Renormalization-group approach to a random resistor network

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A renormalization-group approach for calculating the critical exponent t for the electrical conductance of a two-dimensional random resistor network is developed. The value $t = 1.359$ (for a triangular lattice) is obtained.

Random resistor networks (RRN's) can be used as a model to represent some features of transport in disordered systems and have been widely studied (see Refs. 1–3 and references therein). The bond-percolation problem provides a particularly simple example of a RRN,[1] in which the conductance between two adjacent sites of a lattice may assume either some finite value σ or zero (with probabilities p and $1-p$, respectively). Such a RRN provides a simple model system, in which a metal-insulator transition occurs: the total conductance G of the network is zero for $p<p_c$ (percolation threshold) and increases with p for $p>p_c$. Slightly above the percolation threshold, G is proportional to $(p-p_c)^t$, and this defines the critical exponent t. This exponent is believed to be a universal constant (depending only on the dimensionality of the network). For two-dimensional networks numerical studies of Kirkpatrick[1] gave $1<t<1.3$. Watson and Leath[4] experimentally obtained $t=1.38\pm0.12$.

The conductance G is more complex than quantities such as the percolation probability $P(p)$ or the mean cluster size $S(p)$ which are usually studied in the percolation problem.[5] G depends on the topology of the "percolation channels"[1] and [unlike $P(p)$ or $S(p)$] does not have a thermodynamic analog in the Ashkin-Teller-Potts (ATP) model.[6] In what follows we shall use the term percolation problem only in its narrow sense, i.e., in connection with quantities of "thermodynamic" nature (and their critical exponents). The calculation of G and of the exponent t will be referred to as the RRN problem.

There have been some attempts[7] to derive a scaling relation which connects t to the percolation-problem exponents. These derivations are based on some specific assumptions about the structure of the "percolation channels," and the obtained relations differ from one another. No such assumptions are made in this work: we perform a direct calculation of the exponent t, using the renormalization-group ideas.[8,9]

Two renormalization-group approaches to the percolation problem have been developed recently.[10,11] In the first,[10] the renormalization of the ATP Hamiltonian is performed, and the equivalence between the ATP model and the percolation problem is used. In the second approach,[11] the original lattice is rescaled by a decimation procedure[12] and the renormalized probability p' is calculated directly. For example,[11] considering a four-site cluster $A1B2$ in a square lattice (Fig. 1) and calculating the probability p' of sites A and B being connected, one obtains

$$p' = 2p^2 - p^4 . \tag{1}$$

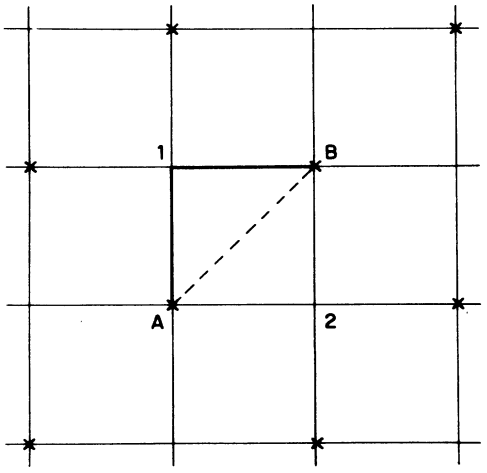

FIG. 1. Decimation procedure for a square lattice (the scaling factor $b=\sqrt{2}$). The sites remaining after scaling are indicated by crosses. An example of a nearest-neighbor (nn) bond AB on the new lattice is shown by a dashed line. One possible way in which this bond may be formed is via the nn bonds $A1, 1B$ (shown by the heavy lines) on the old lattice.

The nontrivial fixed point of Eq. (1) is $p^* = 0.618$; near the fixed point $\delta p' = \lambda \delta p$ ($\delta p = p - p^*$) with $\lambda = 1.528$ (Ref. 11).

Our extension of the ideas of Ref. 11 to the RRN may be demonstrated in the simple example of Fig. 1. We shall require the average conductance between points A and B to remain the same after the decimation of sites 1 and 2. Then, in addition to Eq. (1), we have

$$p'\sigma' = 2p^2(1-p^2)\tfrac{1}{2}\sigma + p^4\sigma = p^2\sigma . \tag{2}$$

Here σ and σ' are the conductances assigned to a nearest-neighbor (nn) bond in the original and in the rescaled lattices, respectively.

