^{3}He phase transition in dilute mixtures of ^{3}He in superfluid ^{4}He

Douglas Fay
Abteilung für Theoretische Festkörperphysik, Universität Hamburg, Hamburg, West Germany
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A theory of pairing, which includes many-body effects and is exact in the limit of low density, is employed to calculate a pairing interaction for the ^{3}He component which contains concentration-dependent effects. Using the latest ^{3}He effective potentials the maximum transition temperature at low pressure is estimated to be of order 10^{-4} mK. At high pressure both s- and p-state transitions may occur at temperatures as high as the 0.1-mK range.

I. INTRODUCTION

The recent discovery of the superfluid phase transition in liquid ^{3}He has stimulated renewed interest in the possibility of observing a transition in the ^{3}He component of dilute mixtures of ^{3}He in ^{4}He.[1] The usual model for mixtures at very low temperatures is that of a system of low-density Fermions of effective mass m_0^* interacting via an effective interaction v_{eff}. Within this context the ^{4}He is assumed to be in its ground state and serves only as an "aether" in which the ^{3}He atoms move and which modifies their mass and interaction. A number of authors have used this model to calculate the transition temperature directly from v_{eff} within the BCS pairing theory. We have calculated the pairing interaction and T_c employing the latest phenomenological effective potential[2] and including concentration dependent many-body effects.

Our starting point is a general theory of pairing in low-density systems developed by Layzer and the author.[3,4] In this theory, which will be discussed in more detail in Sec. II, the pairing interaction U_L for pairs with orbital angular momentum L is written as a sum of a "BCS" part U_L^{BCS} and a part δU_L due to many-body contributions to the irreducible particle-particle scattering vertex. U_L^{BCS} results when this vertex is approximated by the bare interparticle potential (or v_{eff} in the application to mixtures). It was shown in Refs. 3 and 4 that δU_L predominates over U_L^{BCS} in the limit of very low density for pairing with $L>0$. Thus, since the ^{3}He component of the mixture is assumed to be low density and $L=1$ pairing may be appropriate at certain concentrations, the δU correction should be included. It turns out, however, that at the relevant concentrations, δU_L is a relatively small correction which tends to suppress pairing in s states and enhance it in p states.

δU_L depends explicitly on the ^{3}He concentration and can be calculated exactly to leading order in the concentration. Unfortunately, due to our present lack of knowledge of the effective interaction, the concentration dependence of U_L^{BCS} cannot yet be taken into account in a consistent way. The leading contribution to U_L^{BCS} is directly related to the "free-space" scattering phase shift calculated from v_{eff}. Since v_{eff} is obtained from experiments at several concentrations[2] the phase shifts implicitly contain concentration dependent effects. In addition, for $L=0$, we have also included an explicitly concentration-dependent term in U_0^{BCS} [see Eq. (12)] which has been neglected in most other calculations and tends to suppress s state pairing. Owing to these difficulties the magnitudes of the resulting T_c's should be viewed with caution.

Briefly, the results are as follows. At zero pressure the p state provides the highest T_c for Fermi momentum $k_F > 0.24$ Å^{-1} (^{3}He concentration $> 1.9\%$). The transition temperatures are discouragingly low: the maximum s-state T_c is 1.3×10^{-5} mK at $k_F = 0.15$ Å^{-1} (0.6% ^{3}He) and the highest p-state T_c of 4×10^{-4} mK occurs at the maximum concentration of about 6.5%. This experimentally unfortunate situation may improve at higher pressure. We have also done calculations at 10 and 20 atm using the "speculative" effective potentials of Landau *et al.*[5] and find the highest T_c's at 20 atm.[6] Indeed, both an s-state transition at low concentration ($\sim 1\%$) and a p-state transition near the limiting concentration may occur at temperatures that should soon be experimentally accessible. It must be stressed however that the high-pressure results are only tentative; further experimental investigation of the high-pressure region is necessary not only to seek the transition itself but also to obtain enough additional data to accurately determine the effective potential.

