Spin models and cluster distributions for bond and site percolation models*

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(Received 11 August 1977)

The bond and site percolation models are formulated as the limit of spin models and spin Hamiltonians are obtained for these models. The order parameters in each case provide generating functions for the site and bond distributions in the clusters. The form of the site-bond distribution is discussed and shown to be extremely narrow near p_c. Critical exponents are obtained near one dimension and shown to be the same for the two models.

I. INTRODUCTION

Percolation behavior can occur in a variety of physical systems, e.g., random magnets at low temperatures or conducting networks. Two percolation models are often considered theoretically—the bond and site models. In the bond model, bonds connecting nearest-neighbor sites of a lattice are placed at random to form clusters. All bonds are independent and p and $1-p$ is the probability of a bond being present or absent, respectively. In the site model, the sites of the lattice are randomly occupied and p and $1-p$ is the probability of a site being occupied or unoccupied, respectively. Nearest-neighbor occupied sites are connected to form clusters. Reviews of percolation problems have been given by Shante and Kirkpatrick[1] and by Essam.[2]

In this paper the bond and site percolation models are formulated as the limit of interacting spin models. The order parameters in each case is related to the generating functions for the distribution of sites and bonds in the connected clusters. The generating function is defined by

$$C(h,g)=\sum_{s,b}P_{s,b}e^{-hs-gb}\ . \tag{1.1}$$

$P_{s,b}$ is the probability that a site (occupied in the site model) lies in a cluster of s sites and b bonds. Only finite clusters contribute to the sum in (1.1) so that

$$C(0,0)=\begin{cases}1, & p<p_c,\\ 1-P, & p>p_c,\end{cases} \tag{1.2}$$

where p_c is the critical concentration and P is the probability that a site (occupied in the site model) is in the infinite cluster. Also of interest is the distribution of bonds in clusters of size s and for this purpose we define a function

$$D(s,g)=\sum_{b}P_{s,b}e^{-gb}\ . \tag{1.3}$$

In this paper we study the form of the generating functions C and D and the distribution function $P_{s,b}$ near the critical concentration p_c.

Insight into percolation models is obtained by formulating them as the limit of a spin model. This was first done by Kasteleyn and Fortuin[3] for the bond model and their results were generalized in Ref. 4 to give information on the distribution of sites and bonds in the clusters. It is shown here that the site percolation model can also be formulated as the limit of a spin model and information on the distribution of sites and bonds in clusters can also be obtained.

We first discuss the spin representation of these percolation models and derive the spin Hamiltonians in Sec. II. In Sec. III the mean-field theory of these models is presented. In Sec. IV the Migdal method is used to discuss critical exponents near one dimension and continuum models are derived and used to discuss critical exponents near six dimensions. The results are discussed in Sec. V.

II. SPIN MODELS

A. Bond model

On each site of a regular lattice of N sites and $\frac{1}{2}zN$ nearest-neighbor bonds (z is the coordination number) we attach a q-component spin λ_i and define a partition function (we use a subscript B for the bond model)

$$Z_B=\mathrm{Tr}_\lambda\prod_{nn}\{1+v\delta_{\lambda_i,\lambda_j}\exp[g(\delta_{\lambda_i,1}-1)]\}$$
$$\times\exp\left[h\sum_i(\delta_{\lambda_i,1}-1)\right]\ , \tag{2.1}$$

where $v=p/(1-p)$ and the product is over all nearest-neighbor pairs of sites. The two terms in the curly brackets denote the absence or presence of a bond connecting nearest-neighbor sites i and j, respec-

tively. Thus if i and j are connected by a bond the weight factor is

$$v\delta_{\lambda_i,\lambda_j}\exp[g(\delta_{\lambda_i,1}-1)] ,$$

while if no bond connects them the weight factor is 1. These factors then require that all the spins in a connected cluster take on the same value. There is a factor e^{-g} for each bond in a connected cluster if the spins $\lambda_i \neq 1$ and 1 otherwise. The field h acts on spins in the states $\lambda_i \neq 1$. After taking the trace on λ a cluster of s sites and b bonds is weighted by a factor $v^b[1+(q-1)e^{-hs-gb}]$.

