Vortices and the low-temperature structure of the *x-y* model

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(Received 23 May 1977)

An exact duality transformation is applied to the partition function Z for the *x-y* model in two and three dimensions. The fields which appear in the dual representation of Z are integer valued and represent the topological excitations, or vortices, of the *x-y* model. Furthermore, this form of the partition function is particularly simple at low temperatures. In two dimensions, the dual representation of Z at low temperatures describes a two-dimensional Coulomb gas in which the point charges are vortices. In three dimensions, the dual form of Z describes a locally invariant gauge theory, analogous to QED, and coupled to integer-value, conserved currents which represent the line vortices of the three-dimensional *x-y* model. Qualitative comments about the low-temperature behavior of the theories are made. The meaning of vortices on a lattice is also discussed.

I. INTRODUCTION

Topological excitations are certain stable or metastable configurations of fields which exist because the system in question has a compact symmetry. These excitations which in general represent rather complicated field (or spin) configurations can have a profound effect on the behavior of the system. Because the relevant field configurations are so complex, ordinary perturbation theory (or the high-temperature expansion) is not useful for describing these excitations.

Recently, a new method was proposed for dealing with these objects.[1-3] The method is applicable to a class of theories in d dimensions which possess a U(1) symmetry and consists of transforming the usual partition function of the theory into a new form using a duality transformation. The fields which appear in this new form are integer-valued fields which directly represent the topological excitations of the theory. Moreover, this new form is particularly simple at low temperatures (or, in the language of field theory, small coupling constant).

The simplest member of the class of theories considered in Ref. 1 is the *x-y* model. In this paper we will apply our methods to the two- and three-dimensional *x-y* models. The extension to higher dimensions will be clear.[1,3] According to the usual homotopy arguments,[4,5] the globally U(1)-invariant theory in d dimensions should have topologically stable excitations of dimension $d-2$. In two dimensions, therefore, we expect the *x-y* model to have point vortices, while when $d=3$ we expect line vortices. (For $d=1$, localized topological singularities are not important at low temperatures, although smoothly varying spin configurations which go through several revolutions from one end of the lattice to the other are. Formally, we can consider this an excitation of dimension -1.) After applying our transformation to these theories we will obtain expressions for the partition functions in terms of the vortex degrees of freedom. This exercise will also demonstrate in detail the steps involved in the general duality transformation described in Refs. 1–3.

For the $d=2$ *x-y* model, we will be able to write the partition function in terms of point vortices. The low-temperature limit of our expression will coincide with the low-temperature approximations to Z derived by other authors,[2,3,5,6] and is just the partition function of a neutral Coulomb gas in two dimensions.

In three dimensions, we will show that the *x-y* model is equivalent to a locally invariant gauge theory for which the symmetry group is Z_∞, the additive group of integers. At low temperatures, this partition function becomes identical to the generating functional for photons coupled to integer-valued conserved currents in three Euclidean dimensions. These conserved currents represent the vortex-line filaments (actually, vortex rings) which appear in this theory.

In Sec. II, we will present the duality transformation for the two-dimensional *x-y* model. We will also discuss in what sense the integer-valued fields of our dual representation may be regarded as vortices. This point requires some elaboration. The physical degrees of freedom of the *x-y* model are the spins, $s_i = e^{i\theta_i}$. Physically, we associate a vortex with a spin configuration in which the spins rotate through 2π some nonzero number of times as we move along a contour which surrounds the vortex. To describe this vorticity mathematically, it is common to say that the phase integral $\int d\theta(x)$ (or, on a lattice, a phase sum $\sum \Delta\theta_i$ around the closed path) is nonzero, which means that θ_i is multivalued. (Of course, the physical degrees of freedom, s_i, are single valued.) However, in the usual formulation of the x-y model, (2.1), this criterion is difficult to implement since the range of θ is only $-\pi$ to π. Moreover, performing the duality transformation, one obtains integer-valued fields only after integrating over all spin configurations, and so it is difficult to uniquely identify a given configuration of dual integer-valued fields with a configuration of spins. It is possible, however, to define another model[5] whose low-temperature behavior is the same as that of the x-y model. The vortex excitations of this model can be simply described in terms of a multivalued angle variable, τ, and can be related in a unique way to configurations of the spins, $e^{i\tau}$. Our integer-valued, dual fields will be put in a one-to-one correspondence with the vortices of this model, and in this sense can be regarded as the vortices of the x-y model. In Sec. III we apply our duality transformation to the three-dimensional x-y model. We discuss problems of gauge invariance and demonstrate that the currents of the gauge theory represent the line vortices of the model. Conclusions and a summary are presented in Sec. IV.

II. TWO-DIMENSIONAL x-y MODEL

A. Partition function at low temperatures

We now present the details of our duality transformation for the two-dimensional x-y model. This operation will result in an expression for the partition function which is simple at low temperatures. Some of the steps involved are well known and have been described by others in different contexts, but are repeated here for the sake of completeness. In the Sec. II B we will discuss the interpretation of the integer-valued fields that will appear in our final expression.

The usual form of the x-y partition function is

$$Z=\int_{-\pi}^{\pi}\prod_k \frac{d\theta_k}{2\pi}\exp\left[\beta\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)\right] , \qquad (2.1)$$

where the limits on the θ integrations follow from the fact that Z includes a sum over all (physical) configurations of the spins, $U_k=\exp(i\theta_k)$. The sum in the exponent runs over all nearest-neighbor pairs on a two-dimensional square lattice. Using the character expansion for the interaction,

$$\exp[\beta\cos(\theta_i-\theta_j)]=\sum_{n=-\infty}^{\infty} I_n(\beta)\exp[in(\theta_i-\theta_j)] , \qquad (2.2)$$

(2.1) can be rewritten

$$Z=\sum_{\{n\}}\prod_{i,\mu} I_{n_{\mu;j}}(\beta) \times\int_{-\pi}^{\pi}\prod_k\frac{d\theta_k}{2\pi}\exp\left[i\sum_j\theta_j(\vec{\Delta}\cdot\vec{n}_j)\right] . \qquad (2.3)$$

