Renormalization-group treatment of the random resistor network in $6-\epsilon$ dimensions*

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We consider a hypercubic lattice in which neighboring points are connected by resistances which assume independently the random values $\sigma_>^{-1}$ and $\sigma_<^{-1}$ with respective probabilities p and $1-p$. For $\sigma_<=0$ the lattice is viewed as consisting of irreducible nodes connected by chains of path lenght L. This geometrical length is distinct from the characteristic length L_r which sets a scale of resistance in the random network or L_m which sets a scale of effective exchange in a dilute magnet. Near the percolation concentration p_c one sets $L \sim |p-p_c|^{-\zeta}$, $L_r \sim |p-p_c|^{-\zeta_r}$, and $L_m \sim |p-p_c|^{-\zeta_m}$. Stephen and Grest (SG) have already shown that $\zeta_m = 1+o(\epsilon^2)$ for spatial dimensionality $d=6-\epsilon$. Here we show in a way similar to SG that $\zeta_r = 1+o(\epsilon^2)$. Thus it is possible that $\zeta_m=\zeta_r=1$ for a continuous range of d below 6. However, increasing evidence suggests that this equality does not hold for $d<4$, and in particular a calculation in $1+\epsilon$ dimensions analogous to that of SG for ζ_m does not seem possible.

I. INTRODUCTION

In this paper we report the first field-theoretic formulation for the macroscopic conductivity Σ of a random network of resistors. This formulation enables us to give a renormalization-group treatment of the random network in the critical region near the percolation threshold.

We consider the network formed by connecting pairs of sites x and x' of a d-dimensional hypercubic lattice by resistances $[\sigma(x,x')]^{-1}$. In the model we treat here, $\sigma(x,x')$ vanishes unless x and x' are nearest-neighbor sites, in which case each random variable $\sigma(x,x')$ independently assumes the values $\sigma_>$ and $\sigma_<$ with respective probabilities p and $1-p$. Except for some concluding remarks we consider only the case $\sigma_>=\sigma$, $\sigma_<=0$. We may define clusters to be groups of sites connected with respect to $\sigma_>$. The mean-square cluster size is then the generalized susceptibility of the percolation problem.[1-5] Although field-theoretic renormalization-group treatments of the percolation problem have been given,[5-7] until now no such scheme has been used to calculate Σ. At present the only renormalization-group approaches are those[8,9] based on finite-cluster recursion relations.

This paper is organized as follows. Section II summarizes the current scaling arguments relevant to dilute networks. In particular, we distinguish between different length scales appropriate to different properties of the dilute network. In Sec. III we describe the calculation of the length scale appropriate to the randomly diluted resistor network. The method of calculation is similar to that used by Stephen and Grest[10] for the dilute magnet near the percolation threshold. In Sec. IV we present our conclusions concerning the existence of different length scales and comment on extensions to to other related problems.

II. SUMMARY OF SCALING ARGUMENTS

In this section we review the scaling arguments concerning the behavior of random networks near the percolation threshold at $p=p_c$. In particular, de Gennes[11] and Skal and Shklovskii[12] suggest that for $p>p_c$, the infinite cluster, a section of which is shown in Fig. 1, should be viewed as a superlattice of "nodes" defined as points where there are more than two disjoint conduction paths to infinity. This construction is shown in Fig. 2 where we represent schematically a section of a random array of resistors or bonds with nodes indicated. Two nodes are "adjacent" if there is a path connecting them which does not pass through another node. The length L_{ij} is then defined as being the smallest number of bonds in a path connecting adjacent nodes i and j. A length L may then be defined as being a typical value of L_{ij}. Evidently, the real space distance between the nodes of the superlattice is of order the pair connectedness correlation length ξ, whereas L is greater than or equal to ξ.[12] For spatial dimensionality d greater than 6, the critical behavior of the percolation statistics is mean-field-like, i.e., there are no fluctuation corrections, so that $L \propto \xi^2$ for $p \to p_c^+$.

More generally, one writes $\xi \sim |p-p_c|^{-\nu_p}$ for $p \to p_c$ and $L \sim (p-p_c)^{-\zeta}$ for $p \to p_c^+$. There are several possible regimes for this superlattice. It seems likely that for sufficiently high d, i.e., $d > d_*$, the properties of the random network can be represented as those of a homogeneous superlattice of nodes connected by chains of length L, with no corrections caused by

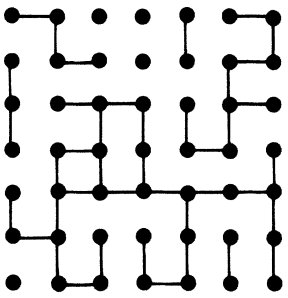

FIG. 1. Section of a lattice with bonds occupied randomly with probability $p = \frac{1}{2}$.

