Quantum theory of acoustic superradiance

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(Received 7 December 1976)

We develop a quantum description of acoustic superradiance for spins interacting with longitudinal phonons in a one-dimensional system. The analysis is performed by looking at the second-order Heisenberg equations of motion for the phonon population in each mode. These equations are solved in the superradiant case and the results are compared with those obtained in the theory of avalanche relaxation previously developed.

I. INTRODUCTION

Recently,[1] the possibility of acoustic superradiance has been investigated and a semiclassical theory has been developed for paramagnetic spins interacting with lattice vibrations. As is well known, in a paramagnetic crystal the populations of electronic levels of paramagnetic impurities at equilibrium are determined by the temperature of the black-body acoustic radiation present in the crystal. If the populations are modified by an external cause and the energy of the impurities is increased, the system may recover its equilibrium configuration by releasing the excess of energy in the form of elastic vibrations of the atoms surrounding the impurities. At low temperature this relaxation takes place mainly through direct processes. In such processes each electronic transition leads to the emission of a phonon having energy equal to the spacing between the levels involved. In simple cases recovery of the equilibrium situation is exponential and its characteristic time is T_1. This time is referred to as spin-phonon relaxation time and, strictly speaking, it is meaningful only when each impurity can be assumed to relax without being influenced by the phonon field emitted by the others.

In Ref. 1 the relaxation process was studied when the populations of the levels are initially inverted and the temperature of the thermal bath is 0 °K. According to that theory, under appropriate conditions, the impurities decay simultaneously within a narrow time interval T_R much smaller than T_1. Their energy is converted into elastic energy and a highly directional acoustic pulse is emitted. This process takes place after a delay time T_D from initial excitation, one or two orders of magnitude larger than T_R.

The directionality of the pulse is characteristic for this type of relaxation and is a direct consequence of phase correlations in the time evolution of the states of different impurities. The direction coincides with a diffraction maximum for the radiation and therefore depends on the relative positions of the active centers. This directionality does not seem to play an important role in different forms of collective relaxation such as phonon avalanche. In fact, some of the descriptions of the latter phenomenon that have been proposed[2,3] and that explain the experimental results, at least qualitatively, do not take into account phase correlations in the motion of different spins. Coherent relaxation cannot be excluded but it can be trusted not to play a fundamental role in Brya and Wagner's experiments. However, it is likely that in some cases both acoustic superradiant emission and phonon avalanche occur during the same relaxation process, although at different stages. Therefore it seems desirable to develop a theoretical formulation which includes both ways of collective decay. Interesting contributions in this direction have appeared recently.[4,5]

In this paper we present a fully quantum-mechanical description of acoustical superradiance. Second-order Heisenberg equations are derived for the operators associated with the number of phonons in each vibrational mode. These equations together with first-order Heisenberg equations for some appropriate spin operators lead to a solvable system of equations. The solutions are found and compared with those obtained in the semiclassical approximation as well as with those obtained in the case of phonon avalanche. In this analysis we take into account the interaction of paramagnetic spins with the phonon field but we neglect spin-spin and phonon-phonon interaction. In this approximation the spin-phonon Hamiltonian, in the notation of number operators, is formally identical to the Hamiltonian used to study the optical case. Therefore most of the conclusions reached in the quantum theory of optical superradiance[6-9] can be applied to the case considered here. Nevertheless the method of solution adopted here presents some interesting features, since it allows a direct connection with the theory of phonon avalanche previously developed,[3] and it is an attempt at an unitary description in which a possible transition between the two decay regimes can be analyzed.

In the following we first describe the model used and derive the equations for the phonon operators. Second we formally solve these equations in the superradiant case and derive the equations of motion for the spin operators. Then we solve the resulting system of equations. Comparison with previous results concludes the paper.

