PHYSICAL REVIEW B VOLUME 17, NUMBER 3 1 FEBRUARY 1978

Temperature dependence of quasiparticle sound velocity in a Bose gas

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(Received 3 January 1977)

The low-temperature dependence of quasiparticle sound velocity in a nonideal Bose gas is investigated. Besides a $T^4 \ln T^{-1}$ increase with temperature — formally identical to that obtained by Khalatnikov and Andreyev for He II by the kinetic-equation approach — a T^4 decrease of the sound velocity is found. Assuming the theoretical expressions to remain formally valid for superfluid helium, comparison is made with experimental results on the variation of sound velocity with temperature.

I. INTRODUCTION

Near the absolute zero of temperature, collisions between elementary excitations in superfluid helium become too infrequent for hydrodynamics to provide a valid description of kinetic phenomena. Khalatnikov and Andreyev[1] (KA) have consequently used a collisionless kinetic equation for the excitations together with an equation for the superfluid velocity V_s to calculate the effect of temperature on the velocity of sound in liquid helium. They found a $T^4 \ln T^{-1}$ increase in sound velocity with temperature, the prediction being in semiquantitative agreement with experiment.[2]

The question we wish to consider is whether the result obtained by KA can be shown to exist in the microscopic theory of a Bose fluid. It has been known for some time[3,4] that, at zero temperature, the macroscopic sound velocity in a Bose fluid is identical to the quasiparticle sound velocity defined through the long-wavelength poles of the single-particle Green's function. Although at finite temperatures no such exact result is known, approximations such as the Bogolubov weak-interaction approximation[5] or the Bogolubov-Hartree-Fock pair approximation[6] predict a decrease in the phonon sound velocity with temperature. The purpose of this paper is to show that while an exact calculation of the Green's functions of a Bose liquid at nonzero temperatures is not possible, a consistent weak-interaction calculation of the long-wavelength quasiparticles of a Bose gas gives formally the result derived by KA. In addition, one obtains a T^4 decrease in the sound velocity with temperature.

An outline of the paper is as follows: In Sec. II a few basic results concerning the temperature-dependent Green's functions are recalled. The inconsistency of the Bogolubov approximation as regards the temperature-dependent terms is pointed out in Sec. III, and a consistent approximation for the self-energy parts is introduced. Section IV contains the calculation of the temperature-dependent part of the quasiparticle sound velocity. Although the results are derived explicitly for a weakly interacting Bose gas, we assume them to remain formally valid for liquid helium and compare them with the experimental results in Sec. V.

II. TEMPERATURE GREEN'S FUNCTIONS

For an assembly of bosons of mass m, enclosed in a volume V and interacting through a two-body potential $v(r)$, the Hamiltonian (in units such that $\hbar = m = 1$) is

$$H = \sum_{K} \frac{1}{2} K^2 a_K^\dagger a_K + \frac{1}{4V} \sum_{K_1,\ldots,K_3} [v(K_1 - K_3) + v(K_2 - K_4)] \times a_{K_1}^\dagger a_{K_2}^\dagger a_{K_3} a_{K_1+K_2-K_3} , \tag{1}$$

where $a_K, a_K^\dagger$ denote the creation and annihilation operators for the single-particle state of momentum K and $v(K)$ is the Fourier transform of $v(r)$.

The thermodynamic potential of the system at temperature T is given by

$$\Omega = -(1/\beta) \ln(\mathrm{Tr}\, e^{-\beta H'}) , \tag{2}$$

$$H' = H - \mu N , \tag{3}$$

$$N = \sum_K a_K^\dagger a_K , \tag{4}$$

where μ denotes the chemical potential and $\beta = (k_B T)^{-1}$, k_B denoting the Boltzmann constant.

