tfrac{1}{2}\int_{-\infty}^{+\infty}d\zeta'\sin(\zeta-\zeta')e^{-|q|\cdot|\zeta-\zeta'|}\alpha_{\vec{q}}(\zeta')\,. \qquad (31)$$

Thus

$$\psi^{\zeta}_{\vec q}(\zeta) = \alpha_{\vec q}(\zeta) - \frac{|q|}{2}\int_{-\infty}^{+\infty} d\zeta' e^{-|q|\cdot|\zeta-\zeta'|}\alpha_{\vec q}(\zeta') \quad (32)$$

and

$$\gamma_d = \sum_{\vec q}\int_{-\infty}^{+\infty} d\zeta \int_{-\infty}^{+\infty} d\zeta' \left(\delta(\zeta-\zeta') - \frac{|q|}{2}e^{-|q|\cdot|\zeta-\zeta'|}\right) \times \alpha_{\vec q}(\zeta)\alpha_{-\vec q}(\zeta') , \quad (33)$$

where $\delta(\zeta)$ is the Dirac delta function. Further,

$$\gamma = \sum_{\vec q}\int_{-\infty}^{+\infty} d\zeta[-\alpha'_{\vec q}\alpha'_{-\vec q} + (-q^2+2+3\cos\phi)\alpha_{\vec q}\alpha_{-\vec q}] . \quad (34)$$

Substituting Eqs. (33) and (34) into Eq. (28a), we obtain, for a given $\vec q$,

$$\frac{4\pi M}{H_b(\vec q)} = \max\left(\int d\zeta[-|\alpha'_{\vec q}|^2 + (2+3\cos\phi - q^2)|\alpha_{\vec q}|^2] \Big/ \int d\zeta\int d\zeta'[\delta(\zeta-\zeta') - (\tfrac12|q|)\exp(-|q|\cdot|\zeta-\zeta'|)]\alpha_{\vec q}(\zeta)\alpha_{\vec q}(\zeta')\right) . \quad (35)$$

Equation (35) gives us the instability threshold field H_b for any buckling-instability wave vector $\vec q$.

While we have not been able to obtain an exact solution of Eq. (35), an upper bound on the buckling mode threshold field can be found by evaluating the various energy terms with any arbitrary $\alpha_{\vec q}(\zeta)$ that satisfies the boundary conditions. We shall now do this, using the exact solution for the $\vec q = 0$ mode given in Eq. (20). Substituting Eq. (20) into Eq. (35), we obtain for a given $\vec q$

$$\gamma = (3-q^2)\int_{-\infty}^{+\infty} d\zeta/\cosh^4\zeta = \tfrac43(3-q^2) . \quad (36)$$

To evaluate γ_d, we transform Eq. (33) into a more suitable form. Introducing new variables

$$\zeta_1 = \zeta' + \zeta, \quad \zeta_2 = \zeta' - \zeta , \quad (37)$$

we obtain

$$\gamma_d = \tfrac43 - 2q\int_0^{\infty} d\zeta_2\, e^{-q\zeta_2}\chi(\zeta_2) , \quad (38a)$$

where[9]

$$\chi(\zeta_2) = \int_{-\infty}^{+\infty} d\zeta_1(\cosh\zeta_1 + \cosh\zeta_2)^{-2} = 2\frac{\zeta_2\cosh\zeta_2 - \sinh\zeta_2}{\sinh^3\zeta_2} . \quad (38b)$$

Expanding $\chi(\zeta_2)$ in exponentials gives for γ_d the series expression (convergent for all $\vec q$)

$$\gamma_d = \tfrac43 - 32q\sum_{n=1}^{\infty}\frac{n(n+1)}{(2n+q)^2(2n+2+q)^2} . \quad (39)$$

Substituting Eqs. (36) and (39) into Eq. (28a), we obtain an upper bound $H_b(\vec q)_{max}$ on the buckling-mode threshold for given $\vec q$

$$H_b(\vec q)_{max} = (4\pi M)\left(1 - 24q\sum_{n=1}^{\infty} n(n+1)(2n+q)^{-2} \times (2n+2+q)^{-2} \Big/ (3-q^2)\right) . \quad (40)$$

The sum in Eq. (40) has been evaluated by computer[10] for q in the range $0 \le q < \sqrt3$. The results are shown in Fig. 2. We see immediately that the threshold field will be considerably less than the $\frac43\pi M$ value found for $q=0$. The minimum value of $H_b(\vec q)_{max}$ given by Eq. (40) is 0.149 $(4\pi M)$ and occurs at $q \simeq 1$.