II. THE MODEL

The model used to describe paramagnetic spins interacting with lattice vibrations is the same of Ref. 1. It describes an assembly of spins $S=\frac{1}{2}$, placed in an external magnetic field H and coupled to a one-dimensional harmonic lattice. The Hamiltonian is

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{\rm int}\,, \tag{1}$$

where

$$\mathcal{H}_0=\mathcal{H}_{\rm ph}+\mathcal{H}_s = \sum_j\left[\tfrac{1}{2}\mathcal{P}_j^2/m+\tfrac{1}{2}K(U_j-U_{j+1})^2\right]+g\mu_B H\sum_j S_z^{(j)} \tag{2}$$

is the part concerning the lattice and the spins, without their mutual interaction. $\mathcal{P}_j$ is the momentum operator, U_j is the dispacement operator of the atom j in the x direction, and K is the nearest-neighbors force constant. $S_z^{(j)}$ is the z component of the spin in the j site, and $g\mu_B H=E$ is the energy splitting due to an uniform magnetic field H in the z direction. The interaction is described by

$$\mathcal{H}_{\rm int}=\hbar\epsilon\sum_j (U_{j+1}-U_{j-1})S_x^{(j)}\,, \tag{3}$$

where ϵ is the coupling constant between the spin and the strain at position j.

In the notation of number operators the Hamiltonian of the unperturbed lattice has the form

$$\mathcal{H}_{\rm ph}=\sum_k \hbar\omega_k a_k^* a_k\,, \tag{4}$$

where

$$a_k=(2m\hbar\omega_k N)^{-1/2}\sum_j e^{-ikx_j}(\mathcal{P}_j-im\omega_k U_j)\,. \tag{5}$$

N is the number of atoms in the chain and ω_k is the lattice frequency associated with the k mode. The interaction can be written

$$\mathcal{H}_{\rm int}=V_1+V_2\,, \tag{6}$$

where

$$V_1=\sum_{j,k}(\epsilon_k a_k S_+^{(j)}e^{ikx_j}+\text{H.c.})\,, \tag{7}$$

$$V_2=\sum_{j,k}(\epsilon_k a_k S_-^{(j)}e^{ikx_j}+\text{H.c.})\,, \tag{8}$$

and

$$\epsilon_k=-\epsilon(\hbar^3/2mN\omega_k)^{1/2}\sin(ka)\,. \tag{9}$$

We note that of the two parts of the interaction only the first describes real transition processes. V_2 does not conserve energy and primarily causes a shift in the energy levels.[10] In the following we will neglect its effects completely.

III. EQUATIONS FOR THE PHONONS

We study the time evolution of the system looking at the behavior of the phonon population operators. The first derivative of the phonon population in the k mode is

$$\frac{da_k^*a_k}{dt}=-\left(\frac{i}{\hbar}\right)[a_k^*a_k,(\mathcal{H}_0+V_1)]=-\left(\frac{i}{\hbar}\right)[a_k^*a_k,V_1] =\left(\frac{i}{\hbar}\right)\sum_j(\epsilon_k a_k S_+^{(j)}e^{ikx_j}-\text{H.c.}) \tag{10}$$

where we use

$$[a_k^*,a_{k'}^*]=[a_k,a_{k'}]=0;\quad [a_k,a_{k'}^*]=\delta_{k,k'}\,.$$

The second time derivative is given by

$$\begin{aligned}\frac{d^2a_k^*a_k}{dt^2}&=-\left(\frac{i}{\hbar}\right)\left[\frac{da_k^*a_k}{dt},(\mathcal{H}_0+V_1)\right]\\
&=\hbar^{-2}(\hbar\omega_k-E)\sum_j(\epsilon_k a_k S_+^{(j)}e^{ikx_j}+\text{H.c.})+2\hbar^{-2}|\epsilon_k|^2\sum_j(S_z^{(j)}+\tfrac{1}{2})\\
&\quad+\hbar^{-2}|\epsilon_k|^2\sum_{\substack{j,j'\\ j\neq j'}}(S_+^{(j)}S_-^{(j')}e^{ik(x_j-x_{j'})}+\text{H.c.})+4\hbar^{-2}|\epsilon_k|^2a_k^*a_k\sum_j S_z^{(j)}\\
&\quad+2\hbar^{-2}\sum_{k'\neq k}\left(\epsilon_k\epsilon_{k'}^*a_k a_{k'}^*\sum_j S_z^{(j)}e^{i(k-k')x_j}+\text{H.c.}\right),\end{aligned} \tag{11}$$