For studying states of the system characterized by the presence of a condensate in the zero-momentum single-particle state, one can, following Bogolubov,[5] replace $(a_0/V)^{1/2}$ and its conjugate by $n_0^{1/2}$, where n_0

denotes the density of condensed particles. The Hamiltonian H' can then be written

$$H' = \frac{1}{2} n_0^2 v_0 V - \mu n_0 V + H_2 + H_3 + H_4 - \mu N' \ , \tag{5}$$

$$\begin{aligned} H_2 = & \sum_K [\tfrac{1}{2}K^2 + n_0 v_0 + n_0 v(K)] a_K^\dagger a_K \\ & + \frac{1}{2} \sum_K n_0 v(K) (a_K^\dagger a_{-K}^\dagger + a_{-K} a_K) \ , \end{aligned} \tag{6}$$

$$\begin{aligned} H_3 = & \frac{1}{2} \left(\frac{n_0}{V} \right)^{1/2} \sum_{K_1, K_2} [v(K_1) + v(K_2)] \\ & \times (a_{K_1}^\dagger a_{K_2}^\dagger a_{K_1+K_2} + a_{K_1+K_2}^\dagger a_{K_2} a_{K_1}) \ ; \end{aligned} \tag{7}$$

H_4 is the same as the second term in (1) except that none of the K's can be zero; N' denotes the number operator for particles not in the zero-momentum state. In (6) and (7) each K summation excludes the point $K=0$.

At a given temperature, n_0 is determined by the requirement that Ω/V be minimum with respect to variations in n_0, i.e.,

$$\frac{\partial \Omega}{\partial n_0} = 0 \ . \tag{8}$$

The single-particle imaginary-time Green's functions of the system are defined by

$$G(K, \tau) = -\langle T a_K(\tau) a_K^\dagger(0) \rangle \ , \tag{9}$$

$$\hat{G}(K, \tau) = -\langle T a_K(\tau) a_{-K}(0) \rangle \ , \tag{10}$$

where $-\beta \leq \tau \leq \beta$, $a_K(\tau)$ denotes the imaginary-time Heisenberg operator

$$a_K(\tau) = e^{\tau H'} a_K e^{-\tau H'} \ , \tag{11}$$

T is the time-ordering operator, and the symbol $\langle \cdots \rangle$ denotes thermodynamic average calculated with Hamiltonian (5).

The Green's functions (9) and (10) can be expanded as a Fourier series in the interval $-\beta \leq \tau \leq \beta$, viz.,

$$G(K, \tau) = \frac{1}{\beta} \sum_l e^{-iZ\tau} G(K, Z) \ , \tag{12}$$

$$\hat{G}(K, \tau) = \frac{1}{\beta} \sum_l e^{-iZ\tau} \hat{G}(K, Z) \ , \tag{13}$$

$$Z = 2\pi i l/\beta, \quad l = 0, \pm 1, \pm 2, \ldots \ . \tag{14}$$

Using perturbation theory[7] one can derive the following equations for $G(K,Z)$ and $\hat{G}(K,Z)$:

$$G(K,Z) = \frac{Z + \frac{1}{2}K^2 - \mu + \Sigma_{11}(-K, -Z)}{D(K,Z)} \ , \tag{15}$$

$$\hat{G}(K,Z) = -\Sigma_{02}(K,Z)/D(K,Z) \ , \tag{16}$$

$$\begin{aligned} D(K,Z) = & \{Z - \tfrac{1}{2}[\Sigma_{11}(K,Z) - \Sigma_{11}(-K, -Z)]\}^2 \\ & - \{\tfrac{1}{2}K^2 - \mu + \tfrac{1}{2}[\Sigma_{11}(K,Z) + \Sigma_{11}(-K, -Z)] \\ & - \Sigma_{02}(K,Z)\} \\ & \times \{\tfrac{1}{2}K^2 - \mu + \tfrac{1}{2}[\Sigma_{11}(K,Z) + \Sigma_{11}(-K, -Z)] \\ & + \Sigma_{02}(K,Z)\} \ , \end{aligned} \tag{17}$$

where Σ_{11} and Σ_{02} denote, respectively, the normal and anomalous self-energy parts.

The analytic continuation of $G(K,Z)$ onto the real axis gives the Fourier transform $G(K,\omega)$ of the retarded real-time Green's function.[8] The poles of $G(K,\omega)$ by definition, represent the quasiparticle excitations at a given temperature.