For buckling-type nucleation modes we can also calculate a lower bound to the threshold field $H_b(\vec q)$. The method is as follows. Brown[8] has shown that the energy term

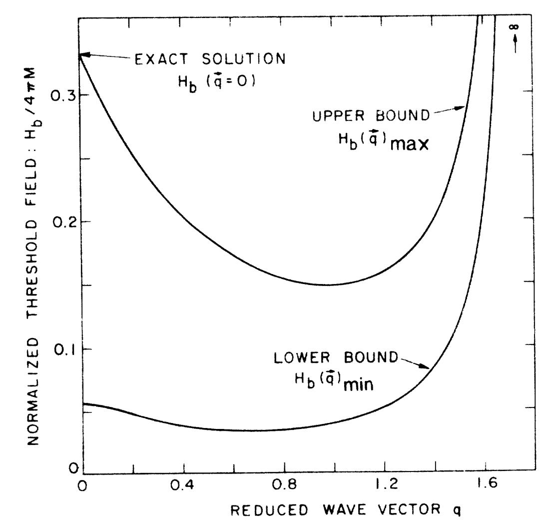

FIG. 2. Upper and lower bounds to the buckling-mode instability threshold field H_b as a function of the reduced wave vector $\vec q$. The exact solution for $\vec q = 0$ is noted.

$$E_d' = \frac{\delta}{\sigma}\left(-(8\pi)\int H_1^2\, d\Omega - \int \vec{M}\cdot\vec{H}_1\, d\Omega\right), \quad (41)$$

where $\vec{H}_1$ is any irrotational vector, is always less than or equal to the true magnetostatic self-energy E_d. It thus follows that, if we replace E_d by E_d', the resulting threshold field $(H_b)_{min}$ will always be less than (or equal to) the actual threshold field for a buckling-mode type instability. We now use this technique to obtain $(H_b)_{min}$ as a function of $\vec{q}$.

We shall take the field $\vec{H}_1$ for a given buckling wave vector $\vec{q}$, as $H_1 = -\nabla\Phi$, where

$$\Phi = (a/q)F(\zeta)e^{i\vec{q}\cdot\vec{\rho}}. \quad (42)$$

Here a is a variational parameter, and we continue to use the reduced coordinates introduced earlier. The field $\vec{H}_1$ is

$$\vec{H}_1 = (a/q)[i(\vec{u}_1 q_\xi + \vec{u}_2 q_\eta)F + \vec{u}_3 q f]e^{i\vec{q}\cdot\vec{\rho}}, \quad (43a)$$

where $\vec{u}_i$ $(i = 1, 2, 3)$ are unit vectors and

$$qf(\zeta) = F'(\zeta) = \frac{dF(\zeta)}{d\zeta}. \quad (43b)$$

Using Eqs. (1), (21a), and (43), and keeping only terms to second order in α, Eq. (41) becomes

$$E_d' = -a^2 b\delta/8\pi - Ma\delta I, \quad (44a)$$

where

$$b = \int_{-\infty}^{+\infty} (F^2 + f^2)\, d\zeta. \quad (44b)$$

We maximize E_d' by setting $\partial E_d'/\partial a = 0$. This gives

$$(E_d')_{max} = 2\pi M^2\delta(I^2/b). \quad (45)$$

Comparing Eqs. (27a) and (45), we see that we are replacing the true magnetostatic self-energy term γ_d in Eq. (25) by $\gamma_d' = I^2/b$. When this is done, the resultant variational equation becomes