where the commutators of the spin operators are

$$[S_x^{(j)}, S_y^{(j')}] = i\delta_{j,j'} S_z^{(j)}$$

and

$$S_\pm^{(j)} = S_x^{(j)} \pm i S_y^{(j)} .$$

It may be useful at this point to make some qualitative considerations on the role played by the various terms that appear in Eq. (11). We note that the second term depends linearly on the number of impurities, while the last three terms depend on the square of this number. This difference is essential in distinguishing the terms that describe usual spontaneous emission from those describing superradiance or avalanche relaxation. Moreover, the last two terms are proportional to the square of the coupling times phonon operators and should represent stimulated emission due to the phonons emitted by the spin system, if the lattice at $t=0$ is in a phononless state. In fact the fourth term describes incoherent phonon avalanche.[3] The third term does not explicitly contain phonon operators and it depends on the relative phases of the spins at different sites. As we shall see, this term plays a central role in superradiant emission where, according to the formulation of Dicke,[6] the radiation emitted is assumed to immediately leave the active region.

The first term needs a little work to be interpreted. The operator $(\sum_k a_k^* a_k + \sum_j S_z^{(j)})$ commutes with $(\mathcal{H}_0 + V_1)$ and therefore is a constant of motion. Then we can write

$$\sum_k a_k^* a_k + \sum_j S_z^{(j)} = \left(\sum_k a_k^* a_k + \sum_j S_z^{(j)} \right) \Big|_{t=0} , \qquad (12)$$

and consequently we have

$$\sum_k \hbar\omega_k a_k^* a_k + E \sum_j S_z^{(j)} = \sum_k (\hbar\omega_k - E) a_k^* a_k + E \left(\sum_k a_k^* a_k + \sum_j S_z^{(j)} \right) \Big|_{t=0} . \qquad (13)$$

The total energy of the system is also a constant of motion. Therefore,

$$\sum_k \hbar\omega_k a_k^* a_k + E \sum_j S_z^{(j)} + \sum_{k,j} (\epsilon_k a_k S_+^{(j)} e^{ikx_j} + \text{H.c.}) = \left(\sum_k \hbar\omega_k a_k^* a_k + E \sum_j S_z^{(j)} + \sum_{k,j} (\epsilon_k a_k S_+^{(j)} e^{ikx_j} + \text{H.c.}) \right) \Big|_{t=0} . \qquad (14)$$

Then, by using Eqs. (13) and (14), we have

$$\sum_k \left((\hbar\omega_k - E) a_k^* a_k + \sum_j (\epsilon_k a_k S_+^{(j)} e^{ikx_j} + \text{H.c.}) \right) = \sum_k \left((\hbar\omega_k - E) a_k^* a_k + \sum_j (\epsilon_k a_k S_+^{(j)} e^{ikx_j} + \text{H.c.}) \right) \Big|_{t=0} . \qquad (15)$$

If this equality can be assumed to hold for each k mode separately, we finally get

$$\sum_j (\epsilon_k a_k S_+^{(j)} e^{ikx_j} + \text{H.c.}) = -(\hbar\omega_k - E) a_k^* a_k + (\hbar\omega_k - E) a_k^* a_k |_{t=0} + \sum_j (\epsilon_k a_k S_+^{(j)} e^{ikx_j} + \text{H.c.}) |_{t=0} . \qquad (16)$$

A different derivation of Eq. (16) can be found in Ref. 11. Direct substitution shows that the first term in Eq. (11) distinguishes between the resonant modes and those out of resonance.

IV. SUPERRADIANT CASE

We wish to restrict this analysis to the acoustic analog of the Dicke superradiance.[6] Therefore we do not take into account the effects of the interaction of the emitted phonons with the spins by neglecting the terms that in Eq. (11) depend on phonon operators times the square of the coupling. With this condition Eq. (11) reduces to

$$\frac{d^2 a_k^* a_k}{dt^2} + \hbar^{-2} (\hbar\omega_k - E)^2 a_k^* a_k = \hbar^{-2} (\hbar\omega_k - E) \sum_j (\epsilon_k a_k S_+^{(j)} e^{ikx_j} + \text{H.c.}) |_{t=0} + \hbar^{-2} (\hbar\omega_k - E)^2 a_k^* a_k |_{t=0} + 2\hbar^{-2} |\epsilon_k|^2 \sum_j (S_z^{(j)} + \tfrac{1}{2}) + \hbar^{-2} |\epsilon_k|^2 \sum_{\substack{j,j' \\ j \neq j'}} (S_+^{(j)} S_-^{(j')} e^{ik(x_j - x_{j'})} + \text{H.c.}) , \qquad (17)$$