parallel paths. Then the superlattice of Fig. 1 could be represented by that of Fig. 2(a). As we shall discuss below, there is evidence that d_* may be equal to 4. A second regime may occur for $d_c < d < d_*$ when parallel paths become important, so that nodes are connected not by single strands but rather by complicated parallel networks[13] as depicted in Fig. 2(b). In this regime one could distinguish between various values of L. For instance, in addition to the purely geometrical length introduced above, one might also consider the effective resistance between adjacent nodes R_{ij}, whose typical value will be denoted $R_{\rm eff}$. For $d > d_*$, it is clear that $R_{\rm eff} = L/\sigma$, whereas for $d < d_*$, we set $R_{\rm eff} = L_r/\sigma$, where the effective length L_r is less than L due to the occurence of parallel paths. For the dilute Ising model one can likewise introduce an effective exchange interaction between adjacent nodes J_{ij}, whose typical value is $J_{\rm eff}$. For $d > d_*$ one has, by considering only chains, that

$$\tanh(J_{\rm eff}/kT) = [\tanh(J/kT)]^L \ , \tag{1}$$

whereas for $d < d_*$, one would take account of parallel paths by replacing L in this relation by L_m, and again one has $L_m < L$. Finally, for $d < d_c$ it may not be possible to give a local specification of nodes. This regime is shown schematically in Fig. 2(c). General arguments[14] indicate that nodes cannot be defined in two dimensions and therefore, that $d_c = 2$ is the lower critical dimensionality for the random network problem. One can derive expressions for $J_{\rm eff}$ analogous to Eq. (1) for XY, Heisenberg, and higher-component models.

Using the above superlattice picture, the resistance between nodes is of order L_r/σ and the effective lattice constant is ξ, compared to the respective values σ^{-1} and a_0 for the original lattice. Since $\Sigma \propto \sigma a_0^{2-d}$, the above superlattice picture yields the relation $\Sigma \sim (\sigma/L_r)(\xi)^{2-d}$. If we write $\Sigma \sim (p-p_c)^{\mu}$, $p \to p_c^+$, and $L_r \sim (p-p_c)^{-\zeta_r}$, $p \to p_c^+$, then we have[11]

$$\mu = \zeta_r + (d-2)\nu_p \ . \tag{2}$$

Similarly, use of relation (1) enables one to determine $T_c(p)$ for $p \to p_c^+$ by setting $J_{\rm eff}/kT_c(p)$ equal to a constant. Thereby one finds

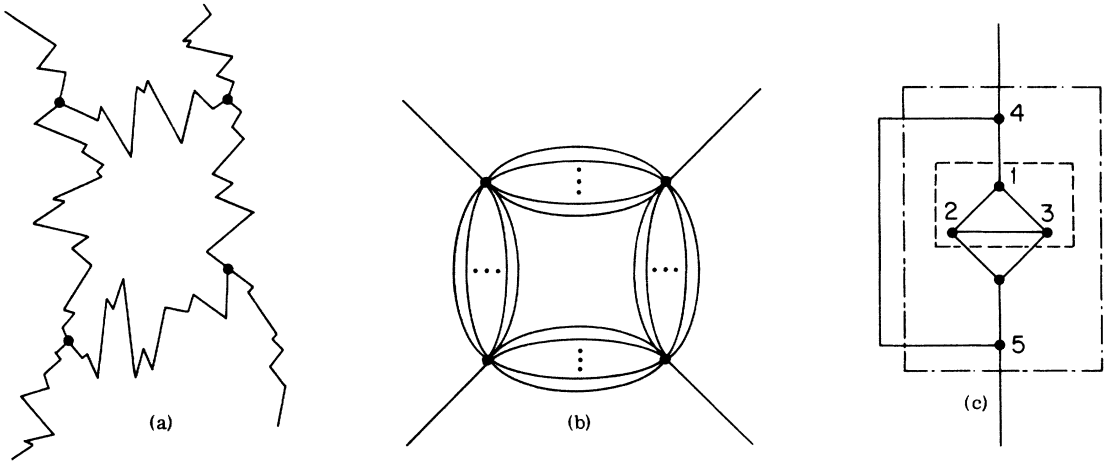

FIG. 2. (a) Schematic representation of a randomly dilute network near the percolation threshold. A path length between nodes (•) is of order L whereas the real space separation between nodes is of order ξ. This picture holds for $d > d_*$ when parallel paths between nodes can be neglected. (b) As in (a) except that this picture holds for $d < d_*$ when parallel paths between nodes cannot be neglected. (c) Here we illustrate the difficulty in defining nodes in two dimensions by the criterion that from a node there are more than two independent paths to "infinity." Considering only the inner rectangle (dashed line) vertices 1, 2, and 3 appear to be nodes. Examination of the larger rectangle (dot-dashed line) shows that 1, 2, and 3 are not nodes but 4 and 5 appear to be nodes. Finally, from consideration of the entire diagram we see that even these are not nodes. For a two-dimensional graph it is believed that this argument can be extended indefinitely (Ref. 14).