III. WEAK-INTERACTION APPROXIMATION FOR SELF-ENERGY PARTS

Our aim is to make a consistent weak-interaction calculation of the self-energy parts Σ_{11} and Σ_{02} so as to be able to work out the long-wavelength quasiparticle spectrum. We start by considering the first-order (Bogolubov-approximation) graphs of Σ_{11} and Σ_{02} shown in Fig. 1. Their contributions are

$$\Sigma_{11}^{(1)}(K,Z) = 2 n v_0 \ , \tag{18}$$

$$\Sigma_{02}^{(1)}(K,Z) = n_0 v_0 = (n - n') v_0 \ , \tag{19}$$

where n denotes the total particle density and n' is the density of excited particles given by

$$n' = -\int d\bar{p} \sum_l e^{iZ\delta} G(p,Z) \ . \tag{20}$$

Here as well as in what follows we use the notation

$$d\bar{p} = d^3p/(2\pi)^3 \ . \tag{21}$$

The interaction $v(K)$ is assumed to be a positive constant v_0. The approximations (18) and (19) for the self-energy parts yield

$$G^{(1)}(K,Z) = [Z + \tfrac{1}{2}K^2 + (n - n')v_0]/[Z^2 - E^2(K)] \ , \tag{22}$$

$$\hat{G}^{(1)}(K,Z) = -(n - n')v_0/[Z^2 - E^2(K)] \ , \tag{23}$$

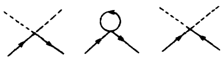

FIG. 1. First-order diagrams of Σ_{11} and Σ_{02}.

where the quasiparticle energy $E(K)$ is given by

$$E^2(K) = \tfrac{1}{4}K^4 + K^2 n v_0 - K^2 n' v_0 \ . \tag{24}$$

For small K, we have

$$E^2(K) \simeq (n - n') v_0 K^2 \ . \tag{25}$$

Since n' increases with increase in temperature, (25) implies that the quasiparticle sound velocity decreases with temperature. The above approximation, however, is inconsistent as regards the last (temperature-dependent) term in (24). To see this we substitute (22) into (20) and use the relation[4]

$$\mu = \Sigma_{11}(0,0) - \Sigma_{02}(0,0) \ , \tag{26}$$

to obtain

$$n' \simeq \int d\vec{\mathrm{p}} \left(\frac{\frac{1}{2}p^2 + n_0 v_0}{E(e^{\beta E} - 1)} + \frac{\frac{1}{2}p^2 + n_0 v_0 - E}{2E} \right] \ . \tag{27}$$

On the other hand, the contribution of the "pair" diagram shown in Fig. 2 to Σ_{02} is

$$\begin{aligned} \Sigma_{02}^{(p)} &= -\frac{v_0}{\beta} \int d\vec{\mathrm{p}} \sum_l \hat{G}(p,Z) \\ &\simeq v_0 \int d\vec{\mathrm{p}} \left(\frac{n_0 v_0}{E(e^{\beta E} - 1)} + \frac{n_0 v_0}{2E} \cdots \right] \ . \end{aligned} \tag{28}$$

At low temperatures specified by

$$\beta n v_0 >> 1 \ , \tag{29}$$

the main contribution to the temperature-dependent terms in (26) and (27) comes from momenta $p \sim (\beta^2 n v_0)^{-1/2}$. Consequently, $n' v_0$ is of the same order of magnitude as $\Sigma_{02}^{(p)}$. Contributions of the same order arise from certain higher-order graphs as well. It is, in fact, known that the contribution of all two-line graphs[9] to $\Sigma_{11}(0,0) - \Sigma_{02}(0,0)$ is exactly twice the value of $\Sigma_{02}^{(p)}$. One must collect all contributions of the same order in order to arrive at a consistent result.

To carry out the calculation in a systematic fashion, we write

$$\Sigma_{11}(K,Z) = 2nv_0 + Y_1(K,Z) \ , \tag{30}$$

$$\Sigma_{02}(K,Z) = nv_0 + Y_2(K,Z) \ , \tag{31}$$

$$\begin{aligned} Y_2(K,Z) = -n' v_0 - \frac{v_0}{\beta} \int d\vec{\mathrm{p}} \sum_l \hat{G}(p,Z) \\ + Y_2'(K,Z) \ , \end{aligned} \tag{32}$$

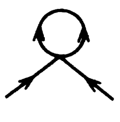

FIG. 2. Pair diagram of Σ_{02}.