In this expression, we are instructed to sum over a set of integer-valued fields, $\{n\}$, one for each link of the lattice. Since the links can be labeled by a site i and a direction μ, the $n_{i,\mu}$ can be thought of as a collection of two-vectors, $\vec{n}_i$. The coefficient of θ_j in the exponent is the discrete divergence of $\vec{n}$ at the site j. Specifically,

$$\vec{\Delta}\cdot\vec{n}_{\vec{j}}=n_{\vec{j},\hat{x}}-n_{\vec{j}-\hat{x},\hat{x}}+n_{\vec{j},\hat{y}}-n_{\vec{j}-\hat{y},\hat{y}} , \qquad (2.4)$$

where we recall that the site labels are two-dimensional vectors. (We will usually omit the vector symbol from site labels for notational convenience.) Since the fields, n, are integer valued, integrating over the θ_j in (2.3) will just give us a set of Kronecker δ functions which enforce the condition

$$\vec{\Delta}\cdot\vec{n}_j=0 \qquad (2.5)$$

at all sites j. This condition is automatically satisfied if we write

$$n_{\mu,j}=\epsilon_{\mu\nu}\Delta_\nu\phi_j , \qquad (2.6)$$

where $\{\phi_j\}$ is a set of integer-valued fields located at the sites of the dual lattice. For a d-dimensional hypercubical lattice, the dual lattice can be obtained by shifting the original lattice by half a lattice spacing in each direction. (See Fig. 1.) Using (2.6) in (2.3) (and neglecting overall constants), we have

$$Z=\sum_{\{\phi\}}\prod_{j,\mu} I_{\epsilon_{\mu\nu}\Delta_\nu\phi_j}(\beta) =\sum_{\{\phi\}}\exp\left[\sum_{j,\mu}\ln[I_{\epsilon_{\mu\nu}\Delta_\nu\phi_j}(\beta)]\right] . \qquad (2.7)$$

Now, for integer n, $I_n(\beta)$ has the representation

$$I_n(\beta)=\frac{1}{\pi}\int_0^{\pi}d\omega\, e^{\beta\cos\omega}\cos n\omega . \qquad (2.8)$$

We insert this representation in (2.7) and expand $\cos n\omega$ in powers of n. The partition function can then be written

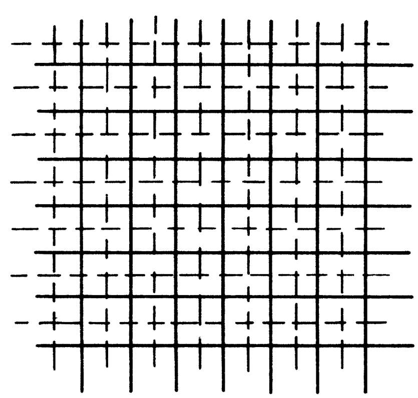

FIG. 1. Original square lattice and its dual in two dimensions.

$$Z=\sum_{\{\phi\}}\exp\left[\sum_{j,\mu}\sum_{p=1}^{\infty}\frac{D_p(\beta)}{p!}(\Delta_\mu\phi_j)^{2p}\right] , \tag{2.9}$$

where we have dropped an overall multiplicative factor of $I_0^{2N}(\beta)$, N being the number of sites of the lattice. $D_p(\beta)$ is the pth cumulant of a set of functions, $T_p(\beta)$ defined as

$$T_p(\beta)=(-1)^p\frac{p!}{(2p)!}\frac{1}{\pi I_0(\beta)}\int_0^\pi d\omega\,\omega^{2p}e^{\beta\cos\omega} . \tag{2.10}$$

So, for example,

$$D_1=T_1 ,$$
$$D_2=T_2-T_1^2 , \tag{2.11}$$
$$D_3=T_3-3T_1T_2+2T_1^3,\ \text{etc.}$$

Using the identity

$$\sum_{k=-\infty}^{\infty}\delta(\phi-k)=\sum_{m=-\infty}^{\infty}e^{i2\pi m\phi} \tag{2.12}$$

we can rewrite (2.9) as

$$Z=\int\delta\phi\sum_{\{m\}}\exp\left[\sum_{j,\mu,p}\frac{D_p(\beta)}{p!}(\Delta_\mu\phi_j)^{2p}+i2\pi m_j\phi_j\right] , \tag{2.13}$$

where now the ϕ_j's are treated as continuum fields, $-\infty<\phi_j<\infty$.

Now, let us estimate the behavior of the $D_p(\beta)$. For very small β, the $T_p(\beta)$ have the form

$$T_p(\beta)\simeq(-1)^p p!\left[\frac{\pi^{2p}}{(2p+1)!}+\frac{\beta}{\pi(2p)!}\times\int_0^\pi d\omega\,\omega^{2p}\cos\omega+\cdots\right] \tag{2.14}$$

so that even though the expansion in (2.13) converges for all real $\Delta\phi$, all $D_p(\beta)$ will have contributions which are of low order in β and so cannot be neglected. consider now large β. We want to show that for $\beta>>1$, Z is well approximated by keeping only the $p=1$ term in the expansion (2.9) or (2.13). To do this we first examine the coefficients $D_p(\beta)$ for $\beta>>1$ and $\beta>>p$. In this limit, the behavior of the $T_p(\beta)$ can be calculated using a saddle-point estimate of the integral in (2.10), or, more conveniently, we can replace one power of ω in the integrand by $\sin\omega$ and do integration by parts. This procedure will pick up the leading contribution in β^{-1}, since for large β the major contribution to the integral comes from small ω. Repeated use of this trick will let us determine higher-order corrections. Doing this, we find

$$T_p(\beta)=(-1)^p\frac{p!}{(2p)!}\frac{(2p-1)!!}{\beta^p}+O(\beta^{-p-1});\quad \beta>>1,p . \tag{2.15}$$

Using (2.15), we see that $D_1(\beta)=-1/2\beta$. Furthermore, in the calculation of $D_p(\beta)$ for $p>1$, there are cancellations among terms of low order in β^{-1}, so that, for example, $D_2(\beta)\sim O(1/\beta^3)$. Now, the expansion in the exponent of (2.9) [that is, the expansion of $\ln I_n(\beta)$ in powers of n] is certainly convergent for all real $(\Delta\phi)^2$. Moreover, if $\beta>>1$, and $(\Delta\phi)^2\lesssim\beta$, we make errors only of order β^{-1} in the exponent of (2.9) by keeping just the $p=1$ term. Thus to show that keeping only the term with $p=1$ is in fact a good large β approximation to Z, we need to show that configurations of $\{\phi\}$ such that $(\Delta\phi)^2>\beta$ make a negligible contribution to Z. This is not completely obvious since the series in the exponent (2.9) alternates in sign, and so the $p=1$ term which becomes large and negative when $(\Delta\phi)^2>\beta$ could be canceled by terms of higher order in p. That this does not happen can easily be shown using a uniform asymptotic form for $I_n(\beta)$ valid as n and β both go to infinity.[7] The quadratic term in (2.9) is therefore a good low-temperature approximation to Z. A systematic expansion about this approximation is possible by retaining higher and higher terms in p [and more and more accurate approximations for the $D_p(\beta)$]. Such an expansion is related to (but is not exactly) a perturbation series in powers of T. It will be sufficient for our purposes to keep only the term with $p=1$, which we shall do the rest of this paper. It is important to remember, however, that each term in the exponent of (2.9) (or in the analogous expression in higher dimensions) has the same symmetry properties, and so we will never introduce artificial symmetries by retaining only the quadratic term.