The total conductance G may be written in the form

$$G = \sigma L^{d-2} f(p) , \tag{3}$$

where d is the dimensionality of the network, L its linear dimension (measured in lattice units), and $f(p)$ is some function which near (from above) the fixed point is proportional to $(\delta p)^t$. The requirement that G should not change by scaling leads to

$$\sigma'(L/b)^{d-2}(\delta p')^t = \sigma L^{d-2}(\delta p)^t , \tag{4}$$

where b is the scaling factor. Thus, using $\delta p' = \lambda \delta p$, we find

$$t = [\ln(b^{d-2}\sigma/\sigma')]/\ln\lambda . \tag{5}$$

According to Eq. (2), at the fixed point $\sigma/\sigma' = 1/p^*$, which [with the above values of p^* and λ (Ref. 11)] leads to a reasonable value $t = 1.134$.

Alternatively, Eq. (5) may be written as

$$t = (d - 2 + \ln(\sigma/\sigma')/\ln b)\nu \tag{6}$$

where $\nu = \ln b/\ln\lambda$ is the correlation length exponent.[8,11]

The simple percolation problem discussed above (with a single parameter p) is not "renormalizable" because scaling introduces bonds between distant sites and correlations between bonds. These correlations may be taken into account by introducing more complex, multisite, bonds, which connect more than two sites.[11] Thus one arrives at a generalized percolation problem which, in addition to the bond probability p, is characterized by a set $\{p_i\}$ of independent probabilities for other types of bonds (including multisite bonds). For the same reasons the simple RRN discussed above has also to be generalized, and a set of additional conductances $\{\sigma_i\}$ has to be introduced. The way in which these conductances will be assigned to different bonds will be discussed below. Instead of Eq. (3), we now have

$$G = \sigma L^{d-2} f(p, p_1, p_2, \ldots ; \ \alpha_1, \alpha_2, \ldots) , \tag{7}$$

where $\alpha_i = \sigma_i/\sigma$. Equation (7) has a form which obviously satisfies the requirement that if all conductances $\sigma, \sigma_1, \ldots$ are multiplied by a given factor, so is G. Using arguments similar to those of Ref. 8 (pp. 110 and 111) one obtains from Eq. (7) the expression for the critical exponent t given by Eq. (5) [or Eq. (6)], where λ is now the biggest eigenvalue of the linearized renormalization-group transformation matrix. At the fixed point $\alpha_i^* = \sigma_i/\sigma = \sigma_i'/\sigma'$; hence the ratio σ/σ' in Eqs. (5) and (6) may be replaced by any of the ratios σ_i/σ_i'. Thus σ does not play any special role—any of the conductances σ_i can be singled out instead of σ in Eq. (7).

Let us represent any bond (including multisite bonds) by a line. To every type of bond we assign a different color. A microconfiguration is defined by a set of lines with their color assignment (a pair of sites can be connected directly by a number of lines of different colors). A configuration $\mathcal{G}$ is defined by a set of lines irrespective of their color assignment (two or more lines directly connecting a pair of sites are replaced by one line only). The probability $W_{\mathcal{G}}$ of the configuration $\mathcal{G}$ is the sum of the probabilities of all microconfigurations compatible with $\mathcal{G}$. The probability of a microconfiguration is easily calculated in terms of the set of independent probabilities p, $p_1, \ldots$.

Let $\mathcal{G}$ and $\mathcal{G}'$ be configurations of the initial and new lattices respectively. In the generalized percolation problem, the renormalized probabilities can be obtained from the requirement

$$W_{\mathcal{G}'}(p', p_1', \ldots) = \sum_{\mathcal{G} \in R_{\mathcal{G}'}} W_{\mathcal{G}}(p, p_1, \ldots) , \tag{8}$$

where $R_{\mathcal{G}'}$ denotes the set of configurations in the old lattice which are compatible with $\mathcal{G}'$. Equations (8) guarantee that the total probability of two or more given sites (common to both lattices) being connected (via any path) is the same in both lattices.

To obtain the renormalized conductances in the generalized RRN we shall require, in addition to Eq. (8),

$$W_{\mathcal{G}'}(p', p_1', \ldots) G_{\mathcal{G}'}^{AB}(\sigma', \sigma_1', \ldots) = \sum_{\mathcal{G} \in R_{\mathcal{G}'}} W_{\mathcal{G}}(p, p_1, \ldots) G_{\mathcal{G}}^{AB}(\sigma, \sigma_1, \ldots) . \tag{9}$$

Here, $G_{\mathcal{G}'}^{AB}$ and $G_{\mathcal{G}}^{AB}$ are the total conductances between sites A and B in the configurations $\mathcal{G}$ and $\mathcal{G}'$, respectively. Eqs. (9) guarantee the electrical equivalence of both lattices in the sense that the average conductance between two given sites is preserved in the renormalization transformation.