$$kT_c(p) = \frac{2J}{\zeta_m |\ln(p - p_c)|}, \quad p \to p_c^+ \ . \tag{3}$$

where ζ_m is defined by $L_m \sim (p - p_c)^{-\zeta_m}$, $p \to p_c^+$. Bergstresser[15] has given the rigorous asymptotic result

$$kT_c(p) \sim 2J/|\ln(p - p_c)|, \quad p \to p_c^+ \ , \tag{4}$$

from which $\zeta_m = 1$ independent of d, at least for $d \geq 2$. de Gennes[11] and Stauffer[16] have suggested that $\zeta_r = 1$ independent of d, but, as discussed below, current numerical evidence suggests that this is not the case for $d < 4$. Levinshtein *et al.*[17] have suggested that $\zeta_r = \nu_p$. This gives somewhat better agreement near $d = 2$, although it is clearly not valid near $d = 6$.

We may deduce upper and lower bounds for L from which we may conclude the existence of the regime where L, L_r, and L_m are not all identical. For example, as mentioned above $L \geq \xi$ or $\zeta \geq \nu_p$. Another bound on ζ can be obtained as follows.[18] We define ν_{SAW} so that the real space length of a self-avoiding walk of length N steps connecting lattice points is of order $N^{\nu_{\mathrm{SAW}}}$ for large N. The real space length of the path between nodes of length L steps is larger than $L^{\nu_{\mathrm{SAW}}}$ because the constraint that the path not intersect itself or any other path is even stronger than the self-avoidance constraint. Hence $\xi > L^{\nu_{\mathrm{SAW}}}$, or $\zeta \leq \nu_p/\nu_{\mathrm{SAW}}$, so that

$$\nu_p \leq \zeta \leq \nu_p/\nu_{\mathrm{SAW}} \ . \tag{5}$$

Since $\nu_p \approx 1.3$ for $d = 2$, we see that ζ cannot be equal to $\zeta_m = 1$. Since both L_r and L_m are less than or equal to L, Eq. (5) implies that both ζ_r and ζ_m are less than or equal to ν_p/ν_{SAW}.

We next review the scaling relations involving L for the randomly dilute resistor network and the randomly dilute magnet. It has previously been suggested[19] that an appropriate "resistive susceptibility" $\chi^{(r)}$ for the randomly dilute network is

$$\chi^{(r)} = \sum_{x'} [g(x,x')]_{\mathrm{av}} \ , \tag{6}$$

where $[\ ]_{\mathrm{av}}$ indicates an average over random configurations and

$$g(x,x') = \begin{cases} R(x,x') & \text{for } C(x,x') = 1, \\ 0 & \text{for } C(x,x') = 0 \end{cases} \ , \tag{7}$$

where $R(x,x')$ is the resistance between lattice points x and x', and $C(x,x')$ equals unity if x and x' are in the same cluster and zero otherwise. If one writes $\chi^{(r)} \sim |p - p_c|^{-\gamma_r}$, $p \to p_c$, then scaling arguments[19] yield

$$\gamma_r = \gamma_p + \mu - (d-2)\nu_p \ ; \tag{8}$$

or, using Eq. (2),

$$\gamma_r = \gamma_p + \zeta_r \ . \tag{9}$$

Lubensky[18] has given scaling relations for dilute magnets based on the above node picture. He argues, following Stauffer,[20] that the point $p = p_c$ and $T = 0$ in a magnet randomly diluted with nonmagnetic impurities (or bonds of zero strength) can be viewed as a multicritical point, and in order to describe crossover behavior in the temperature variable, it is appropriate to compare the one-dimensional spin-correlation length $\xi_1(T)$ with L_m. This argument leads to a scaling form for the spin susceptibility

$$\chi(T) \sim |p - p_c|^{-\gamma_p} f_1[\xi_1(T)/L_m] \ , \tag{10}$$

in agreement with the form proposed by others[21] and with calculations in $6 - \epsilon$ dimensions by Stephen and Grest.[10] In Ref. 18, the possibility that $L_m = L_r$ was considered but rejected because it apparently disagreed with experiments[22] on $RbMn_xMg_{1-x}F_4$. Stephen and Grest[10] have shown that ζ_m may be calculated as the crossover exponent φ' which describes finite temperature deviations from the percolation fixed point at $p = p_c$, $T = 0$. In this regime the free energy $F(T,p)$ is of the form