where Eq. (16) has been used. Equation (17) can be formally integrated by considering the terms on the right-hand side as a source operator $S(t)$. The general solution is

$$a_k^* a_k = f_1(t) \left(C_1 - \int_0^t \frac{S(t') f_2(t') \, dt'}{\Delta(f_1, f_2)} \right) + f_2(t) \left(C_2 + \int_0^t \frac{S(t') f_1(t') \, dt'}{\Delta(f_1, f_2)} \right) , \qquad (18)$$

where C_1 and C_2 are two time-independent operators to be determined. $f_1(t)$ and $f_2(t)$ are linearly independent solutions of the corresponding homogeneous equation and $\Delta(f_1, f_2)$ is the Wronskian of these two solutions. It is

$$f_1(t)=\sin\frac{|(\hbar\omega_k-E)|t}{\hbar}\,,\quad f_2(t)=\cos\frac{|(\hbar\omega_k-E)|t}{\hbar}\,; \tag{19}$$

and the Wronskian gives

$$\Delta(f_1,f_2)=f_1\frac{df_2}{dt}-f_2\frac{df_1}{dt}=-\frac{|(\hbar\omega_k-E)|}{\hbar}\,. \tag{20}$$

Equation (18), at $t=0$, gives

$$a_k^*a_k|_{t=0}=C_2\,, \tag{21}$$

where the operator C_2 is shown to coincide with the operator $a_k^*a_k$ at $t=0$ in the Heisenberg representation. Moreover, derivation of Eq. (18) with respect to time gives

$$\frac{da_k^*a_k}{dt}=\frac{df_1}{dt}\left(C_1-\int_0^t\frac{S(t')f_2(t')\,dt'}{\Delta(f_1,f_2)}\right)+\frac{df_2}{dt}\left(C_2+\int_0^t\frac{S(t')f_1(t')\,dt'}{\Delta(f_1,f_2)}\right), \tag{22}$$

and, at $t=0$,

$$\left.\frac{da_k^*a_k}{dt}\right|_{t=0}=\frac{|(\hbar\omega_k-E)|}{\hbar}\,C_1\,. \tag{23}$$

By using Eq. (10) we may write

$$C_1=\frac{i}{|(\hbar\omega_k-E)|}\sum_j\left.(\epsilon_k a_k S_+^{(j)}e^{ikx_j}-\text{H.c.})\right|_{t=0}\,. \tag{24}$$

At $t=0$ our operators in the Heisenberg representation are equal to the corresponding ones in the Schrödinger representation and their average values at $t=0$ give

$$\langle C_1\rangle=\langle C_2\rangle=0\,. \tag{25}$$

These values are calculated using Eqs. (21) and (24), assuming the system in the phononless state with all the spins in the upper level. Equation (18) for the average values give

$$\begin{aligned}\langle a_k^*a_k\rangle_t&=f_2(t)\int_0^t dt'\,\frac{\langle S(t')\rangle f_1(t')}{\Delta(f_1,f_2)}-f_1(t)\int_0^t dt'\,\frac{\langle S(t')\rangle f_2(t')}{\Delta(f_1,f_2)}\\&=\hbar^{-2}|\epsilon_k|^2\int_0^t dt'\left(\left\langle 2\sum_j(S_z^{(j)}+\tfrac{1}{2})\right\rangle_{t'}+\left\langle\sum_{\substack{j,j'\\ j\neq j'}}(S_+^{(j)}S_-^{(j')}e^{ik(x_j-x_{j'})}+\text{H.c.})\right\rangle_{t'}\right)\frac{\sin[|(\hbar\omega_k-E)|(t-t')/\hbar]}{|(\hbar\omega_k-E)|/\hbar}\,,\end{aligned} \tag{26}$$

where the time-independent operators on the right-hand side of Eq. (17) do not appear since their average values vanish.