When the product in (2.1) is multiplied out all possible configurations of clusters on the lattice are obtained. It is shown in Ref. 4 that the generating function (1.1) is given by

$$C_B(h,g) = -\frac{1}{N}\left(\frac{\partial^2}{\partial h\,\partial q}\ln Z_B\right)_{q=1} . \tag{2.2}$$

The partition function can be written in the form $Z_B = \mathrm{Tr}_\lambda e^{-H'_B}$, where

$$H_B' = -\sum_{nn}(K_B\delta_{\lambda_i,\lambda_j} + k\delta_{\lambda_i,1}\delta_{\lambda_j,1}) - h\sum_i(\delta_{\lambda_i,1}-1) , \tag{2.3}$$

and $e^{K_B} = 1+ve^{-g}$, $e^{K_B+k} = 1+v$. We introduce a representation[4] where the spins take on the values of the q roots of unity ($\lambda = e^{2\pi is/q}$, $s = 1, \ldots, q$)

$$q\delta_{\lambda_i,\lambda_j} = 1 + \sum_r \lambda_i^r\lambda_j^{q-r} ,$$
$$q\delta_{\lambda,1} = 1 + \sum_r \lambda^r , \tag{2.4}$$

where $\sum_r \equiv \sum_{r=1}^{q-1}$. The Hamiltonian can be written

$$H_B' = -\frac{zN}{2q^2}(K_Bq+k) - \frac{hN}{q}(1-q) - \frac{K_B}{q}\sum_{nn}\sum_r \lambda_i^r\lambda_j^{q-r} - \frac{k}{q^2}\sum_{nn}\sum_{r_1,r_2}\lambda_i^{r_1}\lambda_j^{r_2} - \frac{1}{q}\left(h+\frac{zk}{q}\right)\sum_{i,r}\lambda_i^r . \tag{2.5}$$

The generating function is

$$C_B(h,g) = 1 - \left(\frac{\partial M_B}{\partial q}\right)_{q=1} , \tag{2.6}$$

where

$$M_B = \left\langle \sum_r \lambda_i^r \right\rangle = \mathrm{Tr}_\lambda \sum_r \lambda_i^r e^{-H_B}/\mathrm{Tr}_\lambda e^{-H_B} . \tag{2.7}$$

The Hamiltonian H_B is obtained from H_B' by setting $q=1$ and omitting constants

$$H_B = -K_B\sum_{nn}\sum_r \lambda_i^r\lambda_j^{q-r} - k\sum_{nn}\sum_{r_1,r_2}\lambda_i^{r_1}\lambda_j^{r_2} - (h+zk)\sum_{i,r}\lambda_i^r . \tag{2.8}$$

M_B is the order parameter for the bond model. The field $k \simeq pg$ enters the spin Hamiltonian both as a field acting on the spins and in the coupling between the spins.

B. Site model

The site percolation model can also be formulated as the $q=1$ limit of a spin model. We again consider a regular lattice of N sites and coordination number z. We label the bonds attached to site i by subscripts im with $m = 1, \ldots, z$. A q-component spin λ_{im} is placed on each bond. This notation for the bond spins is convenient but not unique as there are two ways of labeling each spin depending on which site we choose of the two attached to a bond. The collection of all the spins on the bonds attached to site i is denoted by $\{\lambda_i\}$. The partition function is now given by (we use a subscript S for the site model)

$$Z_S = \mathrm{Tr}_\lambda \prod\left[1 + v\Delta(\{\lambda_i\})\exp\left(\frac{h}{z}\sum_m(\delta_{\lambda_{im},1}-1)\right)\right] \times\exp\left(\frac{g}{2}\sum_{im}(\delta_{\lambda_{im},1}-1)\right) , \tag{2.9}$$

where $v = p/(1-p)$, the product over all sites, and $\Delta(\{\lambda_i\}) = 1$ if all the spins $\{\lambda_i\}$ at site i are the same and is zero otherwise. The two terms in the large square brackets denote whether a site is unoccupied or occupied, respectively. Thus an unoccupied site is weighted by 1 and an occupied site is weighted by a factor

$$v\Delta(\{\lambda_i\})\exp\left(\frac{h}{z}\sum_m(\delta_{\lambda_{im},1}-1)\right) .$$

Thus all the spins surrounding an occupied site must have the same value and all the spins in a connected cluster take on the same value. This condition follows because nearest-neighbor sites have a spin in common

which then serves to connect up all the sites in a cluster. The field h acts on spins in the states $\lambda \neq 1$ and is conjugate to the number of sites in a cluster. The effect of the field g is a bit different than in the bond model. There are three types of nearest-neighbor bonds: (i) bonds connecting two unoccupied sites (free bonds), (ii) bonds connecting an unoccupied and an occupied site (external bonds), and (iii) bonds connecting two occupied sites (internal bonds). Free bonds are weighted by a factor $1+(q-1)e^{-g}$ (after taking the trace on λ). A cluster of s sites, b internal bonds, and b_e external bonds is weighted by $v^s\{1+(q-1)\exp[-hs-g(b+b_e)]\}$.