$$F(T,p) \sim |p - p_c|^{2-\alpha_p} f_2(q\,|p - p_c|^{-\varphi}) \ , \tag{11}$$

where $q = \exp(-2J/kT)$ for the Ising model and $q = kT/J$ for the X-Y or Heisenberg models. Their result is that for $d \to 6^-$, $\varphi' = 1 + o[(6-d)^2]$. Also, for $d \to 1^+$, it was found[10,23] that $\varphi' = 1 + o(d-1)$, where $a = o(x)$ means $a/x \to 0$ as $x \to 0$. Since $\xi_1(T) \sim q^{-1}$, these results suggest that $\zeta_m \equiv \phi' = 1$ independent of $d \geq 1$, as one would expect from Bergstresser's result of Eq. (4) and using the node construction. One of the principal results of this paper is that for $d \to 6^-$, $\zeta_r = 1 + o[(6-d)^2]$, and thus that ζ_r and ζ_m are probably equal for some range of d near 6. This equality probably does not hold for d near 2.

III. RENORMALIZATION-GROUP CALCULATION OF ζ_r

The resistance between two points in a pure resistor network is known[3] to be related to a two-point correlation function of the s-state Potts model in the limit $s \to 0$. We may write the Hamiltonian for the s-state Potts model as

$$\frac{-H^{(s)}}{kT} = J\sigma(s-1) \sum_{\langle x,x' \rangle} [\vec{V}(x) \cdot \vec{V}(x') - 1] \ , \tag{12}$$

where the sum $\langle x,x' \rangle$ is over nearest-neighbor pairs of lattice sites. For each site x, the vector $\vec{V}(x)$ is constrained to point from the center of the $(s-1)$-dimensional simplex to one of its s vertices, so that $\vec{V}(x) \cdot \vec{V}(x') = 1$ if $\vec{V}(x) \| \vec{V}(x')$ and $\vec{V}(x) \cdot \vec{V}(x') = -1/(s-1)$ otherwise. It can then be shown that for large J

$$\lim_{s\to 0}\langle \vec{V}(x)\cdot\vec{V}(x')\rangle = 1-(1/J)R(x,x')+\cdots \text{ (higher order in } 1/J) \ , \tag{13}$$

where $R(x,x')$ is the resistance between sites x and x' in a resistor network in which each nearest neighbor pair of sites is connected by a resistance σ^{-1}. We have used this relation to obtain a Hamiltonian formulation for the random resistor network. For a particular configuration of the resistor network, we consider the following Hamiltonian for the s-state Potts model:

$$\frac{-H^{(s)}}{kT} = J(s-1)\sum_{\langle x,x'\rangle}\sigma(x,x')[\vec{V}(x)\cdot\vec{V}(x')-1] \ . \tag{14}$$

It is clear that the canonical average of $[\vec{V}(x)\cdot\vec{V}(x')]$ with respect to this Hamiltonian will vanish if the sites x and x' are not in the same cluster. Therefore, we have

$$\lim_{s\to 0}\langle \vec{V}(x)\cdot\vec{V}(x')\rangle = C(x,x')-\frac{1}{J}g(x,x'), \quad J\gg 1 \ , \tag{15}$$

where $g(x,x')$ is defined in Eq. (7).

In order to facilitate averaging over the random configurations we consider n identical replicas of the system. This procedure[23] gives an effective Hamiltonian of the form

$$\frac{-H_{\text{eff}}}{kT} = \sum_{\langle x,x'\rangle}\ln\left[1-p+p\exp\left(J\sigma(s-1)\times\sum_{\alpha=1}^{n}(\vec{V}^{\alpha}(x)\cdot\vec{V}^{\alpha}(x')-1)\right)\right] , \tag{16}$$

where the index α labels n replicas. For this effective Hamiltonian it is quite straightforward to show that

$$\lim_{s\to 0}\lim_{n\to 0}\langle \vec{V}^{\alpha}(x)\cdot\vec{V}^{\alpha}(x')\rangle = [C(x,x')]_{\text{av}}-\frac{1}{J}[g(x,x')]_{\text{av}}, \quad J\gg 1 \ , \tag{17}$$

where $[\ ]_{\text{av}}$ denotes an average over random configurations. If we define a susceptibility χ for the effective spin system as

$$\chi = \lim_{s\to 0}\lim_{n\to 0}\sum_{x'}\langle \vec{V}^{\alpha}(x)\cdot\vec{V}^{\alpha}(x')\rangle \ , \tag{18}$$

then it follows from Eqs. (6) and (17) that

$$\chi^{(s)} = -\left.\frac{\partial\chi}{\partial J^{-1}}\right|_{J^{-1}=0} \ . \tag{19}$$

In order to get the right behavior in the $J\to\infty$ limit it is necessary to take the limits in the order indicated in Eq. (17), i.e., $n\to 0$ before $s\to 0$.