In Sec. II we discuss pairing in low-density systems in general and in Sec. III application is made to ^{3}He-^{4}He mixtures. In Sec. IV the numerical results are presented.

II. PAIRING IN LOW-DENSITY FERMION SYSTEMS

Before considering the application to mixtures we outline the general theory of pairing in low-density systems.[3,4] This theory provides exact results at low density for arbitrary L-state pairing in systems with short-range (including hard-core) interactions. The starting point is the exact gap equation linearized at $T = T_c$

$$\varphi(\vec{p}, \omega_n) = -T_c \sum_{\omega_m} \sum_{k} I(\vec{p}, \vec{k}; \omega_n, \omega_m) G(\vec{k}, \omega_m) \times G(-\vec{k}, -\omega_m) \varphi(\vec{k}, \omega_m) \ . \quad (1)$$

G is the exact normal-state single particle propagator and $I(p, k)$ is the irreducible interaction vertex for an antiparallel-spin particle pair scattering from $(p, -p)$ to $(k, -k)$. We note that for calculation of T_c in a system without long-range magnetic order in the unpaired state it is sufficient to consider only the pairing amplitude for pairs in antiparallel spin states.[7] Singlet and triplet amplitudes are obtained as the appropriate combinations of the antiparallel-spin amplitude.

It was shown in Refs. 3 and 4 that in the low-density limit Eq. (1) reduces to the BCS "weak-coupling" form with

$$I \to v'(\vec{p}, \vec{k}) = v(\vec{p}, \vec{k}) + \mathrm{Re}\delta I(\vec{p}, \vec{k}; 0, 0) \ , \quad (2)$$

where v is the primary interatomic interaction and δI is the many-body contribution. This reduction to weak-coupling form is expected to be a reasonable approximation also at higher density so long as T_c/T_F is small and $I(p, k)$ is not strongly energy dependent. Following Emery and Sessler[8] (ES) a pairing interaction U_L can now be defined such that $U_L < 0$ is the transition criterion and

$$T_C^L = 4.56(m/m^*) T_F \exp(\pi m/2m^* U_L) \ . \quad (3)$$

In the ES formalism U_L is obtained from the following set of equations:

$$U_L = (m/k_F)\langle k_F, L | v' | \psi(k_F, L)\rangle \ , \quad (4)$$

$$|\psi(k_F, L)\rangle = |k_F, L\rangle - \bar{G}_L v' |\psi(k_F, L)\rangle \ , \quad (5)$$

$$\langle r | k, L\rangle = k r j_L(kr) \ , \quad (6)$$

$$\bar{G}_L = \frac{1}{\pi} \int_0^\infty dk \frac{\tanh(\frac{1}{2}\beta_c \epsilon_k)}{\epsilon_k} (|k\rangle\langle k| - |k_F\rangle\langle k_F|) , \quad (7)$$

where $\beta_c = (k_B T_c)^{-1}$ and the energy ϵ_k is measured from the Fermi level.

A convenient expansion parameter at low density is $x \equiv k_F f_0 \propto (\text{density})^{1/3}$, where f_0 is the zero-energy s-wave scattering amplitude (the negative of the scattering length). In a hard-sphere gas, for example, $f_0 = -R$. For the ^{3}He component of mixtures x lies in the range $0 < x < 0.4$.

One now writes

$$U_L = U_L^{\mathrm{BCS}} + \delta U_L \ , \quad (8)$$

with U_L^{BCS} defined as the pairing interaction for $\delta I = 0$. The principal and perhaps somewhat surprising result[3,4] is that in the low-density limit, $\delta U_L \propto \delta I_L \propto x^2$, while $U_L^{\mathrm{BCS}} \propto x^{2L+1}$. In this limit δU_L is repulsive for $L = 0$ and attractive for $L > 0$, independent of the sign or strength of v. Thus, for $L > 0$, δU_L predominates at low enough density. Indeed, a hard-sphere gas should be condensed with p-state pairing at sufficiently low density and temperature.[3]