For $\beta>>1$, we can therefore write (2.13) as

$$Z=\int\delta\phi\sum_{\{m\}}\exp\left[\sum_{\mu,j}-\frac{1}{2\beta}(\Delta_\mu\phi_j)^2+i2\pi m_j\phi_j\right] , \tag{2.16}$$

where we have used the approximation of (2.15) for $T_1(\beta)$. The functional integral over ϕ can be done by introducing the representations

$$\phi_{\vec{j}}=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}d^2q\,e^{-i\vec{q}\cdot\vec{j}}\eta(\vec{q}) , \tag{2.17a}$$

$$m_{\vec{j}}=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}d^2q\,e^{-i\vec{q}\cdot\vec{j}}l(\vec{q}) , \tag{2.17b}$$

where the q's are vectors in the first Brillouin zone. If we have periodic boundary conditions, this representation will diagonalize the Hamiltonian. Inserting (2.17) in (2.16) and summing over the lattice sites, we can write

$$Z=\sum_{\{m\}}\int \delta\eta(q)\exp\left[\int_{-\pi}^{\pi}d^2q\left[-K(q)|\eta(q)|^2+\frac{i}{2\pi}l^*(q)\eta(q)\right]\right] \tag{2.18}$$

where

$$K(q)\equiv\frac{-T_1(\beta)}{\pi^2}\Gamma(q)=\frac{-T_1(\beta)}{\pi^2}\left[1-\tfrac{1}{2}\sum_\mu\cos\vec{q}\cdot\hat{\mu}\right]\simeq\frac{1}{2\beta\pi^2}\left[1-\tfrac{1}{2}\sum_\mu\cos\vec{q}\cdot\hat{\mu}\right] \tag{2.19}$$

and where we have dropped an inessential Jacobean and have used the fact that since ϕ_i and m_i are real, $\eta(-q)=\eta^*(q)$, and $l(-q)=l^*(q)$. The Gaussian integrals over $\eta(q)$ can now be carried out with the result

$$Z=\sum_{\{m\}}\exp\left[-\int_{-\pi}^{\pi}d^2q\,\tfrac{1}{2}\ln\left[\frac{2K(q)}{\pi}\right]+\frac{|l(q)^2|}{16K(q)\pi^2}\right] . \tag{2.20}$$

Inverting the transformation (2.17b), we have

$$l(\vec{q})=\sum_{\vec{j}}m_{\vec{j}}e^{i\vec{q}\cdot\vec{j}} , \tag{2.21}$$

which we insert in (2.20). Integrating over $\vec{q}$, and dropping an overall constant, we finally have

$$Z=Z^{(0)}\sum_{\{m\}}\exp\left[\frac{1}{16T_1(p)}\sum_{i,j}m_iV_{ij}m_j\right] , \tag{2.22}$$

where

$$Z^{(0)}=\exp\left[-\int_{-\pi}^{\pi}d^2q\,\tfrac{1}{2}\ln\left[\frac{2K(q)}{\pi}\right]\right]$$

and

$$V_{ij}=\int_{-\pi}^{\pi}d^2q\,\frac{e^{i\vec{q}\cdot(\vec{i}-\vec{j})}}{\Gamma(q)} .$$

$Z^{(0)}$ is the partition function of free, noninteracting spin waves in two dimensions and, as we shall see describes the x-y model in the absence of vortices. The second factor in (2.22) is just the partition function of a neutral Coulomb gas in two dimensions. This can be seen as follows: There is a divergence in V_{ij} which can be removed by writing

$$V_{ij}=\int_{-\pi}^{\pi}\frac{d^2q}{\Gamma(q)}-U_{ij} , \tag{2.23}$$

$$U_{ij}=\int_{-\pi}^{\pi}\frac{d^2q\,1-e^{i\vec{q}\cdot(\vec{i}-\vec{j})}}{\Gamma(q)} . \tag{2.24}$$

The first term in (2.23) is divergent. Its coefficient in the Hamiltonian is proportional to $-(\sum_i m_i)^2$. Hence, the total net charge of the system must be zero.[8] For $i=j$, $U_{ij}=0$. For $i\neq j$, U_{ij} is well approximated[9] by

$$U_{ij}=8\pi(\ln|i-j|+\tfrac{1}{2}\ln 8+\gamma)$$
$$\equiv 8\pi(\ln|i-j|+c) , \tag{2.25}$$

γ being Euler's constant. Using (2.25) and (2.15) we can display the low-temperature form of (2.22):

$$Z=Z^{(0)}\sum_{\{m\}}\exp\left[\frac{-1}{16T_1(\beta)}\sum_{i,j}m_iU_{ij}m_j\right]\simeq Z^{(0)}\sum_{\{m\}}{}'\exp\left[\pi\beta\sum_{i\neq j}m_i\ln|i-j|m_j-\pi\beta c\sum_i m_i^2\right] , \tag{2.26}$$

where the sum over $\{m\}$ is understood to include only those configurations satisfying $\sum_i m_i=0$. We therefore have the partition function of a neutral gas of integer charges interacting through a logarithmic potential. Note also that there is a chemical potential which one must overcome to excite the m's.

B. Interpretation of the m fields

The expression (2.26) looks the same as the approximate low-temperature form of the partition function for the two-dimensional x-y model derived by Berezinski, Kosterlitz, and Thouless, Polyakov, and others.[5,6] In their works, the fields $\{m\}$ have been interpreted as vortex excitations, but strictly speaking, (2.26) has a somewhat different interpretation.

To understand the difference, we need to briefly summarize what other authors have done.[10] Although the details of the analyses differ from author to author, the general approach has been to construct a compact, (really, periodic) quadratic theory which at low temperatures agrees with the small-T behavior of the x-y model. The first step is to approximate $\cos(\theta_i-\theta_j)$ by its quadratic term. Next, the range of integration over θ is allowed to extend from $-\infty$ to ∞. One then essentially solves the equation of motion[8] $\nabla^2\theta=0$, but remembers that since θ is really an angle, discontinuities of $2\pi n$ are allowed in the solution since this discontinuity in θ will still correspond to a mostly smooth configuration of spins. The classical solution is then

$$\theta_c(r)=\sum_k q_k\,\mathrm{Im}\,[\ln(r-r_k)] , \tag{2.27}$$

where the $\{q_k\}$ is a set of integers. One then writes $\theta=\theta_c+\psi$, and inserts this into the quadratic Lagrangian. The result is a partition function which is equal to (2.26). ψ, the perturbation about the classical solution, gives rise to the factor $Z^{(0)}$ and represents a noninteracting spin wave, or free boson field (on the lattice). In this approach the integers q_k are naturally interpretable as vortices at the positions r_k. That is, a small contour integral $\oint d\theta_c$ about the point r_k has the value $2\pi q_k$. Moreover, in this formulation, the vortices emerge as pieces of the original angles θ, namely the classical parts.