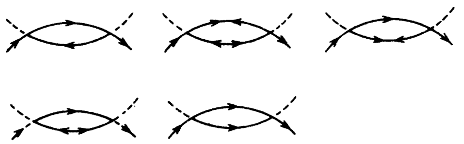

FIG. 3. Second-order two-line graphs of Σ_{11}.

and treat the temperature-dependent parts Y_1 and Y_2 as small compared to $2nv_0$ and nv_0, respectively. Equation (25) for the chemical potential is written in the form

$$\mu = nv_0 + \mu^{(2)} \ , \tag{33}$$

$$\mu^{(2)} = Y_1(0,0) - Y_2(0,0) \ . \tag{34}$$

Substituting from (30) and (31) into (17), and ignoring products of small quantities, one obtains for $D(K,Z)$ the approximate expression

$$D(K,Z) = Z^2 - (c^2 K^2 + \tfrac{1}{4}K^4) - D_1(K,Z) \ , \tag{35}$$

$$\begin{aligned} D_1(K,Z) = Z[Y_1(K,Z) - Y_1(-K,-Z)] \\ + (2c^2 - K^2) Y_3(K,Z) + K^2 Y_2(K,Z) \ , \end{aligned} \tag{36}$$

$$\begin{aligned} Y_3(K,Z) = \tfrac{1}{2}[Y_1(K,Z) + Y_1(-K,-Z)] \\ - Y_2(K,Z) - \mu^{(2)} \ , \end{aligned} \tag{37}$$

$$c^2 = nv_0 \ . \tag{38}$$

The "unperturbed" Green's functions to be used in calculating Y_1 and Y_2 are obtained from (21) and (22) on setting n' equal to zero.

A consistent weak-interaction approximation for Y_1 and Y_2' is now represented by the second-order graphs shown in Figs. 3 and 4. The characteristic feature of these graphs is that they are built from vertices in the Hamiltonian representing scattering of an excited particle by a particle of the condensate and the conjugate process. At low temperatures, the density of excited particles in a weakly interacting Bose gas is small compared to the condensate density. It is, therefore, reasonable to ignore all graphs of Y_1 and Y_2' having vertices corresponding to the scattering of one excited particle by another excited particle.

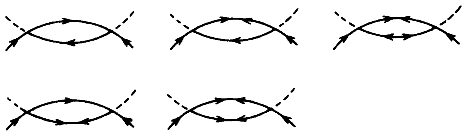

FIG. 4. Second-order two-line graphs of Σ_{02}.

The contributions of the graphs in Figs. 3 and 4 to Y_1 and Y_2' are

$$Y_1(K,Z)=\frac{1}{\beta}\int d\vec{p}\sum_{l}\{G(K+p,Z+Z')[G(p,Z')+2\hat{G}(p,Z')+\tfrac{1}{2}G(-p,-Z')]$$
$$+\hat{G}(K+p,Z+Z')\hat{G}(p,Z')\}\ , \tag{39}$$

$$Y_2'(K,Z)=\frac{1}{\beta}\int d\vec{p}\sum_{l'}\{G(K+p,Z+Z')[G(p,Z')+\hat{G}(p,Z')]+\hat{G}(K+p,Z+Z')[\tfrac{3}{2}\hat{G}(p,Z')+G(p,Z')]\}\ . \tag{40}$$

On performing the frequency summations, we obtain for the temperature-dependent parts of Y_1 and Y_2' the expressions

$$Y_1(K,Z)=n_0v_0^2\int\frac{d\vec{p}}{2E_1E_2}\left[\frac{A_1n_1-C_1n_2}{E_1-E_2+Z}+\frac{B_1n_1-D_1n_2}{E_1-E_2-Z}-\frac{A_1n_1+D_1n_2}{E_1+E_2+Z}-\frac{B_1n_1+C_1n_2}{E_1+E_2-Z}\right]\ , \tag{41}$$

$$Y_2'(K,Z)=n_0v_0^2\int\frac{d\vec{p}}{E_1E_2}\left[\frac{A_2n_1-C_2n_2}{E_1-E_2+Z}+\frac{B_2n_1-D_2n_2}{E_1-E_2-Z}-\frac{A_2n_1+D_2n_2}{E_1+E_2+Z}-\frac{B_2n_1+C_2n_2}{E_1+E_2-Z}\right]\ , \tag{42}$$

$$A_1=\tfrac{1}{2}E_1^2+E_1[\tfrac{3}{2}f_1-2c^2+\tfrac{1}{2}(Z+f_2)]$$
$$+(Z+f_2)(\tfrac{3}{2}f_1-2c^2)\ , \tag{43}$$

$$C_1=\tfrac{1}{2}E_2^2+E_2[\tfrac{3}{2}f_1-2c^2-\tfrac{1}{2}(Z-f_2)]-\tfrac{1}{2}Zf_2$$
$$+f_2(\tfrac{3}{2}f_1-2c^2)\ , \tag{44}$$