We are interested in the behavior of quantities such as the total magnetization or the acoustic intensity. To this end we sum Eq. (26) over all the modes and have

$$\begin{aligned}&\int_0^{\hbar\omega_{\max}}d(\hbar\omega_k)\rho_l(\hbar\omega_k)\langle a_k^*a_k\rangle_t\\&=\hbar^{-2}\int_0^t dt'\left\langle 2\sum_j(S_z^{(j)}+\tfrac{1}{2})\right\rangle_{t'}\int_0^{\hbar\omega_{\max}}d(\hbar\omega_k)\rho_l(\hbar\omega_k)|\epsilon_k|^2\,\frac{\sin[|(\hbar\omega_k-E)|(t-t')/\hbar]}{|(\hbar\omega_k-E)|/\hbar}\\&\quad+\hbar^{-2}\int_0^t dt'\int_0^{\hbar\omega_{\max}}d(\hbar\omega_k)\rho_l(\hbar\omega_k)|\epsilon_k|^2\left\langle\sum_{\substack{j,j'\\ j\neq j'}}(S_+^{(j)}S_-^{(j')}e^{ik(x_j-x_{j'})}+\text{H.c.})\right\rangle_{t'}\frac{\sin[|(\hbar\omega_k-E)|(t-t')/\hbar]}{|(\hbar\omega_k-E)|/\hbar}\,,\end{aligned} \tag{27}$$

where $\rho_l(\hbar\omega_k)$ is the density of vibrational modes in the linear chain. We have replaced the sum over the modes by an integral, assuming that the energy of the modes forms a continuum. The integration in energy involves integration over k values both negative and positive. Since some quantities in the integrals depend on the sign of k we will perform the integration separately, starting with the positive k values. Let us note that the function

$$\frac{\sin[|(\hbar\omega_k-E)|(t-t')/\hbar]}{|(\hbar\omega_k-E)|/\hbar}$$

peaks at $\hbar\omega_k=E$ and rapidly decreases when we move away from this value. In the interval where the above function is appreciably different from zero, the quantitiy $|\epsilon_k|^2\rho_l(\hbar\omega_k)$ can be assumed to be constant and equal to the value $|\epsilon_{k_0}|^2\rho_l(E)$ that it takes at resonance. As far as the second integral in Eq. (27), the quantity

$$\langle(S_+^{(j)}S_-^{(j')}e^{ik(x_j-x_{j'})}+\text{H.c.})\rangle_{t'}$$

can be taken out of the integral and replaced by its value at $k=k_0$, provided that $e^{ik(x_j-x_{j'})}$ do not vary substantially as long as k runs over the effective integration region. For this to be true it should be

$$x_j-x_{j'}<l,\quad l=1/\Delta k\approx v/\Delta\omega_k\,. \tag{28}$$

The quantity Δk is the effective range of integration over k values, v is the velocity of sound, and $\Delta\omega_k$ is the width of the line. We note also that the superradiant contribution due to two impurities whose distance is greater than l results to be very small.[12] We assume in the following that in our case the distance between any two impurities is short enough to comply with condition (28).[13] However, it is likely that the limitations that this condition imposes in the acoustic case are more severe than those involved in the optical case, where condition (28) should also be satisfied in Dicke superradiant emission (see Ref. 6, p. 105). The reason for this lies essentially in the different order of magnitude of the velocities in the two cases.

With this assumption the integration that appears in both terms on the right-hand side of Eq. (27) reduces to

$$\int_0^{\hbar\omega_{\max}} \frac{\sin[|(\hbar\omega_k - E)|(t-t')/\hbar]}{|(\hbar\omega_k - E)|/\hbar}\, d(\hbar\omega_k) \cong \pi\hbar, \qquad (29)$$

and Eq. (27) becomes

$$\begin{aligned}\int_0^{\hbar\omega_{\max}} & d(\hbar\omega_k)\rho_l(\hbar\omega_k)\langle a_k^* a_k\rangle_t \\ &= \pi\hbar^{-1}|\epsilon_{k_0}|^2\rho_l(E)\int_0^t dt'\left\langle 2\sum_j (S_z^{(j)}+\tfrac{1}{2})\right\rangle_{t'} \\ &\quad + \frac{\pi}{2}\hbar^{-1}|\epsilon_{k_0}|^2\rho_l(E)\int_0^t dt'\left(\left\langle \sum_{\substack{j,j'\\ j\neq j'}} (S_+^{(j)}S_-^{(j')}e^{ik_0(x_j-x_{j'})} + \text{H.c.})\right\rangle_{t'} + \left\langle \sum_{\substack{j,j'\\ j\neq j'}} (S_+^{(j)}S_-^{(j')}e^{-ik_0(x_j-x_{j'})} + \text{H.c.})\right\rangle_{t'}\right),\end{aligned} \qquad (30)$$

where the integral relative to the negative k values has been added explicitely.