In our approach, on the contrary, the discrete fields ϕ_i are a complete set of fields replacing the angles which have disappeared from the problem. These fields are then represented in terms of the m fields and the continuum ϕ's, just as in the approach outlined above θ was represented by $\theta_c+\psi$. Furthermore, in our formalism, the m's cannot be interpreted directly as vortices, if by a vortex on the lattice we mean some spin configuration in which the spin rotates through some multiple of 2π around a closed contour. This difficulty has two sources: First, in performing the duality transformation we have summed over all spin configurations, and the spin degrees of freedom have been replaced by another complete set of variables which include the m fields. It is therefore difficult to associate a configuration of m's unambiguously with a configuration of spins. Second, since in the expression (2.1) for the *x-y* model, $-\pi<\theta\leq\pi$, the phase sum $\sum\Delta\theta_i$ will be zero around any closed path, and so this quantity cannot be used as a measure of vorticity. One might suppose that if we extend the range of integration $-\pi L<\theta\leq\pi L$ and let $L\to\infty$, we will be able to identify the m's as vortices. One way to implement this would be to let $\theta_i\to\theta_i+2\pi p_i$ in the original partition function and sum over the $\{p_i\}$ as well as integrating over the $\{\theta_i\}$ from $-\pi$ to π. If we do this, Eq. (2.9) will read

$$Z=\frac{1}{L^N}\sum_{\{\phi\}}\sum_{\{V\}}\exp\left(\sum_{l,\mu}\sum_{p=1}^{\infty}\frac{D_p(\beta)}{p!}(\Delta_\mu\phi_l)^{2p}+i2\pi V_l\phi_l\right)\,, \tag{2.28}$$

where

$$V_j=\sum\Delta p$$

around the plaquette surrounding the jth site of the dual lattice, and the limit $L\to\infty$ should be taken. However, this is the same as (2.9) since $\exp(i2\pi\phi_l V_l)=1$, so in this formulation the partition function is independent of the "vorticity," V_j.

To see this another way, keep the last term in (2.28) to a later stage in the calculation. Then we see that the effect of this term is simply to redefine $m_l\to m_l+V_l$ in (2.13), but we still sum over both m_l and V_l separately, so m_l cannot be identified with any particular vortex.

The problem, of course, is that we have been too careful in retaining the periodic structure of the Hamiltonian. Once we have integrated over the principal part of the angle in (2.28) we will have integrated over all *spin* configurations of the system. Summing over the set $\{V\}$ merely repeats the process, but does not give any new contributions to Z. To put it another way, the variable θ which appears in the *x-y* model [or in the paragraph preceding (2.28)] does not have the same meaning as the variable θ associated with the compact quadratic theory.[5,6] The final low-temperature form of Z derived by both methods is the same, but in the compact quadratic approach leading to (2.27) one is able to associate a unique vortex distribution with each distribution of θ's (or spins), whereas that is not possible in the formalism leading to (2.28).

Nevertheless, even though our m fields cannot, strictly speaking, be identified as vortices, it is clear that they do play the role of vortices in the partition function. The connection between our m fields and the *bona fide* vortices of the compact quadratic theory can be made somewhat clearer by the following considerations: Look at Eq. (2.28) and set all the $V_l=0$ and for simplicity keep only the $p=1$ term. The discreteness of the sum over ϕ is a reflection of the periodicity of the original Hamiltonian. If we make the substitution

$$\frac{1}{L^N}\sum_{\{\phi\}}\to\int\delta\phi\,, \tag{2.29}$$

then, with $V_l=0$ (2.28) will just be a functional Gaussian integral which, when transformed back to the θ variables is seen to be just the partition function with only the quadratic term of the cosine interaction and with $-\infty<\theta<\infty$. This nonperiodic Gaussian form has no vortices—that is, only smoothly varying configurations of θ_l have a significant weight in the partition function. If we make the substitution (2.29) in the full expression (2.28), then (2.28) will be the same as (2.13) with the m's replaced by V's. On the other hand, remembering that we must sum over $\{V\}$, the substitution (2.29) does not change the numerical value of (2.28). All it does is reshuffle the configurations of spins among the configurations of V's, so that it is possible to assign a definite distribution of V's to a given spin configuration. The substitution (2.29) is essential to this reshuffling since it ensures that in the absence of vortices there is no periodicity, and that only smooth configurations of θ's are important.

To summarize, our understanding of vortices is as follows: To talk sensibly about vortices one needs: (i) criterion for measuring vorticity, and (ii) a method of unambiguously associating a vortex distribution with a given spin configuration. A common criterion for measuring vorticity, namely the value of a phase integral, is not directly applicable to the *x-y* model defined by (2.1). In addition our duality transformation does not allow us to uniquely associate a distribution of m fields with a given configuration of x-y spins. On the other hand, these objections do not apply to the compact quadratic theory as treated in Refs. 5 and 6. Now, at very low temperatures, a configuration of spins in the compact quadratic model will have essentially the same energy as the same configuration of spins in the x-y model. There is therefore a one-to-one correspondence between the spin configurations in the two models. Since vortices can be identified in the compact quadratic model, the corresponding spin configurations of the x-y model can be assigned the same vortex distribution. However, the partition function of the compact quadratic model written in terms of the vortex degrees of freedom is the same as (2.26), and so our m fields can be thought of as representing the vortices of the x-y model.

One might wonder what effect the fact that the lattice is spatially discrete plays in this discussion. After all, homotopy arguments concern continuous maps which are not possible on a lattice. For the purpose of this discussion, the continuity can be though of as a criterion for associating a vortex distribution with a spin configuration. However, at low temperatures, at least in the compact quadratic model, there is approximate continuity for the spin configurations, since only those configurations in which the spins vary smoothly from site to site will be important. Hence, it is compactness of the symmetry, and not spatial continuity which is important for the existence of vortexlike excitations. (Of course, the continuum limit in the sense of the renormalization group could *a priori* have different physics, but that is another question.)

We have made some effort to explain in what sense our m fields are vortices. However, this nomenclatural discussion notwithstanding, it is important to remember that in terms of the original spin variables, the low-temperature partition function of the x-y model is still dominated by vortexlike spin configurations as described by other authors.[5,6] Bearing all this in mind, in what follows we shall sometimes gloss over these distinctions and simply refer to our integer-valued fields as vortex excitations.