We have treated H_{eff} by using a procedure similar to the one used by Stephen and Grest[10] in their study of the dilute Ising model. We introduce a complete set of functions $\mu^{(t)}(x,x')$, $t=1,2,\ldots,n$, defined by

$$\mu^{(t)}(x,x') = \sum_{\alpha_1<\alpha_2<\cdots<\alpha_t}[\vec{V}^{\alpha_1}(x)\cdot\vec{V}^{\alpha_1}(x')]\times[\vec{V}^{\alpha_2}(x)\cdot\vec{V}^{\alpha_2}(x')]\cdots\times[\vec{V}^{\alpha_t}(x)\cdot\vec{V}^{\alpha_t}(x')] \ . \tag{20}$$

In terms of these functions, Eq. (16) can be written as

$$\frac{-H_{\text{eff}}}{kT} = \sum_{\langle x,x'\rangle}\sum_{t=1}^{n}A_t(s-1)^t\mu^{(t)}(x,x') \ , \tag{21}$$

where

$$A_t = \frac{1}{s^n}\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\frac{p}{1-p}\right)^l \times[1+(s-1)e^{-sJ\sigma l}]^{n-t}(1-e^{-sJ\sigma l})^t \ . \tag{22}$$

In the limit $n\to 0$, $s\to 0$, $J\gg 1$, Eq. (22) reduces to

$$A_t = -[\ln(1-p)+Ct/J\sigma] + \cdots \text{ (higher order in } 1/J)\cdots \ , \tag{23}$$

where

$$C = \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l^2}\left(\frac{p}{1-p}\right) \ .$$

If we put $J^{-1}=0$, the effective Hamiltonian becomes

$$\frac{-H_{\text{eff}}}{kT} = -\ln(1-p)\sum_{\langle x,x'\rangle}[s^nP^0(x,x')-1] \ , \tag{24}$$

where

$$P^0(x,x') \equiv \prod_{\alpha}\frac{1}{s}[1+(s-1)\vec{V}^{\alpha}(x)\cdot\vec{V}^{\alpha}(x')]$$

is the projection operator onto the s^n states in which $\vec{V}^{\alpha}(x)\,||\,\vec{V}^{\alpha}(x')$ for all α. The right-hand side of Eq. (24) has the same structure as that of the Hamiltonian of an s^n-state Potts model. Thus, for $J^{-1}=0$, if we let $n\to 0$ before taking the $s\to 0$ limit, H_{eff} reduces to the Hamiltonian of the one state Potts model which describes percolation.[2,5]

We have done an ϵ expansion in $6-\epsilon$ dimensions to study the cirtical behavior of H_{eff}. In order to develop a continuum theory we write the functions $\mu^{(t)}(x,x')$ defined in Eq. (20) as[24]

$$\mu^{(t)}(x,x') = \sum_{\alpha_1<\alpha_2<\cdots<\alpha_t} \sum_{i_1,i_2,\ldots,i_t=1}^{s-1} \sum_{\sigma_1,\sigma_2,\ldots,\sigma_t=1}^{s} \sum_{\sigma_1',\sigma_2',\ldots,\sigma_t'=1}^{s} e_{i_1}^{\sigma_1} e_{i_2}^{\sigma_2} \cdots e_{i_t}^{\sigma_t} P_{\sigma_1}^{\alpha_1}(x) P_{\sigma_2}^{\alpha_2}(x) \cdots P_{\sigma_t}^{\alpha_t}(x)$$

$$\times e_{i_1}^{\sigma_1} e_{i_2}^{\sigma_2} \cdots e_{i_t}^{\sigma_t} P_{\sigma_1}^{\alpha_1}(x') P_{\sigma_2}^{\alpha_2}(x') \cdots P_{\sigma_t}^{\alpha_t}(x') \quad , \tag{25}$$

where $P_\sigma^\alpha(x)=1$ if $\vec{V}^\alpha(x)$ is in the σ direction [i.e., if $\vec{V}^\alpha(x)$ points towards the σ vertex of the $(s-1)$-dimensional simplex] and $P_\sigma^\alpha(x)=0$ otherwise, and e_i^σ is the ith Cartesian component of the vector $\vec{V}$ when it points in the σ direction. Then we have the relations

$$\sum_\sigma e_i^\sigma = 0, \quad \sum_i e_i^\sigma e_i^{\sigma'} = \frac{s\delta_{\sigma\sigma'}-1}{s-1}, \quad \sum_\sigma e_i^\sigma e_j^\sigma = \frac{s}{s-1}\delta_{ij} \quad . \tag{26}$$