Our assignment for the set of conductances

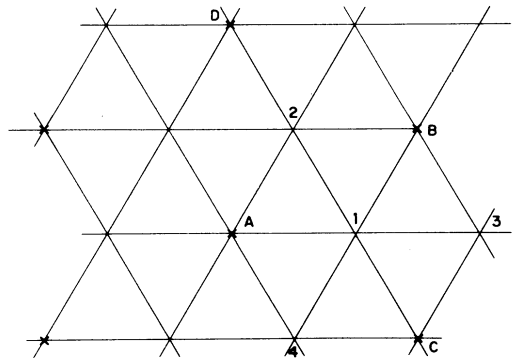

FIG. 2. Decimation procedure for a triangular lattice (the scaling factor $b=\sqrt{3}$). The remaining sites are indicated by crosses.

$\{\sigma, \sigma_t\}$ is done as follows: to each bond l in the configuration $\mathcal{G}$ we assign a conductance $\sigma_{\mathcal{G}}^{(l)}$ (some of the bond conductances may be equal if the bonds are symmetrically situated).

Equations (8) are sufficient for the evaluation of the fixed point probabilities $\{p^*, p_i^*\}$ and the biggest eigenvalue λ of the renormalization-group transformation matrix. The critical exponent t is then determined by the ratio σ'/σ, which is evaluated from Eqs. (9), where all probabilities assume their fixed-point values.

We now perform an approximate renormalization calculation for a triangular lattice (Fig. 2). We shall limit ourselves to only three independent probabilities: of a nn bond—p, of a next-nn bond—q, and of a three-site bond (connecting three sites on an elementary triangle like $A12$)—r. In terms of the ATP model this corresponds to a Hamiltonian with nn, next nn, and three-spin (in an elementary triangle) interactions. We shall also limit ourselves to three conductances only, and assign them as follows: σ to a nn bond (in a configuration in the sense defined above) which is not part of an elementary triangle, σ_Δ to a nn bond which belongs to an elementary triangle, and σ_1 to a next-nn bond.

In the actual evaluation of the sums in Eqs. (8) and (9) we shall further limit ourselves to configurations $\mathcal{G}$ compatible with a given configuration $\mathcal{G}'$ within a small cluster only. For the evaluation of p' we use the cluster $A12B$; for q', the cluster $C12D$, and for r', the cluster $1ABC$, from which the following equations result:

$$\begin{aligned} p' &= q + \bar{q}[r^2 + 2r\bar{r}(2p\bar{p} + p^2) \\ &\qquad + \bar{r}^2(2p^2 - p^4 + 2p^3\bar{p}^2)], \\ q' &= p^3, \\ \bar{r}'p'^3 + r' &= p^3, \end{aligned} \tag{10}$$

where the notation $\bar{x} \equiv 1 - x$ is used. Equations (10) lead to the fixed point values for the probabilities $p^* = 0.4215$, $q^* = 0.0749$, $r^* = 0$. From the linearization of Eqs. (10) near the fixed point one obtains $\lambda = 1.549$, $\nu = 1.256$. This differs slightly from the value $\nu = 1.284$ obtained in Ref. 11, where the cluster $A2B3C4$ (Fig. 2) was considered.

The equations for the conductances evaluated within the above mentioned clusters are

$$\begin{aligned} p'\sigma' &= \tfrac{1}{2}\sigma \times 2p^2\bar{p}^2(3p + \bar{p}) + \tfrac{1}{3}\sigma \times 2p^3\bar{p}^2 + \sigma p^4\bar{p} \\ &\qquad + \sigma_\Delta p^5 + \sigma_1 q + \frac{3\sigma\sigma_\Delta}{2\sigma + 3\sigma_\Delta} 4p^4\bar{p} + O(r), \\ q'\sigma_1' &= \tfrac{1}{3}\sigma p^3, \\ p'^3 \times \tfrac{3}{2}\sigma_\Delta' + O(r') &= \tfrac{1}{2}\sigma p^3. \end{aligned} \tag{11}$$

From Eqs. (10) and (11), the fixed point values of the parameters $\alpha_1 = \sigma_1/\sigma$ and $\alpha_\Delta = \sigma_\Delta/\sigma$ are found to be $\alpha_1^* = \alpha_\Delta^* = \frac{1}{3}\sigma/\sigma'$, where $\sigma/\sigma' = 0.5517$. This gives $t = 1.359$, which is in good agreement with the experimental value $t = 1.38 \pm 0.12$ of Ref. 4.[13]

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[13]Very recently, our attention was drawn to: (i) the recent work of R. B. Stinchcombe and B. P. Watson [J. Phys. C 9, 3221 (1976)] which also deals with a renormalization approach for percolation conductivity, (ii) the works of S. Kirkpatrick [Phys. Rev. B (to be published)] and of J. P. Straley [Phys. Rev. B (to be published)] on related topics, and (iii) more recent computer-simulation calculations by S. Kirkpatrick [Phys. Rev. Lett. 36, 69 (1976)], that put a sharper limit to a value of t: $1 < t < 1.2$.