To provide background for the application to mixtures, we consider now the low-density limit in somewhat more detail. The leading contributions to δI at low density are obtained by replacing v with the Galitskii-Feynman particle-particle t matrix[9] in the second- and higher-order diagrams for I. Note that the first-order I diagram v (which is by definition not included in δI) must not be replaced by t since I must be irreducible in the particle-particle channel. The replacement can be done however in diagrams for δI and has the nice side effect of removing the divergence difficulties associated with a hard-core interaction. As a matter of fact, a hard-core v also causes no problem in the calculation of U_L^{BCS} since Eqs. (4) and (5) with $v' \to v$ are of t-matrix form and wash out the singular effect of the hard core. The actual solution of the equations must of course be carried out in position space when $v(r)$ has no Fourier transform.[8] The diagrammatic representation of the integral equation satisfied by t is shown in Fig. 1. Note that no exchange diagram is included; we do perturbation theory in unantisymetrized form in terms of interaction lines. The diagrams contributing to δI at second order (in t) are shown in Fig. 2.

Galitskii[9] has shown that the t matrix has the following form at low density:

$$(-m/4\pi) t = f + \delta f \ , \quad (9)$$

where f denotes the standard free-space scattering amplitude and δf contains the many-body effects of the exclusion principle and off-the-energy-shell

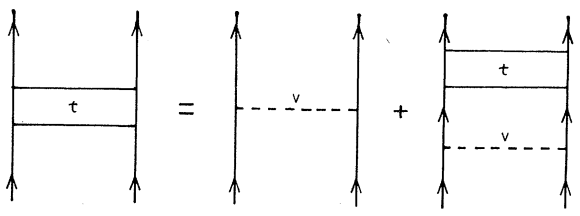

FIG. 1. Particle-particle t matrix.

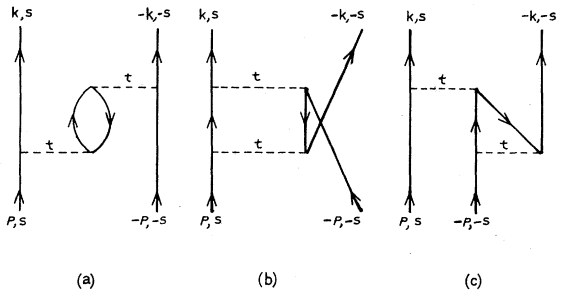

FIG. 2. Second-order (in the particle-particle t-matrix t) contributions to the many-body irreducible pair interaction function $\delta I(p,k)$ for particles with equal and opposite momenta p and spin s.

scattering. δf contains an additional factor of x relative to f. As we here only wish to indicate the origin of the leading terms in the density expansion we have written Eq. (9) in symbolic form. For a detailed account of the Galitskii theory the reader is referred to Ref. 9 or the book of Fetter and Walecka.[10] In the low-density limit t approaches the constant $(-4\pi/m)f_0$.[9] In this limit diagrams (a) and (c) of Fig. 1 cancel exactly, leaving diagram (b) as the sole contribution. This contribution is proportional to $f_0^2 Q(p+k)$, where Q is just one-half the usual polarization bubble. To leading order one can neglect the second term in Eq. (5) which leads to $\delta U_L = (m/k_F)\langle k_F, L|\delta I|k_F, L\rangle$.[4] Thus one easily verifies that the leading term in δU is order x^2.

That diagrams (a) and (c) must cancel to leading order can be seen by the following argument.[11] Since the spin projections of the entering and leaving particles in Fig. 2 are fixed, in both diagrams (a) and (c) one of the t scatterings must occur in a parallel spin state while in diagram (b) both are in antiparallel states. Now in the low-density limit where t reduces to the constant interaction f_0, diagrams (a) and (c) must be omitted since scattering of parallel spin particles by a zero-range contact interaction is forbidden by the Pauli principle.