$$A_2=E_1^2+E_1(f_1+f_2+Z-2c^2)+f_1(Z+f_2)$$
$$+\tfrac{3}{2}c^4-c^2(f_1+f_2+Z)\ , \tag{45}$$

$$C_2=E_2^2+E_2(f_1+f_2-Z-2c^2)+f_2(f_1-Z)$$
$$+\tfrac{3}{2}c^4-c^2(f_1+f_2-Z)\ , \tag{46}$$

$$E_1=E(p)=(c^2p^2+\tfrac{1}{4}p^4)^{1/2}\ , \tag{47}$$

$$E_2=E(p+K)\ , \tag{48}$$

$$n_i=(e^{\beta E_i}-1)^{-1},\quad i=1,2, \tag{49}$$

$$f_1=f(p)=\tfrac{1}{2}p^2+c^2\ , \tag{50}$$

$$f_2=f(p+K)\ , \tag{51}$$

B_1 and B_2 are obtained, respectively, from A_1 and A_2 by replacing E_1 by $-E_1$, while D_1 and D_2 are obtained from C_1 and C_2 by replacing E_2 by $-E_2$.

IV. TEMPERATURE DEPENDENCE OF THE LONG-WAVELENGTH SPECTRUM

The quasiparticle excitations are given by the zeros of the analytic continuation $D(K,\omega)$ of $D(K,Z)$. In view of (35), we have

$$D(K,\omega)=\omega^2-(c^2K^2+\tfrac{1}{4}K^4)-D_1(K,\omega)\ , \tag{52}$$

$$D_1(K,\omega)=D_1(K,Z\rightarrow\omega+i\delta)\ . \tag{53}$$

To obtain $D(K,\omega)$ we thus need to calculate the three quantities

$$Y_1'(K,\omega)=[Y_1(K,Z)-Y_1(-K,-Z)]_{Z\rightarrow\omega+i\delta}\ , \tag{54}$$

$$Y_2'(K,\omega)=Y_2'(K,Z\rightarrow\omega+i\delta)\ , \tag{55}$$

$$Y_3(K,\omega)=Y_3(K,Z\rightarrow\omega+i\delta)\ . \tag{56}$$

We outline below the calculation of $Y_3(K,\omega)$; $Y_1'(K,\omega)$ and $Y_2'(K,\omega)$ are calculated in a similar fashion. We write

$$Y_3(K,\omega)=L_3(K,\omega)+iL_3'(K,\omega)\ , \tag{57}$$

and calculate first the real part L_3. It is given by the sum of two terms, viz.,

$$L_3=M_3(K,\omega)+M_3'(K,\omega)\ , \tag{58}$$

$$M_3(K,\omega)=n_0v_0^2\int\frac{d\vec{p}}{4E_1E_2}\left[\frac{A_3n_1+D_3n_2}{E_1+E_2+\omega}+\frac{B_3n_1+C_3n_2}{E_1+E_2-\omega}\right]\ , \tag{59}$$

$$M_3'(K,\omega)=n_0v_0^2\int\frac{d\vec{p}}{4E_1E_2}\left[\frac{C_3n_2-A_3n_1}{E_1-E_2+\omega}+\frac{D_3n_2-B_3n_1}{E_1-E_2-\omega}\right]-\mu^{(2)}\ , \tag{60}$$

$$A_3=2(f_1^2+\omega E_1-f_1f_2)\ , \tag{61}$$

$$C_3=2(f_2^2-\omega E_2-f_1f_2)\ , \tag{62}$$

B_3 and D_3 are obtained, respectively, from A_3 and C_3 by replacing ω by $-\omega$. In (59) and (60) the principal value of the integrals is implied.