It is worthwhile stressing one other difference between the compact quadratic approach and our duality transformation. Some previous discussions[5,6] of the x-y model involved at some point an explicit sum over an infinite number of minima of the Hamiltonian, all of which had the same energy. This was accomplished either explicitly in the partition function, or by solving the classical equations of motion as described above. Using this procedure, one might *a priori* be concerned with problems of overcounting in the partition function when one calculates very high orders in perturbation theory about these vacuums. Such a problem does not arise in our approach. The duality transformation leading to (2.13) is exact and so there is no possibility of overcounting. We view this as a significant conceptual advantage that might be of real value in more complicated problems.[11]

Finally, we mention that since our analysis yields a low-T partition function which is to a first approximation (i.e., keeping only the $p=1$ term in 2.13) identical to that derived by other methods, our approach predicts the same qualitative features for the low-temperature behavior of this model as have been previously discussed. This is clearly true when describing the theory in terms of its vortex excitations. Furthermore, although the formalism is somewhat different, the expressions for the spin-spin correlation functions calculated by our method (which involves the duality transformation) will be the same as those computed using more traditional methods.

III. THREE-DIMENSIONAL x-y MODEL

We turn now to a discussion of the three-dimensional x-y model. In this case application of the duality transformation shows that the $d=3$ x-y model is equivalent to a locally gauge-invariant theory which resembles QED. In fact, the low-T limit of the partition function is the same as the generating functional for photons which are coupled to conserved, integer-valued currents in three Euclidean dimensions. These currents are the vortex lines of the three dimensional x-y model (in the same sense in which the m fields of Sec. II were the vortex points of the $d=2$ x-y model.)

As before, we begin with the usual expression for the $d=3$ x-y partition function

$$Z=\int_{-\pi}^{\pi}\prod_k \frac{d\theta_k}{2\pi}\exp\left[\beta\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)\right] , \tag{3.1}$$

where now the spins $U_k=\exp(i\theta_k)$ are associated with the vortices of a three-dimensional cubical lattice, and the sum in the exponent runs over all nearest-neighbor pairs. Proceeding as before, we use (2.2) to write (3.1) in a form analogous to (2.3). Integrating over the angles θ_k results in a product of kronecker δ-function constraints which enforce the condition

$$\vec{\Delta}\cdot\vec{n}_j=0 \tag{3.2}$$

at each site j, only now $\vec{n}$ is a three-vector (since there are three links pointing in positive directions associated with each site) and $\vec{\Delta}$ in (3.2) is the three-divergence. To satisfy (3.2) it is necessary and sufficient to write $\vec{n}$ in the form

$$n_{\mu,j}=\epsilon_{\mu\nu\lambda}\Delta_\nu A_{\lambda;j} . \tag{3.3}$$

The $A_{\lambda;j}$ are a set of integer-valued vector fields which are naturally associated with the links of the dual lattice. Referring to Fig. 2, we see that there is a one-to-one correspondence between links of the original lattice and plaquettes (elementary faces) of the dual

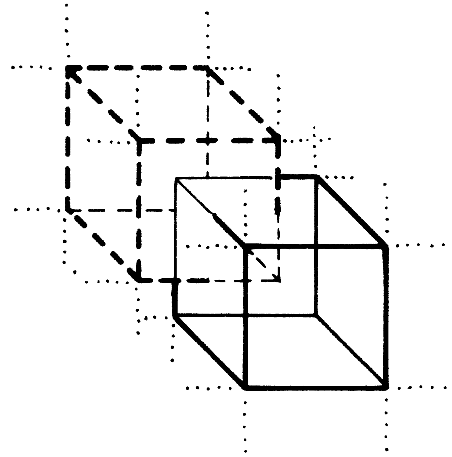

FIG. 2. Simple cubic lattice in three dimensions and its dual lattice.

lattice. Four fields, $A_{\lambda;l}$, are coupled together according (3.3) to yield a value of a given $n_{\mu,l}$. Each of these $A_{\lambda,l}$ is associated with one of the links of the dual lattice which borders the dual-lattice plaquette through which the original lattice link associated with the given $n_{\mu,l}$ passes.

Using (3.3), (3.1) can be written

$$Z=\sum_{\{n\}}\exp\left[\sum_{l,\mu}\ln[I_{\epsilon_{\mu\nu\lambda}\Delta_\nu A_{\lambda;l}}(\beta)]\right] . \qquad (3.4)$$

The sum in the exponent runs over sites and directions of the dual lattice. The sum over $\{n\}$ is understood to be a sum over all distinct configurations of $n_{\mu,l}$ which can be obtained from a set of $A_{\lambda,l}$ via (3.3). Putting it another way, one chooses a set of $A_{\lambda,l}$, then forms from them the set $\{n_{\mu,l}\}$ and calculates a contribution to (3.4). Moving on to another set $\{A_{\lambda,l}\}$ one repeats the process, but one must be careful to avoid counting distinct sets of $A_{\lambda,l}$ which give rise to the same set of integers $\{n_{\mu,l}\}$. Sets of A's related by

$$A_{\lambda,l}\rightarrow A_{\lambda,l}+\Delta_\lambda\rho_l , \qquad (3.5)$$

where ρ_l is an integer-valued field, will give through (3.3) the same set of $n_{\mu,l}$. This problem is just the usual problem associated with defining a functional integral (in our case, a functional sum) in a theory with a local gauge symmetry, expressed here through (3.5). On the dual lattice, the symmetry (3.5) can be visualized as follows: associate a number with each dual-lattice site j. Add that number to each $A_{\lambda,j-\hat{\lambda}}$ and subtract it from each $A_{\lambda,l}$ ($\lambda=1,2,3$). This operation leaves the $n_{\mu,l}$ invariant. Since this transformation can be carried out at each dual-lattice site independently, the symmetry is local.

The symmetry (3.5) is quite similar to the symmetry of the photon field in QED, only in our case the vector potentials and gauge functions take on only integer values. (Actually, an overall, nonintegral constant can also be added, but this is trivial.) In fact, (3.3) shows that the $n_{\mu,l}$ can actually be thought of as $F_{\nu\lambda,l}$, or integer-valued E and B fields in three space-time dimensions.