The expansion of δU_L thus has the form

$$\delta U_L = A_L x^2 + O(x^3) \ . \tag{10}$$

The A_L have been calculated exactly[3] with the result $A_0 = 0.51$ and $A_1 = -0.05$ for $L = 0$ and 1, respectively. The x^3 contribution has been calculated approximately[4] but will not be discussed here because, as will be seen shortly, there are additional x^3 contributions in the case of mixtures which cannot yet be accurately estimated.

The expansion of $U_L^{\rm BCS}$ for $L = 0$ is

$$U_0^{\rm BCS} = -x + (4/\pi)x^2 + O(x^3) \ . \tag{11}$$

This result has been proved by direct calculation[4] from Eqs. (4)–(7) employing a model v consisting of a repulsive hard core plus a δ-function attraction designed to simulate the ^{3}He potential. The result is presumably valid for any short-range potential.

From Eqs. (10) and (11) we see that for[12] $f_0 > 0$ the system is condensed with s-state pairing as $x \to 0$. As k_F increases, the x^2 and higher-order terms will eventually cause U_0 to turn repulsive,[12] at which point p-state pairing must be considered.

III. APPLICATION TO ^{3}He-^{4}He MIXTURES

Several difficulties appear when the above low-density theory is applied to mixtures. In this case we take v to be $v_{\rm eff}$, the *effective* interaction[13,2] between two ^{3}He quasiparticles. We assume for now that $v_{\rm eff}$ is independent of the ^{3}He concentration. This assumption will be discussed shortly. Since $v_{\rm eff}$ includes contributions from the ^{4}He background medium, we must ask whether δI can also be included without "double counting." The δI diagrams in Fig. 2, for example, can be considered classically as three-body collisions in which a particle from the Fermi sea interacts with the original Cooper pair.[3] Now, as pointed out by Emery,[14] three-body collisions are, to some extent, accounted for in $v_{\rm eff}$ since the third ^{3}He must displace a ^{4}He whose effect is already included in $v_{\rm eff}$. However, the important point is that at low density the *leading* contribution to δU comes entirely from diagram (b) in which an *exchange* occurs between one member of the original pair and the third particle. Since this exchange process could not take place if the third particle were a ^{4}He, we conclude that diagram (b) is *not* contained in $v_{\rm eff}$. The same argument holds for diagram (c). Diagram (a) on the other hand is a non-exchange, density-fluctuation-type diagram and is probably, at least partially, included in the part of $v_{\rm eff}$ arising from exchange of a ^{4}He phonon. We have shown however that diagram (a) does not contribute to δU at leading order (x^2). At order x^3 and higher, contributions to δI appear which do "double count." We avoid the difficult problem of exactly how these diagrams should be handled by retaining only the leading term in the expansion of δU. Preliminary calculations show that T_c is not particularly sensitive to the x^3 contributions.

In order to include the rather large contributions to $U_L^{\rm BCS}$ at order x^{2L+3}, it is convenient to introduce the phase shift approximation.[14] This is equivalent to replacing $\bar{G}_L$ in Eq. (5) by a free-space Green's function when calculating $U_L^{\rm BCS}$. Then $U_L^{\rm BCS} \to -\tan\delta_L(k_F)$, where $\delta_L(k_F)$ is the "free-space" phase shift, due to $v_{\rm eff}$, for scattering at relative

momentum k_F. This approximation yields correctly the x^{2L+1} and x^{2L+3} contributions to $U_L^{\rm BCS}$. Concentration dependence arising from exclusion principle restrictions in $\bar{G}_L$ enters first at order x^2 for $L=0$ [see Eq. (11)] and at x^5 for $L=1$.[4] Neglecting, at $L=0$, possible corrections to the weak-coupling from of the gap equation,[15] we have

$$U_0^{\rm BCS} = -\tan\delta_0(k_F) + (4/\pi)x^2 + O(x^4) \ , \tag{12}$$

$$U_1^{\rm BCS} = -\tan\delta_1(k_F) + O(x^5) \ . \tag{13}$$

At $L=0$, the x^2 correction to the phase-shift approximation due to many-body effects leads to a factor e^{-2} in the equation for T_c^0 which has also been noted by Emery.[14] As can be seen from Fig. 3, this correction has a sizable effect at the important densities.