The dominant contribution to L_3 arises from $M_3'(K,\omega)$ since the integrand in (60) has a pole for any value of the momentum p. For small values of $|K|$, one can write

$$n_2=n_1+\delta n\ , \tag{63}$$

$$\delta n\simeq\frac{\partial n_1}{\partial E_1}(E_2-E_1)\ . \tag{64}$$

The expression for $M_3'(K,\omega)$ then simplifies to

$$M_3'(K,\omega)=n_0v_0^2\int\frac{d\vec{p}}{4E_1E_2}\left[n_1(E_2-E_1)+\left(\frac{C_3(p,\omega)}{E_1-E_2+\omega}+\frac{D_3(p,\omega)}{E_1-E_2-\omega}\right)\delta n\right]\ , \tag{65}$$

where we have made use of the relations

$$A_3-C_3=2(E_1+E_2)(E_1-E_2+\omega)\ , \tag{66}$$

$$B_3-D_3=2(E_1+E_2)(E_1-E_2-\omega)\ , \tag{67}$$

and the fact that $Y_3(0,0)$ is zero.

As $M_3'(K,\omega)$ is an even function of K and ω, we replaced it by the average of $M_3'(K,\omega)$ and $M_3'(-K,-\omega)$ to obtain

$$M_3'(K,\omega)=n_0v_0^2\int\frac{d\vec{p}}{4E_1E_2}\left(\frac{(f_2-f_1)^2-\omega(E_2-E_1)}{E_1-E_2+\omega}+\frac{(f_2-f_1)^2+\omega(E_2-E_1)}{E_1-E_2-\omega}-2(E_2-E_1)\right)\delta n\ . \tag{68}$$

To first order in K,

$$f_2-f_1=Kpx,\quad E_2-E_1=vKx\ , \tag{69}$$

where

$$v=\frac{\partial E}{\partial p}=\frac{fp}{x},\quad x=\cos(p,K)\ . \tag{70}$$

Using (69) in (68) and performing integration over the angles, we get

$$M_3'(K,\omega)$$

$$=-n_0v_0^2\int\frac{p^2dp}{2\pi^2}\frac{K^2}{E_1^2}\left(\frac{p^2}{6}-\frac{c^6}{2f^2}+\frac{p^2f^2}{6E_1^2}\right)\frac{\partial n_1}{\partial E_1}$$

$$-n_0v_0^2\int\frac{p^2dp}{8\pi^2}\frac{K^2c^7}{E_1^2f^2v}\frac{\partial n_1}{\partial E_1}\ln\left|\frac{c+v}{c-v}\right|\ . \tag{71}$$

In writing (71) we have replaced ω by cK as we are interested in the poles of $D(K,\omega)$ only. Similarly by writing

$$\omega Y_1'(K,\omega)=L_1(K,\omega)+iL_1'(K,\omega)\ , \tag{72}$$

$$K^2Y_2'(K,\omega)=L_2(K,\omega)+iL_2'(K,\omega)\ , \tag{73}$$

$$L_1=M_1(K,\omega)+M_1'(K,\omega)\ , \tag{74}$$

$$L_2=M_2(K,\omega)+M_2'(K,\omega)\ , \tag{75}$$

the dominant parts $M_1'(K,\omega)$ and $M_2'(K,\omega)$ can be calculated. The result is

$$M_1'(K,\omega)=n_0v_0^2\int\frac{p^2dp}{\pi^2}\frac{c^2K^2}{E_1^2}\left(\frac{c^4}{f}-2c^2\right)\frac{\partial n_1}{\partial E_1}+n_0v_0^2\int\frac{p^2dp}{2\pi^2}\frac{c^2K^2}{E_1^2}\left(2c^3-\frac{c^5}{f}\right)\frac{\partial n_1}{\partial E_1}\frac{1}{v}\ln\left|\frac{c+v}{c-v}\right|\ , \tag{76}$$

$$M_2'(K,\omega)=K^2n_0v_0^2\int\frac{p^2dp}{\pi^2}\frac{1}{E_1^2}\left[2E_1n_1+\left(E_1^2+\frac{p^4}{4}+\frac{c^4}{2}\right)\frac{\partial n_1}{\partial E_1}\right]-n_0v_0^2K^2$$

$$\times\int\frac{p^2dp}{2\pi^2}\frac{1}{E_1^2}\left(E_1^2+\frac{p^4}{4}+\frac{c^4}{2}\right)\frac{\partial n_1}{\partial E_1}\frac{c}{v}\ln\left|\frac{c+v}{c-v}\right|\ . \tag{77}$$