We will return later to a discussion of this gauge symmetry and how one can define the functional sum in Z, but first we wish to extract the leading low-T piece of (3.4). This is easily done in a manner analogous to the treatment of the two-dimensional x-y model. We use (2.8) to expand the exponent of (3.4) in a power series in $n_{\mu,l}$. Doing this we have

$$Z=\sum_{\{n\}}\exp\left[\sum_{l,\mu}\sum_{p=1}^{\infty}\frac{D_p(\beta)}{p!}(\epsilon_{\mu\nu\lambda}\Delta_\nu A_{\lambda,l})^{2p}\right] \qquad (3.6)$$

with $D_p(\beta)$ given by (2.11) and (2.10). The analysis of the coefficients proceeds as in Sec. II, and so it is a good low-T approximation to keep only the term with $p=1$. Doing this we can write

$$Z\cong\sum_{\{F_{\mu\nu}\}}\exp\left[\frac{-1}{2\beta}\sum_{l,\mu,\nu}(\Delta_\mu A_{\nu,l}-\Delta_\nu A_{\mu,l})^2\right];\quad \beta>>1 , \qquad (3.7)$$

where

$$F_{\nu\lambda,l}\equiv\Delta_\nu A_{\lambda;l}-\Delta_\lambda A_{\nu;l}=n_{\mu,l} \qquad (3.8)$$

according to (3.3). (Note that the indices on n refer to sites and directions on the original lattice, while the indices on F and A refer to sites and directions of the dual lattice.)

Equation (3.7) looks exactly like the generating functional for free photons in three-space time dimensions except that the fields $A_{\lambda,l}$ are restricted to take on only integer values. It is interesting to compare (3.6) or (3.7) with the corresponding expression in the $d=2$ case, (2.9). Keeping only the $p=1$ term in (2.9), we see that we have a discrete two-dimensional Gaussian model which would describe a free massless spin-zero boson if the field ϕ took on continuous values. In both cases it is the discreteness of the fields that gives rise to nontrivial topological structures.

To understand (3.7) [or (3.6), since the symmetry properties are the same] we need to rewrite Z as we did in Sec. II by using the identity (2.12). Here, however, we must be somewhat more careful because of the local gauge symmetry. First we present a heuristic derivation of the final form for Z. This will be followed by a more careful discussion in which Z will be defined by choosing a gauge.

The heuristic argument proceeds as follows: In (3.6) or (3.7) we need to sum over a set of variables $\{n_{\mu,l}\}$. Each $n_{\mu,l}$ can take on integer values from $-\infty$ to ∞. We use (2.12) to write

$$\sum_{\{n\}} \to \int \delta \bar{n} \sum_{\{K\}} \exp\left[i2\pi \sum_{j,\mu} K_{\mu;j} n_{\mu;j} \right] . \tag{3.9}$$

Using (3.9) in (3.7), and making use of (3.3), we have

$$Z = \sum_{\{K\}} \int \delta(\epsilon_{\mu\nu\lambda}\Delta_\nu A_{\lambda;j}) \exp\left[\sum_{j;\mu} \frac{-1}{2\beta} (\epsilon_{\mu\nu\lambda}\Delta_\nu A_{\lambda;j})^2 + i2\pi K_{\mu;j} \epsilon_{\mu\nu\lambda}\Delta_\nu A_{\lambda;j} \right] . \tag{3.10}$$

We sum the last term in the exponent by parts and rewrite (3.10) (dropping Jacobeans)

$$Z = \sum_{\{J\}}{}' \int{}' \delta(A_{\mu;j}) \exp\left[\sum_{j,\mu,\nu} \frac{-1}{2\beta} F_{\mu\nu;j} F_{\mu\nu;j} + i2\pi J_{\mu;j} A_{\mu;j} \right] , \tag{3.11}$$

where

$$J_{\mu;j} = -\epsilon_{\mu\nu\lambda}\Delta_\nu K_{\lambda;j} . \tag{3.12}$$

The prime on the functional integral indicates that, as usual, a gauge condition must be imposed in order to define the functional integral. The prime on the sum over J indicates that not any arbitrary set of integers, $\{J\}$ is allowed; in particular, only those J's satisfying the representation (3.12) will appear. This immediately implies that

$$\Delta_\mu J_{\mu;j} = 0 , \tag{3.13}$$

so the currents are conserved. This is heartening since it assures us that every allowed configuration of J's will give a gauge-invariant contribution to Z. Finally, choosing a gauge and performing the functional integral we have

$$Z = Z^{(0)} \sum_{\{J\}}{}' \exp\left[\pi\beta \sum_{\substack{\mu,\nu \\ j,k}} J_{\mu;j} D_{\mu\nu;jk} J_{\nu;k} \right] , \tag{3.14}$$

where $D_{\mu\nu;jk}$ is the three-dimensional photon propagator (on a lattice), and $Z^{(0)}$ represents the contribution to Z from the sourceless $J=0$ sector. The currents $J_{\mu;j}$ which are associated with the links of the dual lattice represent the vortex lines of the three-dimensional x-y model. Before elaborating this point, let us examine the steps from (3.7) to (3.14) more carefully.

This sequence of steps constitutes a good example of proof by notation. The sleight of hand happens in the substitution (3.9). The point is that the variables $n_{\mu;j}$ are not all independent, so it is not precisely clear what is being done in (3.9). To properly carry through the derivation of (3.14), one should identify a complete set of independent variables which will be summed over and then perform the substitution (3.9) on those variables. This can be done by choosing a gauge in the sum (3.6) or (3.7).

The gauge we will consider is an axial gauge defined as follows: First we choose ρ_j in (3.5) so that all the $A_{\lambda;j}=0$ for $\lambda=1$. Referring to Fig. 3, which is a picture of the dual lattice, this corresponds to setting all the $A_{\lambda;j}$ associated with vertical links of the dual lattice to zero. This does not yet completely specify the gauge, however. We can still add a piece to $\rho_j, \bar{\rho}_j$ which satisfies $\Delta_1\bar{\rho}_j=0$ without affecting the condition that $A_{1;j}=0$. We do this by choosing $\bar{\rho}_j$ such that $A_{2;j}=0$ for all j which lie on the bottom plane of the lattice. Finally, we have one last gauge choice to make since we are free to add to $\rho_j+\bar{\rho}_j$ another piece $\tilde{\rho}_j$ such that $\Delta_1\tilde{\rho}_j=\Delta_2\tilde{\rho}_j=0$. This piece we choose by requiring that $A_{3;j}=0$ for all j which lie along the bottom front edge of the lattice as indicated in Fig. 3. We therefore have

$$A_{1;j}=0, \quad \text{all } j , \tag{3.15a}$$

$$A_{2;j}=0, \quad j=(-\infty, j_2, j_3) , \tag{3.15b}$$

$$A_{3;j}=0, \quad j=(-\infty,-\infty,j_3) . \tag{3.15c}$$

Up to a trivial overall constant, this completely specifies a gauge: the remaining $A_{\mu;j}$ are a complete set of variables which independently take on integer values from $-\infty$ to $-\infty$. We can now write (3.7) [or (3.6)] in the form

$$Z = \sum_{[A]} \exp\left[\frac{-1}{2\beta} \sum_{j,\mu,\nu} (\Delta_\mu A_{\nu;j} - \Delta_\nu A_{\mu;j})^2 \right] , \tag{3.16}$$

where the set $[A]$ includes all those variables not fixed by the gauge (3.15).