To actually compute f_0 and the $\delta_L(k_F)$ for mixtures at zero pressure we employed the effective potential (which is really a transition amplitude) $A(q)$ of Kuenhold and Ebner[2]:

$$A(q) = -\frac{V_c}{V}\sum_{n=0}^{4} a_n\left(\frac{q}{q_0}\right)^{2n}\exp\left[-B\left(\frac{q}{q_0}\right)^2\right], \tag{14}$$

where q is the magnitude of the momentum transfer, V is the volume, and $q_0 = 0.753$ Å^{-1}. The partial waves of the standard scattering amplitude $f_L(k)$ as a function of the relative momentum k are given by

$$f_L(k) = -\left(\frac{m_0^* V}{4\pi}\right)\left(\frac{2L+1}{2k^2}\right)\int_0^{2k} q\,dq\,A(q)P_L\left(\frac{1-q^2}{2k^2}\right). \tag{15}$$

The phase shifts required for calculation of $U_L^{\rm BCS}$ are computed from

$$\delta_L(k) = \tfrac{1}{2}\sin^{-1}[2kf_L(k)/2L+1] \ . \tag{16}$$

The zero-energy s-wave scattering amplitude is needed to determine the expansion parameter x:

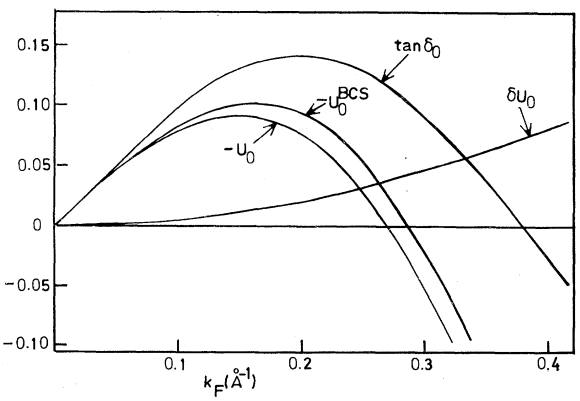

FIG. 3. Contributions to the s-state pairing interaction as functions of the Fermi momentum at zero pressure.

with $m_0^* = 2.28m_3$ in the zero-concentration limit[2] we find from Eq. (15) $f_0 = f_0(0) = 1.00$, in good agreement with Emery's result.[14]

The validity of the application of the low-density theory to mixtures depends not only on the density being low but also on the availability of an effective interaction between an isolated pair of ^{3}He atoms in the ^{4}He medium. The effective interaction we used was obtained from measurements at finite ^{3}He concentration and was assumed to be independent of concentration. A possibly significant concentration dependence was in fact suggested by Kuenhold and Ebner[2] and recently Fu and Pethick[16] have found a concentration-dependent contribution to $v_{\rm eff}$ which varies as $c^{1/3}$, where c is the ^{3}He concentration. This result, together with Eq. (15), indicates that the "free-space" scattering amplitude $f_L(k)$ and hence f_0 are actually functions of the density expansion parameter x and suggests, since $c^{1/3} \sim k_F$, that $f_L(k)$ can be written in the general form

$$f_L(k,x) = f_L(k,0) + \Delta f_L x + \cdots \ , \tag{17}$$

where we now define $x \equiv k_F f_0(0,0)$. Assuming the correctness of Eq. (17), our use of the experimentally determined $f_0(0,x)$ instead of $f_0(0,0)$ in the expansion parameter x does not give rise to error in the leading term of δU_L. The corrections entering δU at order x^3 probably do not have a significant effect on T_c unless Δf_0 turns out to be anomalously large.