$$\alpha_{\vec{q}}'' + [6/\cosh^2\zeta - \lambda^2]\alpha_{\vec{q}} = \lambda' I f/b, \quad (46a)$$

where now

$$\lambda = (1 + q^2)^{1/2}, \quad (46b)$$

and

$$\lambda' = 4\pi M/H_b(\vec{q})_{min}. \quad (46c)$$

Comparing Eq. (46a) with Eq. (15) we see that the required solution may be written in the form

$$\alpha_{\vec{q}} = \lambda'(I/b)[p(\zeta) + c\alpha_1(\zeta)], \quad (47)$$

where $p(\zeta)$ is a particular solution of Eq. (46a) and $\alpha_1(\zeta)$, the (even) solution of the homogeneous equation, is given in Eq. (17). The constant c is to be determined by the requirement that $\alpha_{\vec{q}}(\zeta) \to 0$ as $\zeta \to \infty$.

If we now form the integral

$$I \equiv \int_{-\infty}^{+\infty} f\alpha_{\vec{q}}\, d\zeta = \lambda'\frac{I}{b}\int_{-\infty}^{+\infty} [p(\zeta) + c\alpha_1(\zeta)]\, d\zeta = \lambda'\frac{I}{b}P, \quad (48)$$

we obtain the relation

$$I(1 - \lambda' P/b) = 0. \quad (49)$$

The variational equation (46a) will have a solution other than $I = 0$ if and only if

$$H_b(\vec{q})_{min} = 4\pi M/\lambda' = 4\pi M(P/b). \quad (50)$$

This then is the desired lower bound to the buckling mode instability threshold field.

To obtain a particular solution $p(\zeta)$ of Eq. (46a) we have employed the method of variation of parameters. This gives

$$p(\zeta) = W^{-1}\left(-\alpha_1(\zeta)\int_0^\zeta f(\zeta')\alpha_2(\zeta')\, d\zeta' + \alpha_2(\zeta)\int_0^\zeta f(\zeta')\alpha_1(\zeta')\, d\zeta'\right), \quad (51a)$$

where

$$W = \alpha_1\alpha_2' - \alpha_2\alpha_1' = \tfrac{1}{9}\lambda q^2(q^2 - 3) \quad (51b)$$

is the Wronskian. Equation (17) can be rewritten in the form

$$\alpha_{1,2} = \tfrac{1}{2}[(\tanh^2\zeta - \lambda\tanh\zeta + k)e^{\lambda\zeta} \pm (\tanh^2\zeta + \lambda\tanh\zeta + k)e^{-\lambda\zeta}], \quad (52)$$

where $k = \frac{1}{3}(\lambda^2 - 1)$ and it is convenient to introduce the quantities

$$s \pm t = f\alpha_{1,2}, \quad (53a)$$

$$S - S_0 = \int_0^\zeta s(\zeta')\, d\zeta', \quad (53b)$$

$$T - T_0 = \int_0^\zeta t(\zeta')\, d\zeta'. \quad (53c)$$

Using Eqs. (52) and (53), $p(\zeta)$ can be written

$$p(\zeta) = (2/Wf)(Ts - St - T_0 s + S_0 t). \quad (54)$$

Since p is an even function of ζ, it is sufficient to consider the range $\zeta > 0$. As $\zeta \to \infty$, we see from Eq. (52) that $s/f \to \infty$ while $t/f \to 0$. We thus set $c = (2/Wf)T_0$ and obtain

$$p + c\alpha_1 = (2/Wf)[Ts - St + (S_0 + T_0)t]. \quad (55)$$

This yields, finally

$$H_b(\vec{q})_{min} = \frac{2(4\pi M)}{bW}\left(\int_0^\infty (Ts - St)\, d\zeta - (S_0 + T_0)T_0\right). \quad (56)$$

To evaluate $H_b(\vec{q})_{min}$, we chose, primarily for mathematical convenience,

$$F(\zeta)=q(e^{-\epsilon_1|\zeta|}-e^{-\epsilon_2|\zeta|})\cosh^3\zeta, \quad \epsilon_1>\epsilon_2>3. \tag{57}$$

The evaluation of $H_b(\vec{q})_{min}$ from Eqs. (56) and (57) is straightforward but tedious and will not be given here. The resulting expression for the lower bound, evaluated at the limit $\epsilon_1 \to \epsilon_2 = \epsilon$ is shown in Fig. 2 as a function of q. The value of $H_b(\vec{q})_{min}$ given for each q is the maximum found as a function of the parameter ϵ.