By deriving Eq. (30) with respect to time and using relation (12) we have for the magnetization

$$\frac{d}{dt}\left\langle \sum_j S_z^{(j)}\right\rangle_t = -\frac{1}{T_{1l}}\left\langle \sum_j (S_z^{(j)}+\tfrac{1}{2})\right\rangle_t - \frac{1}{4T_{1l}}\left[\langle S(k_0,t)\rangle + \langle S(-k_0,t)\rangle\right], \qquad (31)$$

where

$$T_{1l} = [(2\pi/\hbar)|\epsilon_{k_0}|^2\rho_l(E)]^{-1} \qquad (32)$$

is the spin phonon-relaxation time in the linear chain and, respectively,

$$S(k_0,t) = \sum_{\substack{j,j'\\ j\neq j'}} (S_+^{(j)}S_-^{(j')}e^{ik_0(x_j-x_{j'})} + \text{H.c.}),$$

$$S(-k_0,t) = \sum_{\substack{j,j'\\ j\neq j'}} (S_+^{(j)}S_-^{(j')}e^{-ik_0(x_j-x_{j'})} + \text{H.c.}).$$

Equation (31) describes the time behavior of the magnetization in the purely superradiant case. The first term on the right-hand side gives the usual spontaneous decay while the second term is the superradiant contribution. To derive the time dependence of this term we examine the Heisenberg equations for $S(k_0,t)$ and $S(-k_0,t)$.

V. EQUATIONS FOR THE SPINS

The equation for $S(k_0,t)$ gives

$$\begin{aligned}\frac{dS(k_0,t)}{dt} &= -\left(\frac{i}{\hbar}\right)[S(k_0,t),(\mathcal{H}_0+V_1)] \\ &= -\left(\frac{i}{\hbar}\right)[S(k_0,t),V_1] \\ &= \left(\frac{2i}{\hbar}\sum_k\sum_{j,j'}(S_z^{(j')}e^{i(k-k_0)x_{j'}}) \times(\epsilon_k a_k S_+^{(j)}e^{ik_0x_j} - \text{H.c.})\right) \\ &\quad + \text{H.c.},\end{aligned} \qquad (33)$$

where the sum is extended to both positive and negative k values. Due to the presence of phonon operators the relevant terms in this sum should be those inside the emission band. Therefore if condition (28) is satisfied by every couple of impurities and k is a positive wave vector which belongs to a mode inside the band we may write

$$\sum_{j'} S_z^{(j')}e^{i(k-k_0)x_{j'}} \cong \sum_{j'} S_z^{(j')}, \qquad (34)$$

where the origin of the x axis is assumed to coincide with the position of one of the impurities.

Let us consider now the modes inside the band but having negative k. Using Eq. (34),

$$\sum_{j'} S_z^{(j')}e^{i(k-k_0)x_{j'}} \cong \sum_{j'} S_z^{(j')}e^{-2ik_0x_{j'}}. \qquad (35)$$

For this expression to be appreciably different from zero the z component of the spins should vary, as we move from one site to the others, within distances comparable with the wavelength of the resonant mode. Although this possibility cannot be *a priori* excluded we consider here only solutions in which the spin-population difference does not vary as rapidly[14] and assume

$$\sum_{j'} S_z^{(j')} e^{-2ik_0 x_{j'}} = 0 . \qquad (36)$$

Taking into account approximations (34) and (36), Eq. (33) becomes

$$\frac{dS(k_0,t)}{dt} = \Big[\frac{2i}{\hbar}\Big(\sum_{j'} S_z^{(j')}\Big) \times \sum_{k>0,j} (\epsilon_k a_k S_+^{(j)} e^{ikx_j} - \text{H.c.})\Big] + \text{H.c.}, \qquad (37)$$

where condition (28) has been used again to replace $e^{ik_0x_j}$ by e^{ikx_j}. By using Eq. (10) we finally get

$$\frac{dS(k_0,t)}{dt} = \Big[2\Big(\sum_{j'} S_z^{(j')}\Big)\sum_{k>0}\frac{da_k^* a_k}{dt}\Big] + \text{H.c.}, \qquad (38)$$

and analogously

$$\frac{dS(-k_0,t)}{dt} = \Big[2\Big(\sum_{j'} S_z^{(j')}\Big)\sum_{k<0}\frac{da_k^* a_k}{dt}\Big] + \text{H.c.} \qquad (39)$$