As regards the "nonsingular" terms M_1, M_2, and M_3 in (74), (75), and (58), the integrands of these terms [see Eq. (59)] possess a pole in the extremely narrow range $|p| < |K| \to 0$. A little manipulation shows that the numerators of these terms are proportional to K^2. In the limit of small $|K|$, their contributions are found to be

$$M_1(K,\omega) = -n_0 v_0^2 \int \frac{p^2 dp}{\pi^2} \frac{c^2 K^2}{E_1^3} \left(\frac{p^2}{2} - c^2 \right) n_1 \ , \tag{78}$$

$$M_2(K,\omega) = -n_0 v_0^2 \int \frac{p^2 dp}{\pi^2} \frac{K^2}{E_1^3} \times \left(E_1^2 + \frac{p^4}{4} + \frac{c^4}{2} \right) n_1 \ , \tag{79}$$

$$M_3(K,\omega) = n_0 v_0^2 \int \frac{p^2 dp}{2\pi^2} \frac{K^2}{E_1^2} \times \left(\frac{(\frac{1}{3}p^2 - c^2) n_1}{2E_1} + \frac{f^2 p^2}{6E_1^2} \frac{\partial n_1}{\partial E_1} \right) . \tag{80}$$

Collecting the above results, we obtain after some simplification the following expression for the real part of $D_1(K,\omega)$:

$$\mathrm{Re} D_1(K,\omega) = -\frac{9c^2K^2}{4n} \int \frac{p^4 dp}{4\pi^2} \frac{\partial n_1}{\partial E_1} \ln \left| \frac{c+v}{c-v} \right| - n_0 v_0^2 K^2 \int \frac{p^6 dp}{\pi^2} \left(\frac{n}{2E_1^3} - \frac{115}{96} \frac{1}{E_1^2} \frac{\partial n_1}{\partial E_1} \right) . \tag{81}$$

One can also calculate without difficulty the imaginary part of $D_1(K,\omega)$. For small $|K|$ the dominant contributions to L_1', L_2', and L_3' are

$$L_1'(K,\omega) = n_0 v_0^2 \int \frac{p^2 dp}{2\pi} \frac{cK}{E_1 E_2} \times [(-A_4 n_1 + C_4 n_2)\delta(E_1 - E_2 + \omega) + (B_4 n_1 - D_4 n_2)\delta(E_1 - E_2 - \omega)] \ , \tag{82}$$

$$L_2'(K,\omega) = n_0 v_0^2 \int \frac{p^2 dp}{2\pi} \frac{K^2}{E_1 E_2} \times [(-A_2 n_1 + C_2 n_2)\delta(E_1 - E_2 + \omega) + (B_2 n_1 - D_2 n_2)\delta(E_1 - E_2 - \omega)] \ , \tag{83}$$

$$L_3'(K,\omega) = n_0 v_0^2 \int \frac{p^2 dp}{4\pi E_1 E_2} \times [(-A_3 n_1 + C_3 n_2)\delta(E_1 - E_2 + \omega) + (B_3 n_1 - D_3 n_2)\delta(E_1 - E_2 - \omega)] \ , \tag{84}$$

$$A_4 = (E_1 + 2c^2)(f_1 - f_2) + \omega(f_1 - 2c^2) \ , \tag{85}$$

$$C_4 = (E_2 + 2c^2)(f_1 - f_2) + \omega(f_2 - 2c^2) \ , \tag{86}$$

B_4 is obtained from A_4 by replacing E_1 by $-E_1$, and D_4 is obtained from C_4 by replacing E_2 by $-E_2$.

Using (63) and relations such as (66) and (67), we get the following expression for the imaginary part of $D_1(K,\omega)$:

$$\mathrm{Im} D_1(K,\omega) = -n_0 v_0^2 \int \frac{p^2 dp}{2\pi E_1 E_2} [C_5 \delta(E_1 - E_2 + \omega) + D_5 \delta(E_1 - E_2 - \omega)] \delta n \ , \tag{87}$$

$$C_5 = \omega C_4 + c^2 C_3 + K^2 C_2 \ , \tag{88}$$

$$D_5 = \omega D_4 + c^2 D_3 + K^2 D_2 \ . \tag{89}$$

For a given value of $|p|$, the δ functions in (87) have singularities at $x = x_0^{\pm}$ determined by

$$E_2(x_0^{\pm}, p) = E_1 \pm \omega \ . \tag{90}$$

On solving these equations one finds

$$x_0^{\pm} = \pm \left(1 - \frac{3p^2}{8c^2} \right) - \frac{3Kp}{4c^2} \left(1 - \frac{p^2}{c^2} \right) + O(K^2) \ . \tag{91}$$