The minimum value of $H_b(\vec{q})_{min}$ for the $F(\zeta)$ of Eq. (57) is 0.034 $(4\pi M)$ and occurs at $q \simeq 0.7$. Note that, for typical parameter values $4\pi M = 10^4 G$ and $H_k = 2K/M = 10$ Oe, this is equivalent to a reduced field $h = H_b(q=0.7)_{min}/H_k = 34$. Thus our initial requirement, $h \gg 1$, is satisfied.

E. Corrugating-mode instability

The buckling-mode instability, as we have seen, reduces to the uniform mode at $\vec{q}=0$. However, for $\vec{q}=0$, there exists another type of mode, with respect to which our initial configuration is neutral. This mode, in which $\alpha=0$ and β is proportional to ϕ_z, simply corresponds to a uniform translation of the wall along z in zero field and is not a true instability, which we define as a change in the wall structure. Suppose, however, that we consider the class of variations satisfying

$$\alpha=\alpha(\xi,\zeta)=i\sum_q \alpha_q(\zeta)e^{iq\xi}; \quad \alpha_{-q}=-\alpha_q^*, \tag{58a}$$

$$\beta=\beta(\xi,\zeta)=\sum_q \frac{\alpha_q'(\zeta)}{q\sin\phi}e^{iq\xi}; \quad \beta_{-q}=\beta_q^*. \tag{58b}$$

Note that the wave vector $\vec{q}$ is along the ξ (or x) axis and that α and β are independent of η. Substituting Eq. (58) into Eq. (27) we immediately obtain the result $\gamma_d=0$. That is, there is no magnetostatic self-energy associated with an instability mode of this type. The variation defined in Eq. (58) will be referred to as a corrugating mode. It corresponds to a periodic translation of segments of the wall in opposite directions, resulting in a corrugated-iron pattern with wave vector $\vec{q}$. We shall now calculate an upper bound on the instability threshold of a corrugating nucleation mode with given wave vector $\vec{q}$. Since $\gamma_d=0$, this reduces to calculating ΔE, as given by Eq. (25), as a function of h. When ΔE becomes negative for some value of q, it follows that the one-dimensional equilibrium wall structure becomes unstable with respect to the corrugating mode.

Since the form of the equation describing the equilibrium configuration is different for $h \gtrsim 1$, let us first treat the case $h<1$. Here it is convenient to use δ_0 rather than δ as the unit of length in terms of which the reduced coordinates are defined and we shall assume that this has been done. The pertinent equations then become

$$\Delta E=\frac{A}{\delta_0}\int d\Omega\{(\nabla\alpha)^2+(\nabla\beta)^2 - [1+h^2+3h\cos\phi-2\sin^2\phi]\alpha^2 - [h\cos\phi+\cos^2\phi-\sin^2\phi]\beta^2\}, \tag{59a}$$

$$\tan(\tfrac{1}{2}\phi)=\tan(\tfrac{1}{2}\phi_0)\tanh u, \tag{59b}$$

$$u=\tfrac{1}{2}\zeta\sin\phi_0, \tag{59c}$$

$$h=-\cos\phi_0. \tag{59d}$$

To ensure that $\Delta E \to 0$ as $q \to 0$, we take β proportional to ϕ_z and write

$$\alpha_q(\zeta)=\alpha_c q(1+\cos\phi_0)/(\cosh 2u+\cos\phi_0), \tag{60}$$

where α_c is a constant. Equation (59a) becomes

$$\Delta E=\frac{A}{\delta_0}\left(\frac{(1+\cos\phi_0)^2}{\sin^2\phi_0}\right)I_2\alpha_c^2q^2\left((q^2+1-2h^2)-\frac{hI_1}{I_2}\right). \tag{61}$$