We observe that position (36) decouples $S(k_0,t)$ from $S(-k_0,t)$ in the sense that the equation for $S(k_0,t)$ does not contain any more terms relative to negative wave vectors and vice versa.

For Eqs. (38) and (39) to be coupled to Eq. (31) we must go from operators to average values. Since (38) and (39) are first-order equations this task can be accomplished by taking the average values of the operators on both sides of the equations. Unfortunately this procedure leads to quantities such as

$$\Big\langle\Big(\sum_j S_z^{(j)}\Big)\frac{da_k^* a_k}{dt}\Big\rangle$$

that are not easy to handle. In fact the exact evaluation of these terms would require the solution of the Heisenberg equation for the product operator and it is likely that in this way one would get an endless chain of equations that never form a closed system. To overcome this difficulty it seems necessary to adopt the so-called decoupling procedure and to replace $\langle(\sum_j S_z^{(j)})da_k^* a_k/dt\rangle$ by $\langle\sum_j S_z^{(j)}\rangle\langle da_k^* a_k/dt\rangle$.[15] Then Eq. (38) becomes

$$\frac{dS(k_0,t)}{dt} = 4\Big\langle\sum_j S_z^{(j)}\Big\rangle\Big\langle\sum_{k>0}\frac{da_k^* a_k}{dt}\Big\rangle = -2\Big\langle\sum_j S_z^{(j)}\Big\rangle d\Big\langle\sum_j S_z^{(j)}\Big\rangle\Big/dt , \qquad (40)$$

where the equality

$$\Big\langle\sum_{k>0}\frac{da_k^* a_k}{dt}\Big\rangle = -\tfrac{1}{2}d\Big\langle\sum_j S_z^{(j)}\Big\rangle\Big/dt \qquad (41)$$

is used. In fact, due to the symmetry of the system and of the initial condition we should have

$$\Big\langle\sum_{k>0}\frac{da_k^* a_k}{dt}\Big\rangle = \Big\langle\sum_{k<0}\frac{da_k^* a_k}{dt}\Big\rangle , \qquad (42)$$

which together with Eq. (12) leads to Eq. (40).

VI. SOLUTION FOR THE COUPLED SYSTEM

Time derivative of Eq. (31) gives

$$d^2\Big\langle\sum_j S_z^{(j)}\Big\rangle\Big/dt^2 = -\frac{1}{T_{1l}}d\Big\langle\sum_j S_z^{(j)}\Big\rangle\Big/dt + \frac{1}{T_{1l}}\Big\langle\sum_j S_z^{(j)}\Big\rangle d\Big\langle\sum_j S_z^{(j)}\Big\rangle\Big/dt , \qquad (43)$$

where $d\langle S(k_0,t)\rangle/dt$ has been replaced using Eq. (40) and the same has been made for $d\langle S(-k_0,t)\rangle/dt$. The boundary conditions for Eq. (43) are

$$\Big\langle\sum_j S_z^{(j)}\Big\rangle_{t=0} = \frac{N}{2}, \quad \frac{d\langle\sum_j S_z^{(j)}\rangle}{dt}\Big|_{t=0} = -\frac{N}{T_{1l}} . \qquad (44)$$

The second of these conditions can be immediately derived from Eq. (31).