Equation (21) can then be written

$$\frac{-H_{\text{eff}}}{kT} = \sum_{\langle x,x'\rangle} \sum_{t=1}^{n} A_t \sum_{\vec{\alpha},\vec{i}} f^{(t)}(x,\vec{\alpha},\vec{i}) f^{(t)}(x',\vec{\alpha},\vec{i}) \quad , \tag{27}$$

where the sum $\sum_{\vec{\alpha},\vec{i}}$ stands for

$$\sum_{1\le\alpha_1<\alpha_2<\cdots<\alpha_t\le n} \sum_{i_1,i_2,\ldots,i_t=1}^{(s-1)}$$

and

$$f^{(t)}(x,\vec{\alpha},\vec{i}) \equiv \sum_{\sigma_1,\sigma_2,\ldots,\sigma_t=1}^{s} (s-1)^{t/2} e_{i_1}^{\sigma_1} e_{i_2}^{\sigma_2} \cdots e_{i_t}^{\sigma_t} P_{\sigma_1}^{\alpha_1}(x) P_{\sigma_2}^{\alpha_2}(x) \cdots P_{\sigma_t}^{\alpha_t}(x) \quad . \tag{28}$$

We now introduce a variable $h^{(t)}(x,\vec{\alpha},\vec{i})$ conjugate to each $f^{(t)}(x,\vec{\alpha},\vec{i})$, write $\exp(-H_{\text{eff}}/kT)$ as a Gaussian integral, and trace over the f's. This procedure yields a continuum version of the effective Hamiltonian which, after Fourier transformation, becomes

$$\frac{H_E}{kT} = \frac{1}{2}\sum_{t=1}^{n}\sum_{\vec{\alpha},\vec{i}}\int_q (r_t+q^2) h^{(t)}(q,\vec{\alpha},\vec{i}) h^{(t)}(-q,\vec{\alpha},\vec{i}) - \frac{1}{6}u \sum_{t,t',t''=1}^{n} {\sum_{\vec{\alpha},\vec{\alpha}',\vec{\alpha}''}}' \sum_{\vec{i},\vec{i}',\vec{i}''} \int_{q_1}\int_{q_2} F(\vec{\alpha},\vec{\alpha}',\vec{\alpha}'',\vec{i},\vec{i}',\vec{i}'') h^{(t)}(q_1,\vec{\alpha},\vec{i})$$

$$\times h^{(t')}(q_2,\vec{\alpha}',\vec{i}') h^{(t'')}(-q_1-q_2,\vec{\alpha}'',\vec{i}'') \quad , \tag{29}$$

where $\int_q$ stands for

$$\frac{1}{(2\pi)^d}\int_{0<|q|<\lambda} d^dq \quad ,$$

$$r_t \sim (1-zA_t) \quad , \tag{30}$$

where z is the number of nearest neighbors and in the sum ${\sum_{\vec{\alpha},\vec{\alpha}',\vec{\alpha}''}}'$ each replica $1,2,\ldots,n$ appears twice, three times or not at all. Thus, the three sets of indices $\vec{\alpha}$, $\vec{\alpha}$ *prime*, and $\vec{\alpha}''$ are, in general, of the form

$$\vec{\alpha} = (\alpha_{a1},\alpha_{a2},\ldots,\alpha_{an_1},\alpha_{b1},\alpha_{b2},\ldots,\alpha_{bn_2},\alpha_{c1},\alpha_{c2},\ldots,\alpha_{cn_3}) \quad ,$$

$$\vec{\alpha}' = (\alpha_{a1},\alpha_{a2},\ldots,\alpha_{an_1},\alpha_{b1},\alpha_{b2},\ldots,\alpha_{bn_2},\alpha_{d1},\alpha_{d2},\ldots,\alpha_{dn_4}) \quad ,$$

$$\vec{\alpha}'' = (\alpha_{a1},\alpha_{a2},\ldots,\alpha_{an_1},\alpha_{c1},\alpha_{c2},\ldots,\alpha_{cn_3},\alpha_{d1},\alpha_{d2},\ldots,\alpha_{dn_4}) \quad ,$$

in which case, the coefficient $F(\vec{\alpha},\vec{\alpha}',\vec{\alpha}'',\vec{i},\vec{i}',\vec{i}'')$ is given by

$$F(\vec{\alpha},\vec{\alpha}',\vec{\alpha}'',\vec{i},\vec{i}',\vec{i}'') = \prod_{j=1}^{n_1}\left(\frac{(s-1)^{3/2}}{s}\sum_\sigma e_{i_{aj}}^\sigma e_{i'_{aj}}^\sigma e_{i''_{aj}}^\sigma\right)\prod_{k=1}^{n_2}\delta_{i_{bk}i'_{bk}}\prod_{l=1}^{n_3}\delta_{i_{cl}i''_{cl}}\prod_{m=1}^{n_4}\delta_{i'_{dm}i''_{dm}} \quad . \tag{31}$$

The recursion relations can now be obtained in the standard way.[25] To first order in ϵ, the recursion relations for the r_l's are

$$r_l' = b^{2-\eta}\left[r_l - \tfrac{1}{2}\bar{u}^2(s^n-2)\lambda^2(1-b^{-2}) - 2\bar{u}^2 \ln b\, r_l + 2\bar{u}^2 \ln b \sum_{l=1}^{n}(s-1)^l \binom{n}{l} r_l\right] , \quad (32)$$

where b is the length rescaling factor, $\bar{u}^2 = \frac{1}{2}K_6 u^2 = u^2/2^7\pi^3$, and $\eta(\bar{u}^2) = \frac{1}{3}\bar{u}^2(s^n-2)$. The recursion relation for u is identical to that for the cubic potential[6] in the s^n-state Potts model and the fixed-point value of $\bar{u}^2$ is

$$\bar{u}^{*2} = \epsilon/(10-3s^n) . \quad (33)$$

For $s=2$, these recursion relations reduce to those obtained by Stephen and Grest[10] for the dilute Ising model.