The effect of concentration dependence on $U_L^{\rm BCS}$ is more serious. As with δU, the leading term ($\sim x^{2L+1}$) is not affected but, for $L=0$, corrections would enter as soon as order x^2, the order at which many-body effects have been retained in Eq. (12). While concentration-dependent effects cannot be consistently taken into account due to our lack of detailed knowledge of the effective interaction, the phase shifts calculated from the phenomenological effective interaction include the concentration dependence in some average sense and may thus provide reasonably good numerical results.

IV. NUMERICAL RESULTS

The various contributions to the pairing interaction for $L=0$ at zero pressure are shown in Fig. 3. Our $\delta_0(k)$ has the same general shape as Emery's[14] but is somewhat less attractive. The other curves in Fig. 3 are obtained from Eqs. (8), (10), and (12). The point at which the s-state pairing interaction turns repulsive is seen to be substantially lowered by the many-body corrections. In the density range of interest, δU_1 is a rather small (and attractive) correction to $U_1^{\rm BCS}$:

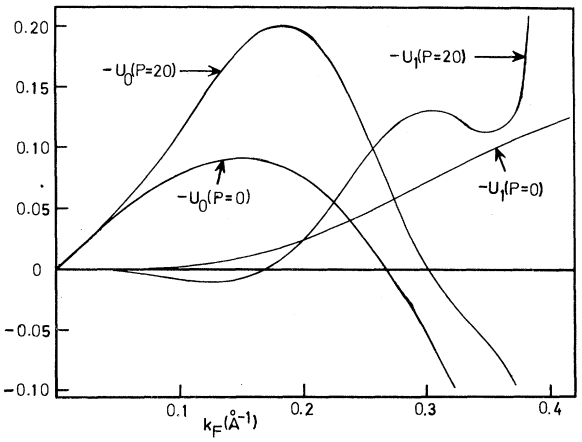

FIG. 4. Pairing interaction U_L for $L=0$ and 1 as functions of Fermi momentum at pressures $P=0$ and 20 atm.

at zero pressure and $k_F = 0.34$ Å^{-1}, $U_1^{BCS} = -0.097$ and $\delta U_1 = -0.006$. U_0 and U_1 are shown in Fig. 4 as functions of k_F. The transition temperatures quoted in the Introduction follow from Eq. (3) with $m^*/m = 1.05$.

At high pressure we employed the potential of Landau *et al.*[5] which has the same form as Eq. (14) but without the exponential factor. At 20 atm with $m_0^* = 2.72 m_3$ we find $f_0 = 0.924$. A calculation identical to that at zero pressure leads to the results shown in Fig. 4. The maximum s-state T_c is 0.24 mK at $k_F = 0.19$ Å^{-1} (~1% ^{3}He), while, for $L=1$, T_c climbs into the mK range for large k_F near the solubility limit. As mentioned in the Introduction, the quantitative high-pressure results should probably not be taken very seriously, particularly at high concentration where U_1 is sensitive to the form of the effective potential at large momentum transfers. Also, in this large k_F region, corrections to the low-density theory may be important. On the other hand, the prediction of the rather high T_c for the s-state transition at high pressure and low concentration should be less sensitive to these effects. The high-T_c values at high pressure arise essentially from the strong momentum-transfer dependence of the effective interaction of Landau *et al.* The enhanced s state T_c is due to the fact that, at high pressure, the minimum of $A(q)$ occurs for $q > 0$ (see Fig. 8 of Ref. 5). This leads to an increase in the integral in Eq. (15) and a corresponding increase in T_c. The very large increase in T_c for $L=1$ at high pressure and large k_F occurs because $A(q)$ becomes positive and large at large q where the P_L in Eq. (15) has turned negative. This behavior of $A(q)$ is best seen in Fig. 1 of Ref. 6. Further experimental investigation is required to determine whether this behavior of $A(q)$ actually occurs.

We note in passing that the predicted p-state transition would most likely occur in a Balian-Werthamer state[17] since the strong spin-fluctuation effects which lead to the stability of the Anderson-Morel state[18] in pure ^{3}He do not play an important role here.

ACKNOWLEDGMENT

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