Let us define the normalized population difference between the levels as

$$u(t) = 2\Big\langle\sum_j S_z^{(j)}\Big\rangle\Big/N . \qquad (45)$$

Then Eq. (41) gives

$$\frac{d^2u}{dt^2} = -\frac{1}{T_{1l}}\frac{du}{dt} + \frac{1}{T_R}u\frac{du}{dt} , \qquad (46)$$

where

$$T_R = 2T_{1l}/N . \qquad (47)$$

Integration on both sides of Eq. (46) gives

$$\frac{du}{dt} = -\frac{1}{T_{1l}}u + \frac{1}{2T_R}u^2 + A . \qquad (48)$$

The constant A can be found by using the second of the boundary conditions and is in the form

$$A = -(1/2T_R + 1/T_{1l}) . \qquad (49)$$

Further integration of Eq. (48) can be performed by separating the variables. The final result that satisfies the first of the boundary conditions is

$$u(t) = \tanh[(T_D - t)/2T_R] , \qquad (50)$$

where the time delay

$$T_D = T_R \ln(\tfrac{1}{2}N) \qquad (51)$$

has been introduced. In the derivation of both Eqs.

(50) and (51), $N \gg 1$ has been assumed. Solution (50) shows typical features of the superradiant emission.[16] T_R measures the duration of the sudden decay after a comparatively slow relaxation which lasts approximately T_D. Both T_D and T_R are much shorter than T_1.

VII. CONCLUSIONS

Let us compare our results with those obtained in the semiclassical approach.[1] To this scope we put in Eq. (46),

$$u(t) = \cos\phi(t) \tag{52}$$

and we have

$$-\frac{d\phi}{dt}\sin\phi = -\frac{1}{T_{1l}}\cos\phi + \frac{1}{2T_R}\cos^2\phi - \left(\frac{1}{2T_R} + \frac{1}{T_{1l}}\right). \tag{53}$$

If we neglect spontaneous relaxation this equation becomes

$$\frac{d\phi}{dt} = \frac{1}{2T_R}\sin\phi \tag{54}$$

which is almost identical to Eq. (15) of Ref. 1, if there the dependence of ϕ on the position is neglected[14] and integration over x is performed. The difference of a factor of 2 should be due to the fact that in the semiclassical treatment stimulated emission due to the phonons emitted by the spin system has not been neglected, as we have made in the derivation of Eq. (17).

Finally let us consider the essential features of the behavior of the system in the phonon-avalanche regime. The role played by T_R in superradiant emission is played in this case by the phonon interruption time defined in our model as

$$\Delta t = T_{1l}\, n/N \tag{55}$$

where n is the number of modes "in speaking terms" with the spins. We note that in the one-dimensional model T_R is shorter than Δt except when $n \leq 2$. Unlike superradiance, the avalanche relaxation stops when the populations of the levels are equal and the band of the emitted phonons narrows during the process.[3] The lack of band-narrowing phonomena in the particular type of superradiance here considered[14] should be explained by the fact that we have neglected the interaction of the emitted phonons with the spins.

ACKNOWLEDGMENT

It is a pleasure to thank F. Persico for his careful reading of the manuscript and much-appreciated comments.

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[12] The length l is the analogy of the "maximum cooperation length" introduced in the optical case by Arecchi and Courtens (Ref. 7).

[13] We note that in the model here used the maximum distance between two paramagnetic impurities coincides with the length of the harmonic chain. This circumstance, which is not essential to the model, could lead the condition (28) to be not satisfied by all couples of impurities but only by the great majority.

[14] The analogy of this approximation in the optical case leads to the behavior indicated as "limited superradiance" [I. P. Herman, J. C. MacGillivray, N. Skribanowitz, and M. S. Feld, *Proceedings of the Vail Conference on Laser Spectroscopy, 1973* (Plenum, New York, 1973)] or "pure superfluorescence" (Ref. 5). Its characteristic lies in the spatial quasiuniformity throughtout the sample of the molecular excitation and in the absence of ringing in the emitted radiation. Our solutions present a similar behavior. The possibility of solutions having a strictly uniform magnetization throughout the linear chain will be discussed elsewhere.

[15] Careful analysis on the validity of this procedure can be found in Ref. 9.

[16] Let us consider, for example, the pioneering analysis of Rehler and Eberly (Ref. 8). Apart from the different mathematical technique, the approach used by these authors seems to be different also for the *a priori* assumption that a well-defined phase correlation exists among the states of different molecules. Nevertheless the results are formally similar to ours [see Ref. 8, Eq. (5.4)] provided that the parameter μ there defined is set equal to 2 as it should be in our one-dimensional case.