Now, for each member of the set $[A]$, we can use the identity (2.12) to write (3.16) in the form

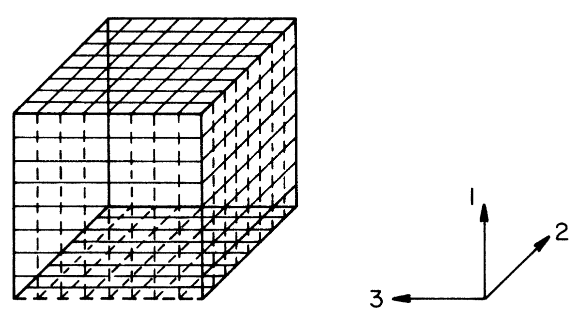

FIG. 3. Representation of the gauge choice (3.15) on the dual lattice. The dashed lines indicate those links along which the field $A_{\mu;j}=0$.

$$Z=\sum_{[H]}\int \delta[A]\exp\left[\sum_{l,\mu,\nu}\frac{-1}{2\beta}(\Delta_\mu A_{\nu,l}-\Delta_\nu A_{\mu,l})^2+i2\pi H_{\mu,l}A_{\mu,l}\right], \tag{3.17}$$

where there is one $H_{\mu;l}$ for each of the fields $A_{\mu;l}$ not fixed by the conditions (3.15). The functional integral over A is understood to be carried out in the gauge specified by (3.15). Note that condition (3.15a) is sufficient to define the inverse of the quadratic form in (3.17), but all conditions (3.15) are necessary to completely specify the gauge.

Carrying out the integral over the A's, (3.17) can be written in the form

$$Z=Z^{(0)}\sum_{[H]}\exp\left[\pi\beta\sum_{\substack{\mu,\nu\\ l,k}}H_{\mu,l}D_{\mu,\nu;lk}H_{\nu;k}\right]. \tag{3.18}$$

D is the three-dimensional photon propagator in the gauge (3.15), and $Z^{(0)}$ is the partition function with all $H_{\mu,l}=0$—that is, the partition function for the free-photon field.

How are we to interpret the fields H? To answer this, note that there is a one-to-one correspondence between the terms in the sum over H in (3.18) and the terms in the sum over J in (3.14). To each configuration of H's (which in general do *not* satisfy the condition $\vec{\Delta}\cdot\vec{H}=0$) we can construct a configuration of J's satisfying $\vec{\Delta}\cdot\vec{J}$ such that the contributions to (3.18) and (3.14) match. This is done as follows: Consider a set of fields $[h]$ complementary to the set $[H]$. The set $[h]$ consists of integer-valued fields, $h_{\mu;l}$, which are associated with the dual-lattice links along which $A_{\mu;l}=0$ according to the gauge choice (3.15). For any configuration of fields H, we can construct in a unique way a configuration of closed contours by filling in the configuration of H fields with h fields. In other words, any line segment corresponding to an allowed configuration of H fields can be closed by tracing lines along the links defined in (3.15). A simple example of this is indicated in Fig. 4. By choosing the h fields to have the correct strength, this closed contour will represent a divergenceless configuration of a current, $G_{\mu,l}=H_{\mu,l}+h_{\mu,l}$. In the gauge (3.15)

$$\sum_{\substack{\mu,\nu\\ l,k}}H_{\mu;l}D_{\mu\nu;lk}H_{\nu;k}=\sum_{\substack{\mu,\nu\\ l,k}}G_{\mu,l}D_{\mu\nu;lk}G_{\nu;k} \tag{3.19}$$

for any configuration of H fields and its attendant configuration of G fields. Since $\vec{\Delta}\cdot\vec{G}=0$ a configuration of G's will give a gauge-invariant contribution to Z. These G's can be identified with the J's of (3.14), so that the H's are just a representation of the J's in a particular gauge. Note that in order to construct the divergenceless current from an arbitrary configuration of H fields, one must recognize that *all* the conditions (3.15) are necessary to completely specify a gauge. Only by fixing all the conditions (3.15) will the set of fields $[h]$ be large enough to uniquely construct closed loops from any configuration of H fields.

We now will demonstrate that the J's represent vortex lines in three dimensions in the same senses in which the m fields of Sec. II represent vortex points in two dimensions. The simplest way to proceed is to first construct an approximate, periodic form of the model, valid at low temperatures, in terms of the spin degrees of freedom. The representation for the case $d=2$ used by Berezinski, Villain, and others[5] is applicable here too and we approximate (3.1) as

$$Z=\int_{-\pi}^{\pi}\prod_k\frac{d\theta_k}{2\pi}\exp\left[\beta\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)\right]$$
$$\simeq\int_{-\infty}^{\infty}\prod_k d\theta_k\sum_{\{l\}}\exp\left[\frac{-\beta}{2}\sum_{i,\mu}(\Delta_\mu\theta_i+2\pi l_{\mu;i})^2\right], \tag{3.20}$$

where, for this discussion we are ignoring overall (infinite) constants. The sum over $\{l\}$ is a sum over a set of integers, one for each link of the lattice, and ensures that the effective Hamiltonian is periodic, i.e., invariant under the operation $\theta_k\to\theta_k+2\pi q_k$, for q_k an integer. The range of the θ integration has been extended from $-\infty$ to ∞. For large β this makes a negligibly small effect in Z. This form for Z demonstrates how one may view the periodicity as arising from a sum over an infinite set of inequivalent minima of the Hamiltonian in θ space.