Here,[11]

$$I_n=\int_0^\infty \frac{dx}{(\cosh x+\cos\phi_0)^n}, \tag{62a}$$

$$I_1=\phi_0/\sin\phi_0, \tag{62b}$$

$$I_2=(1-\cos\phi_0 I_1)/\sin^2\phi_0. \tag{62c}$$

Inspecting Eq. (61), we see that, as $q \to 0$, the algebraic sign of ΔE will be the same as that of

$$f(h)=(1-2h^2)-hI_1/I_2. \tag{63}$$

This function is positive at $h=0$, thus, as we would expect, the usual 180° Bloch wall is stable with respect to a corrugating mode with $q \neq 0$. However, as we increase h, we find that

$$f(h)=0 \quad \text{at } h=0.543. \tag{64}$$

That is, the one-dimensional Bloch wall becomes unstable with respect to a corrugating mode of wave vector $q \to 0$ in an applied field no greater than

$$[H_c(q\to 0)]_{max}=0.543H_k. \tag{65}$$

Again taking $4\pi M=10^4G$, $H_k=10$ Oe, we find that the corrugating-mode nucleation field is less than or equal to 5.43 Oe while the buckling-mode nucleation field is greater than 340 Oe.

III. SUMMARY

In Sec. II, we have calculated threshold instability fields for a Bloch-type domain wall. Two possible types of nucleation modes were considered. The first of these, the buckling mode, is characterized by the constraint that the azimuthal angle ϕ remains at its equilibrium value until the instability threshold for the polar angle θ is reached. For such a mode the instability will occur at a field much greater than the effective uniaxial anisotropy field H_k, and the anisotropy energy of the wall is negligible in comparison with other contributions to the free energy. Thus the equilibrium magnetization configuration has a total turn angle of 360°. For this nucleation mode, we have shown that the instability threshold occurs in the region $0.034 \leq H_b/4\pi M \leq 0.149$ and, further, that the instability mode will not be uniform in the plane of the wall. The x-y plane reduced wave vector associated with the buckling mode will be in the range $0 < q \leq 1.55$.

The second or corrugating mode is characterized by the constraint that its magnetostatic self-energy vanishes. For this type of instability we have obtained only an upper bound on the nucleation field. We find that a corrugating-mode instability occurs at a field $h \leq [H_c(q \rightarrow 0)]_{\max}/H_k = 0.543$. Then the overall magnetization turn angle will never be greater than

$$2\phi_0 = 2 \arccos h = 245.8° \qquad (66)$$

For typical parameter values $4\pi M = 10^4$ G, $H_k = 10$ Oe, we see that a buckling-mode instability of a 360° wall will occur in an applied field of between 340 and 1490 Oe. This is considerably less than the uniform-mode nucleation thereshold which we found would occur at $\frac{4}{3}\pi M = 3330$ Oe. However, on further analysis, we have shown that the one-dimensional 360° wall is, in fact, *never* stable. Before we can succeed in "winding up" the wall to 360°, a corrugating-mode instability threshold is certain to be reached, in a field no greater than 5.43 Oe.

In conclusion, we have shown that the linearized equations of micromagnetics can be used to study the nucleation or instability thresholds of nonuniform magnetization configurations. Since, in general, complete analytic solutions of these equations cannot be found, it is necessary to study particular solutions appropriate to well-defined nucleation modes. As we have seen, even these restricted equations cannot, in many cases, be solved. However, they can be used to calculate rigorous upper and lower bounds to the nucleation threshold field of a particular instability mode.

The vital importance of choosing appropriate instability modes for study is made clear by the results we have obtained for the one-dimensional Bloch wall. By constraining the azimuthal angle to remain at its equilibrium value until an instability was reached in the polar angle, a calculated threshold field at least 50 times greater than that found for an alternative nucleation mode characterized by zero magnetostatic self-energy was obtained. This illustrates the dangers involved in introducing constraints into a variational calculation and the need to compare the effect of different constraints on the physical properties being calculated.

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[11]Ref. 9, p. 107, No. 2.443.