Hard-sphere roton interaction

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Utilizing a roton-roton interaction derived from a hard-sphere model for superfluid helium we obtained a qualitatively correct temperature dependence to the correction of the transition temperature T_λ due to roton-roton interaction as compared to that calculated without such roton-roton interaction.

I. INTRODUCTION

To explain the thermodynamic properties of superfluid ^{4}He, Landau[1] suggested an excitation spectrum as depicted in Fig. 1. For small momenta this spectrum is phononlike, however at high momenta $\hbar k \approx \hbar k_0$ ($k_0 = 1.91$ Å^{-1}) the spectrum deviates from the linear momentum dependence and is called the roton region, which can be approximated by

$$\epsilon = \Delta_0 + (\hbar^2/2\mu)(k-k_0)^2, \tag{1}$$

where $\hbar\vec{k}_0$ and Δ_0 designate the roton momentum and energy, respectively.

At temperatures T higher than 1 °K, the superfluid-mass density ρ_S is depleted primarily by rotons. The roton-mass density ρ_R is given by

$$\rho_R = \int \frac{d^3k}{(2\pi)^3} m(\epsilon) n(\epsilon) = \frac{\hbar^2 k_0^2}{3\beta} N_R, \tag{2}$$

where the effective roton mass

$$m(\epsilon) = (\hbar^2 k^2/3\beta) n(\epsilon) e^{\epsilon/\beta} \tag{3}$$

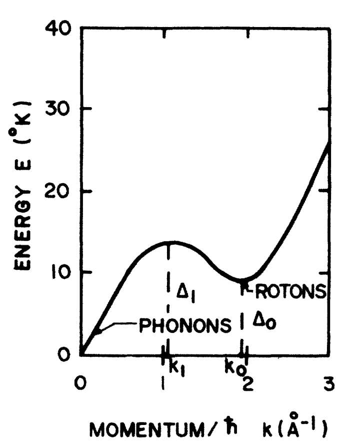

FIG. 1. Excitation spectrum of superfluid helium proposed by Landau.

is given from momentum considerations.[2] In Eqs. (2) and (3), $\beta = k_B T$ is the product of the Boltzmann's constant and the temperature of the liquid, and

$$n(\epsilon) = (e^{\epsilon/\beta} - 1)^{-1} \tag{4}$$

describes the number of rotons with momentum $\hbar\vec{k}$. Thus, the number of rotons per unit volume is given by

$$N_R = \frac{1}{(2\pi)^3} \int n(\epsilon)\, d^3k = \frac{2(\mu B)^{1/2} k_0^2}{(2\pi)^{3/2}\hbar} e^{-\Delta_0/\beta}. \tag{5}$$

From Eqs. (2)–(5) one obtains, setting ρ_R equal to the liquid-helium-mass density ρ, a critical temperature $T_\lambda \approx 2.8$ °K, much too large as compared with the experimental value of 2.17 °K. Furthermore, the superfluid-mass density $\rho_S = \rho - \rho_R$ for $T \approx T_\lambda$ is also an order of magnitude overestimated.[3]

Ruvalds,[4] who postulated a δ-function interaction between the rotons, calculated in the Hartree-Fock approximation the effect of this interaction on the excitation energy, yielding

$$\epsilon' = \epsilon + 2g_4 N_R(T), \tag{6}$$

where g_4 is the attractive strength of his δ-function interaction. Using (6) in (5) and choosing $g_4 = -3.7 \times 10^{-38}$ erg cm^3 he showed that the critical temperature can be brought down to the experimental value. However, it has to be kept in mind that in his model the value for g_4 is an order of magnitude larger than one would need in order to obtain a correct value for the roton-roton binding energy.[5-7] Furthermore, a δ-function interaction cannot explain the value of the roton-roton collision frequency.[8]

Based on a hard-sphere model for liquid ^{4}He,[9,10] we have derived a roton-roton interaction which not only gives the correct value for the roton-roton binding energy but also the correct value for the roton-roton collision frequency.[11] Therefore, we shall use this same interaction potential here to derive its influence on the transition temperature T_λ along the lines as presented in Ref. 4. Our goal is twofold. First, we show that our microscopic model which is consistent with earlier calculations[10] for the binding energy and scattering frequencies of rotons, in fact provides the right order of magnitude correction to the transition temperature. Second, in calculating for the transition temperature, it is also shown explicitly, with our model, that the roton energy decreases with increasing temperature, in contrast to earlier calculations by Parry and ter Harr.[12] This lowering of the roton energy with increasing temperature has been verified conclusively by neutron scattering data,[13] and is the key ingredient in adjusting the transition temperature T_λ to lower values when taken into account.