It can be seen from Eqs. (22) and (30) that for $n=0$,

$$\sum_{l=1}^{n}(s-1)^l \binom{n}{l} r_l = 0 .$$

Thus, in the limit $n \to 0$, Eq. (32) reduces to

$$r_l' = b^{2+\epsilon/21}[r_l + \tfrac{1}{14}\epsilon\lambda^2(1-b^{-2}) - \tfrac{2}{7}\epsilon \ln b\, r_l] , \quad (34)$$

which can be written as

$$r_l' - r^* = b^{1/\nu_p}(r_l - r^*) , \quad (35)$$

where $r^* = -\frac{1}{14}\epsilon\lambda^2$ and $\nu_p = \frac{1}{2} + \frac{5}{84}\epsilon$ is the known result[6] for the percolation exponent. If we now take the limits $s \to 0$, $J \gg 1$, then from Eqs. (23) and (30), we have

$$r_l \sim 1 + z\ln(1-p) + \frac{Czt}{J\sigma} \sim \frac{p_c^{mf}-p}{1-p_c^{mf}} + \frac{Ct}{J\sigma} \quad (36)$$

for p close to $p_c^{mf} = 1-\exp(-1/z)$. Thus, for $J \gg 1$ and p close to p_c, $(r_l - r^*)$ can be written as

$$r_l - r^* = r + tw , \quad (37)$$

where $r \propto (p_c - p)$ and $w \propto (J\sigma)^{-1}$. It then follows from Eq. (35) that

$$r' = b^{1/\nu_p} r, \quad w' = b^{\varphi/\nu_p} w , \quad (38)$$

with $\varphi = 1$. Thus, the percolation fixed point $p = p_c$, $(J\sigma)^{-1} = 0$ is a multicritical point with scaling fields $|p-p_c|$ and $(J\sigma)^{-1}$, and the crossover exponent φ that describes deviations from this fixed point as $(J\sigma)^{-1}$ is turned on is equal to one. We have found that in the $n \to 0$ limit, the recursion relations for the r_l's remain degenerate to second order in ϵ. Thus,

$$\varphi = 1 + o(\epsilon^2) . \quad (39)$$

The susceptibility χ has the following scaling form in the disordered phase near $p = p_c$:

$$\chi \sim |p-p_c|^{-\gamma_p} f_3[(J\sigma)^{-1}|p-p_c|^{-\varphi}] . \quad (40)$$

This, together with Eq. (19), yields

$$\gamma_r = \gamma_p + \varphi , \quad (41)$$

or, from Eqs. (9) and (39),

$$\zeta_r = \gamma_r - \gamma_p = 1 + o(\epsilon^2) . \quad (42)$$

This verifies that $\zeta_r = \zeta_m = 1 + o(\epsilon^2)$ near six dimensions.

IV. DISCUSSION AND CONCLUSIONS

Stephen and Grest[10] and Kirkpatrick[23] were able to calculate ζ_m for the Ising model in $1+\epsilon$ dimensions using the Migdal-Kadanoff[26] bond-moving scheme. This calculation is possible because both the one-state Potts model and the Ising model order in all dimensions above one. Thus, percolation and thermal variables appear on the same footing near one dimension. The pure-zero-state Potts model, on the other hand, orders only at $T = 0 (J = \infty)$ for all dimensions below 2. Its critical properties can be calculated in $2+\epsilon$ dimensions[27] using the Migdal-Kadanoff scheme. Thus the phase diagram for the diluted-zero-state Potts model is qualitatively different above and below two dimensions. This means, in particular, that the crossover exponent φ for turning on J^{-1} cannot be calculated in $1+\epsilon$ dimensions. It, therefore, seems very likely that ζ_r is not equal to ζ_m for $d \leq 2$. It also appears possible that ζ_r and ζ_m may differ for $2 < d < 4$. This view seems to be corroborated by Monte Carlo[28] and series[29] calculations, which indicate that $\zeta_r > 1$, whereas the calculations of Stephen and Grest[10], Kirkpatrick[23] and Bergstresser[15] suggest that $\zeta_m = 1$.