On carrying out the x integration, we find at the end of a somewhat laborious algebraic calculation the result

$$\mathrm{Im} D_1(K,\omega) = -n_0 v_0^2 K^2 \times \int \frac{p^2 dp}{4\pi E_1^2} \frac{\partial n_1}{\partial E_1} \left[-\frac{9}{4} p^4 + O\left(\frac{p^6}{c^2} \right) \right] . \tag{92}$$

The integrals in (81) and (92) can be easily evaluated in the low-temperature limit $\beta c^2 >> 1$. One obtains

$$D_1(K,\omega) = 2cK^2(c_A + c_B + i\Gamma) \ , \tag{93}$$

$$c_A = \frac{9}{4} \frac{\pi^2}{30 n \beta^4 c^4} \ln \left(\frac{2\beta^2 c^2}{53.7\gamma} \right) \ , \tag{94}$$

$$c_B = -127\pi^2/720 n \beta^4 c^4 \ , \tag{95}$$

$$\Gamma = 3\pi^2/40 n \beta^4 c^4 \ , \tag{96}$$

$$\gamma = 1/8c^2 \ . \qquad (97)$$

Substituting (93) into (52) we find that $c_A + c_B$ gives the temperature shift in the phonon sound velocity while $(K\Gamma)^{-1}$ gives the life of a phonon of momentum K.

V. DISCUSSION

In view of (38), we have

$$\frac{9}{4} = \left(\frac{n}{c}\frac{\partial c}{\partial n} + 1\right)^2 \ . \qquad (98)$$

The $T^4 \ln T^{-1}$ increase in quasiparticle sound velocity given by (94) is, therefore, formally identical with the result derived by KA for He II by the phenomenological approach. It should, however, be noted that the argument of the logarithm in (94) is smaller by a factor of about 2 as compared with the result of KA. This discrepancy is due to the fact that, rather than evaluating the integral in the first term in (81) exactly, KA made only an approximate estimate.

The additional T^4 decrease in sound velocity represented by (95) was not found by KA. As seen above, it is a consequence of taking into account the "nonsingular" terms of the self-energy parts. Its effect is to lower the value of the argument of the logarithm in (94) by a factor of about 10. Assuming that this result for the Bose gas remains formally valid for He II, we have plotted in Fig. 5 the increase in sound velocity as a function of temperature using the same value of γ as used by KA ($\gamma = 3\times10^{37}\ \mathrm{g^{-1}\,erg^{-1}}$). The values of the other parameters used are

$$c = 238\ \mathrm{m\,sec^{-1}},\quad \rho = nm = 0.145\ \mathrm{g\,cm^{-3}} \ ,$$

$$\rho/c\,(\partial c/\partial \rho) = 1.8 \ . \qquad (99)$$

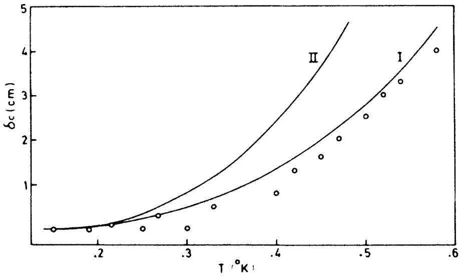

FIG. 5. Increase in sound velocity of He II with temperature. Curve I represents $c_A + c_B$ given by Eqs. (94) and (95), while curve II represents the (corrected) KA result. The circles are the experimental results of Whitney and Chase (Ref. 2).

The result of KA (corrected for the factor 2 referred to above) and the experimental results of Whitney and Chase[2] are also shown in Fig. 5. It will be seen that taking into account the correction (95) gives much better agreement with experiment. The agreement, however, should be viewed with some caution on account of the uncertainty in the experimental value of γ.[10,11]

We note that the T^4 damping of phonons given by (96) is in agreement with a result of Hohenberg and Martin.[4]

Finally, we remark that had we carried out an expansion of the self-energy parts $\Sigma_{11}(K,\omega)$ and $\Sigma_{02}(K,\omega)$ around $\omega = K = 0$ as done by Gavoret and Nozières[3] in the $T = 0$ case, we would have missed the logarithmic term (94) and obtained simply the T^4 decrease given by (95). The reason as to why the expansion procedure gives correct results at $T = 0$ can be traced to relations such as (66) and (67) which eliminate the singularities at $T = 0$.

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