II. THEORY

From the hard-sphere interaction we obtain for the roton-roton interaction in the Hartree-Fock approximation[11]

$$H=\sum_{\vec{k}} \epsilon_k \beta_{\vec{k}}^\dagger \beta_{\vec{k}} + \frac{g}{2V}\sum_{\vec{k},\vec{p}} V(\vec{k},\vec{p})\beta_{\vec{k}}^\dagger \beta_{\vec{k}} \beta_{\vec{p}}^\dagger \beta_{\vec{p}}, \tag{7}$$

where

$$V(\vec{k},\vec{p})=\cos(|\vec{k}-\vec{p}|a)-\frac{a\vec{k}\cdot\vec{p}}{|\vec{k}-\vec{p}|}j_1(|\vec{k}-\vec{p}|a). \tag{8}$$

The $\beta_{\vec{k}}$ represents the quasiparticle operator obeying Bose statistics, and the interaction strength $g=4\pi a\hbar^2/m=4.33\times10^{-38}$ erg cm^3 for a helium atom with a hard-core diameter $a=2.1$ Å and a mass $m=6.7\times10^{-24}$ g. The Hamiltonian given by Eq. (7) represents the lowest order of roton interaction and is, in fact, valid only for momenta k and p around k_0.[11] The first term in Eq. (7) represents the excitation spectrum at $T=0$ °K, and the second term is the Hartree-Fock (HF) exchange. The direct HF term does not appear here because it has already been absorbed into the computation of ϵ_k. The self-consistently corrected energy in this approximation is therefore given by

$$\epsilon_k' = \epsilon_k + \Sigma_k(T), \tag{9}$$

with

$$\Sigma_k(T)=\frac{g}{V}\sum_{\vec{p}} V(\vec{k},\vec{p})n_{\vec{p}}', \tag{10}$$

where $n'(p)$ is given by Eq. (4) with ϵ_p replaced by ϵ_p'. To calculate the self-energy in Eq. (10) approximately, we observed that the experiments yielding information on the roton-roton interaction are the Raman scattering experiments,[14,15] which only reveal the properties of the $l=2$ angular component of the roton-roton interaction.[16] However, measurements on the linewidth of the Raman scattering cross section[17] lead, in fact, to the same collision frequencies as those obtained from viscosity measurments.[18] Thus, we may conclude that the $l=2$ component constitutes the main channel in the roton-roton interaction. With this physical consideration in mind we decompose V into spherical harmonics

$$V(\vec{k},\vec{p})=\sum_l \lambda^{(l)}(k,p)P_l(\cos\theta_{\hat{k}\cdot\hat{p}}). \tag{11}$$

Now, for the calculation of Eq. (10) we shall only retain the $l=2$ component (see Ref. 11 for further computational details). Thus,

$$\Sigma_k(T)=\frac{4\pi}{(2\pi)^3}g\int_0^\infty dq\, q^2 n_q' \Big(-2j_2(ka)j_2(qa)+\tfrac{1}{2}[j_0(|\vec{q}+\vec{k}|a)+j_0(|\vec{q}-\vec{k}|a)]-j_0(ka)j_0(qa) -\frac{k^2+q^2}{2kq}[j_0(|\vec{q}-\vec{k}|a)-j_0(|\vec{q}+\vec{k}|a)-6j_1(ka)j_1(qa)]\Big). \tag{12}$$

Since we are only interested in $\Sigma_k(T)$ for $k\approx k_0$ and since $n(q)$ also peaks strongly around $q=k_0$, Eq. (12) yields

$$\Sigma_{k_0}(T)=-0.422gN_R'(T), \tag{13}$$

where $N_R'(T)$ is given by Eq. (5) with Δ_0 replaced by $\Delta_0+\Sigma_{k_0}(T)$. Expression (13) shows clearly that with increasing temperature the roton energy will decrease contrary to the conclusion drawn by Parry and ter Haar[12] in their calculation. Furthermore, increasing the hard-core diameter a in fact increases the coupling strength g and also the expression inside the large parentheses in Eq. (12), leading to a lowering of T_λ. This actually implies that the model is consistent with the decrease of T_λ at increasing pressures on the liquid.[3]

To calculate for T_λ we substitute $n_R'=N_R'\times10^{-22}$ and $\bar{g}=g\times10^{38}$, then set $\rho_R=\rho$ in Eq. (2), which together with Eqs. (5), (9), and (13) leads to

$$3.586+0.045\bar{g}=\tfrac{1}{2}\ln T+8.65/T, \tag{14}$$

where we have chosen $\Delta_0=8.65$ °K, $\mu=1.06\times10^{-24}$ g, $k_B=1.38\times10^{-16}$, and $\rho=0.145$ g/cm^3. Furthermore, by choosing the hard-core diameter $a=2.1$

Å and the helium mass $m = 6.7 \times 10^{-24}$ g, with the normalized interaction strength $\tilde{g} = 4.33$ erg cm^3, we reduce the value for T_λ from 2.8 to 2.6 °K. To reduce T_λ further to the experimental value of 2.17 °K under the same approximations as considered above, we would have to increase $\tilde{g}$ four times. However, this point should not be disappointing to us. In fact, our interaction strength $\tilde{g}$ as shown by Tüttö[19] has about the maximum value possible under which the HF approximation can still be applied meaningfully. Going to larger values for $\tilde{g}$, contributions from two roton-bound states will become important in the evaluation of the self-energy, thus, reducing its value until it eventually changes sign at sufficiently large $\tilde{g}$. Thus, our value for $\tilde{g}$ is just about the optimal value we can expect. The reason for our failure to obtain a better value for T_λ must lie in the crude approximations that we have made, such as neglecting all scattering channels other than $l = 2$.

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