The more general problem of a random resistor network composed of bonds with conductance $\sigma_>$ present with probability p and bonds with conductance $\sigma_<$ present with probability $1-p$ is of some interest. If $\sigma_> = \infty$, the macroscopic conductivity $\Sigma(p)$ diverges as $(p_c - p)^{-\bar{\mu}}$ as $p \to p_c^-$. Straley[30] and Harris and Fisch[19] conjectured that $\Sigma(p)$ and $\chi^{(r)}(p)$ should obey scaling laws in which $\sigma_</\sigma_>$ appears as a crossover variable. For example, it was argued that $\chi^{(r)}$ should be proportional to

$$\sigma_>^{-1}|p-p_c|^{-\gamma_r} G[(\sigma_</\sigma_>)|p-p_c|^{-\varphi_<}] ,$$

where $G(0) = 1$. From this, it follows that $\bar{\mu} = \varphi_< - \zeta_r$. We now consider this problem in our formalism. Turning on $\sigma_<$ corresponds to turning on bonds with strength $J\sigma_<$ on the formerly unoccupied bonds of the zero-state Potts model. One can easily see that this adds a term $J\sigma_<$ to A_1 only [Eq. (22)] and changes σ to $(\sigma_> - \sigma_<)$ wherever σ appears. Thus, one can show that χ is of the form

$$\chi \sim |p-p_c|^{-\gamma_p} g(J\sigma_< |p-p_c|^{-1}, [J(\sigma_> - \sigma_<)]^{-1}|p-p_c|^{-1}) \quad (43)$$

where $g(0,0)=1$ in the disordered phase near $J\sigma_<=0$, $1/J\sigma_>=0$, and $p=p_c$. If $\sigma_< \neq 0$, there are no disconnected clusters and there is long-range order with every site in the same state at $J=\infty$. Thus, for any $\sigma_> > 0$, there is a critical line separating the ordered and disordered phases. The above mentioned scaling form for χ implies that the equation for the critical line satisfies

$$J^{-1} = J_c^{-1} = \sigma_> (p-p_c) f[\sigma_</\sigma_> (p-p_c)^{-2}] \ ,$$

where $f(0) \sim 1$. Note that this implies $J_c^{-1} \sim (\sigma_< \sigma_>)^{1/2}$ at $p=p_c$. Hence $J_c\sigma_< \sim (\sigma_</\sigma_>)^{1/2}$ and $(J_c\sigma_>)^{-1} \sim (\sigma_</\sigma_>)^{1/2}$ are both much less than unity if $\sigma_< \ll \sigma_>$, and there is a portion of the disordered region for which the above mentioned scaling form applies consistently. Unfortunately, to calculate Σ or $\chi^{(r)}$, we need $\partial\chi/\partial J^{-1}$ evaluated not at the critical line but at $J^{-1}=0$. Thus, these quantities are controlled by the properties near the completely ordered state when $\sigma_< \neq 0$ and not by the properties of the multicritical point $p=p_c$, $J\sigma_<=0$, $J\sigma_>=\infty$. It is, therefore, not evident to us that $\chi^{(r)}$ and Σ satisfy simple scaling relations in $\sigma_</\sigma_>$.

We conclude with some remarks regarding the properties of randomly diluted magnets. First consider the zero temperature ($T=0$) case. The spin-wave stiffness D for a randomly diluted ferromagnet is closely related to the macroscopic conductivity Σ, viz.,[31]

$$D \propto \Sigma/M \ , \quad (44)$$

where M is the magnetization taken to be the fraction of sites in the infinite cluster. Since $M(p) \sim (p-p_c)^{\beta_p}$ for $p \to p_c^+$, we expect $D(p-p_c, T=0)$ to be proportional to $(p-p_c)^{\mu-\beta_p}$. For a randomly diluted antiferromagnet, the spin-wave velocity is proportional to[31] $(\Sigma/\chi_\perp)^{1/2}$, where $\chi_\perp$ is the response to a uniform magnetic field perpendicular to the direction of order. $\chi_\perp$ is expected to diverge as $|p-p_c|^{-\tau}$ as $p \to p_c$ because of uncompensated spin clusters. Heuristic arguments[28] relate τ to β_p and ζ_r: $\tau = \zeta_r - \beta_p$. It would be interesting to extract these results from a field theory of the sort presented here.

At nonzero temperature, the behavior of D and $\chi_\perp$ is different. For $T \neq 0$, the dilute magnet is in the same universality class with the weakly random magnet and the arguments[32] and analysis[33-36] for the critical exponents in the presence of arbitrarily weak random perturbations should be applicable. At present, we cannot calculate the functions describing the crossover between percolative behavior at $T=0$ and weakly random behavior for $T \approx T_c(p) \neq 0$.

Note added in proof. Recently D. J. Wallace and A. P. Young (unpublished) have proven that $\varphi=1$ to all orders in the third-order coupling constant.

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