The argument of (3.20) may be written in terms of its Fourier components. The partition function then becomes

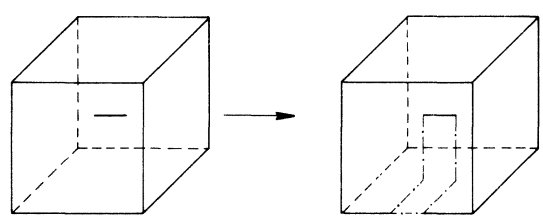

FIG. 4. Construction of a conserved current from the gauge-dependent sources, H. The solid line segment in the space on the left represents a configuration of currents, $H_{\mu,l}$ associated with the links of the dual lattice in the gauge (3.15). The dash-dotted lines in the lattice on the right represent the complementary currents, $h_{\mu,l}$ which lie along the dashed links of Fig. 3 and uniquely complete a closed path.

$$Z \simeq \int_{-\infty}^{\infty} \prod_k d\theta_k \sum_{\{l\}} \int_{-\infty}^{\infty} \prod dt_{\mu;j} \exp\left[\sum_{j,\mu} \frac{-1}{2\beta} t_{\mu;j}^2 + i(\Delta_\mu \theta_j + 2\pi l_{\mu,j}) t_{\mu;j}\right]$$

$$= \sum_{\{l\}} \int_{-\infty}^{\infty} \prod dt_{\mu;j}\, \delta(\Delta_\mu t_{\mu;j}) \exp\left[\sum_{\mu;j} \frac{-1}{2\beta} t_{\mu,j}^2 + i 2\pi l_{\mu,j} t_{\mu;j}\right] , \tag{3.21}$$

where the Dirac δ functions come from the θ integrals, and where we have again neglected overall constants. As before, we can enforce the δ-function constraints by writing

$$t_{\mu,j} = \epsilon_{\mu\nu\lambda} \Delta_\nu A_{\lambda;j} , \tag{3.22}$$

where the $A_{\lambda,j}$ are associated with the links of the dual lattice and take on continuous values from $-\infty$ to ∞. Inserting (3.22) into (3.21), we obtain an expression of the form (3.11) where the $J_{\mu;j}$ are now

$$J_{\mu;j} = -\epsilon_{\mu\nu\lambda} \Delta_\nu l_{\lambda,j} . \tag{3.23}$$

However, from (3.21), $l_{\lambda;j}$ just tells us, loosely speaking, how many revolutions θ has gone through as we move from the site $\vec{j}$ to the site $\vec{j} - \hat{\lambda}$. $J_{\mu;j}$ therefore, tells us how many revolutions θ goes through as we move around an elementary plaquette of the original lattice, and so represents the vorticity.

In accordance with the discussion in Sec. II, this demonstration is, in a sense, heuristic. In Eq. (3.20) we have surreptitiously changed the meaning of the variable θ. As in the two-dimensional case, we cannot associate a configuration of J's uniquely with spin configurations of the theory defined by (3.1). Nevertheless, the J's which exist because of the periodicity of (3.1) faithfully embody all the physics associated with the vortices of the compact quadratic theory.

A determination of the detailed low-T behavior of this system requires more work, but from the form of (3.14) we can make some qualitative statements. At large distances, $D_{\mu\nu;jk} \sim 1/|j-k|$; thus, to produce a vortex ring of linear dimensions $\sim r$ requires an energy $\propto r$. At very low temperatures, Z should be dominated by spin configuration without vortices. As T increases, vortex excitation becomes more likely with increasingly more vortices of increasingly larger size being produced. Finally, it is possible that some sort of phase transition occurs signalled by the appearance of states which are dense with vortices of arbitrarily large size. That there is some simple relationship (not necessarily an equivalence) between such a transition and the standard Wilson-Fisher fixed point is an intriguing possibility, but it is far from obvious. These questions require further study for which the representation (3.14) provides a useful starting point.

IV. COMMENTS

The duality transformation which we have used to recast the x-y model has yielded some very interesting insights. First, we recall that the transformation itself is exact: the low-T approximations which result in the simple forms (2.16) and (3.7) were made on the dual theory after the transfromation. Because the transformation was exact, we did not encounter the conceptual difficulties which exist when using the approach of some previous work which involves summing over inequivalent minima of the Hamiltonian to enforce periodicity. Our transformation involved replacing the complete set of angle variables by a complete set of conjugate variables which are the discrete ϕ or A fields and can be represented by the smooth spin waves plus the vortices. Moreover, since the low-T approximations were made on the dual form of Z, corrections to (2.16) and (3.7) are well defined.

A very interesting feature of the three-dimensional x-y model is that it is equivalent to an Abelian gauge theory with the structure of QED. Since the x-y model is thought to describe superfluid helium, the vortices in the superfluid should be described by the currents in (3.11). This is therefore an example of a system with "extended" excitations which obeys a gauge principle—and a very familiar and important one at that. In any case, this theory is a good one to study further the intriguing equivalence between locally and globally invariant formulations of the same system.[12]

Finally, we mention that since the dual forms of Z are fairly simple at low temperatures, the vortex fields should provide a good basis for investigating low-T properties of these systems. In particular, one might try using (2.16) and (3.7) as the basis for for a renormalization-group calculation. Such an exercise should reveal if there is a phase transition associated with the topological excitations and may help answer the question of its relation (if any) to the usual Wilson-Fisher critical point.[13]

ACKNOWLEDGMENTS

I would like to thank H. Abarbanel, W. Bardeen, L. Brown, M. Peskin, R. Pearson, and M. Roth for stimulating discussions. I am also indebted to M. Peskin for pointing out a mistake in one of my calculations. In addition, I am particularly grateful to J. Lieberman for many useful comments and for discussions about the existence of vortices on a lattice. Finally, I thank the members of the Physics Department of the University of Washington for their hospitality while part of this work was being done.

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[4]See, for example, G. Toulouse and M. Kléman, J. Phys. Lett. (France) 37, p. L149–51 (1976).

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[8]For the purist, this can be done in a more well-defined way by inserting a term $J\phi_j^2$ in (2.16) and in the end letting J approach zero. Neutrality will emerge in the $J=0$ limit.

[9]See F. Spitzer, *Principles of Random Walk* (Van Nostrand, Princeton, 1964) pp. 148–151.

[10]What follows is a description of the treatment presented by Polyakov (Ref. 5). Other authors, for example, Berezinskii or Villain (Ref. 5) sum over a set of inequivalent minima of the quadratic Hamiltonian to enforce periodicity, as discussed below. In either case, the fields in (2.26) are interpreted as vortices. Note that this qualitative description does not apply to Refs. 1–3, which use the duality transformation.

[11]Among the cases we have in mind is the problem of non-Abelian gauge theories in high-energy physics. The suggestion from this analysis is that after summing over all the pseudoparticle (instanton) solutions in the continuum field theory, one will be left with an effective generating functional which resembles the one obtained from the lattice gauge theory. [The original formulation of the lattice gauge theory is in K. Wilson, Phys. Rev. D 10, 2445 (1974).] Note that on the lattice, color confinement is manifest, at least for large coupling.

[12]The $d=3$ Ising model can also be shown to be equivalent to a three-dimensional gauge theory, in this case a Z_2 gauge theory. F. Wegner, J. Math. Phys. 12, 2259 (1971), see also R. Balian, J. Drouffe, and C. Itzykson, Phys. Rev. D 11, 2098 (1975).

[13]Renormalization groups for the $d=2$ x-y model are discussed in Refs. 2 and 6.