The product in (2.9) generates all configurations of occupied sites on the lattice. Thus, in configuration G, if we have $NK_{s,b}(G)$ clusters of s sites and b internal bonds and $NB(G)$ free bonds we can write

$$Z_S=\sum_G[1+(q-1)e^{-g}]^{NB(G)} \times\prod_{s,b}\{v^s[1+(q-1)\times e^{-hs-g(b+b_e)}]\}^{NK_{s,b}(G)} \tag{2.10}$$

where the sum is over all configurations. The following results are then easily obtained:

$$(Z_S)_{q=1}=(1-p)^{-N}\ ,$$

$$\frac{1}{N}\left[\frac{\partial \ln Z_S}{\partial q}\right]_{q=1}=(1-p)^N\sum_G v^{N\sum_{s,b}sK_{s,b}(G)}\left[B(G)e^{-g}+\sum_{s,b}K_{s,b}(G)e^{-hs-g(b+b_e)}\right]$$

$$=Be^{-g}+\sum_{s,b}K_{s,b}e^{-hs-g(b+b_e)}\ , \tag{2.11}$$

where B is the average number of free bonds and $K_{s,b}$ is the average number of clusters (per site) of s sites and b internal bonds. (The external bonds can be eliminated through the relation $zs=2b+b_e$ if desired.) The generating function is given by

$$C_S(h,g)=-\frac{1}{Np}\left[\frac{\partial^2\ln Z_S}{\partial h\,\partial q}\right]_{q=1}=\sum_{s,b}P_{s,b}e^{-hs-g(b+b_e)}\ . \tag{2.12}$$

This is not exactly of the form (1.1) (because of the presence of b_e in the exponent) but contains the same information.

The partition function (2.9) can be written in the form $Z_S=\mathrm{Tr}_\lambda e^{-H_S'}$, where

$$H_S'=-K_S\sum_i\Delta(\{\lambda_i\})-\frac{h'}{z}\sum_i\Delta(\{\lambda_i\})\sum_m(\delta_{\lambda_{im},1}-1)-\frac{g}{2}\sum_{im}(\delta_{\lambda_{im},1}-1)\ , \tag{2.13}$$

with $e^{K_S}=1+v$, $e^{K_S-h'}=1+ve^{-h}$, and $h'\simeq ph$. A convenient representation for the $\Delta(\{\lambda_i\})$ is

$$\Delta(\{\lambda_i\})=\mathrm{Tr}_\mu\prod_m\delta_{\mu,\lambda_{im}}\ , \tag{2.14}$$

where μ is a q-component spin. Using (2.4) the Hamiltonian can be written

$$H_S'=-\frac{1}{q^z}\sum_i\mathrm{Tr}_\mu\times\prod_m\left[1+\sum_r\mu^r\lambda_{im}^{q-r}\right]\times\left[K_S+\frac{h'}{zq}\sum_{m'}\left[(1-q)+\sum_{r'}\lambda_{im'}^{r'}\right]\right]-\frac{g}{2q}\sum_{im}\left[(1-q)+\sum_r\lambda_{im}^r\right]\ . \tag{2.15}$$

The generating function (2.12) then takes the form

$$C_S(h,g)=\frac{e^{-h}}{1-p+pe^{-h}}\left[1-\left[\frac{\partial M_S}{\partial q}\right]_{q=1}\right]\ , \tag{2.16}$$

where

$$M_S=\langle\Delta(\{\lambda_i\})\sum_r\lambda_{im}^r\rangle=\mathrm{Tr}_\lambda\Delta(\{\lambda_i\})\sum_r\lambda_{im}^re^{-H_S}\Big/\mathrm{Tr}_\lambda e^{-H_S}\ . \tag{2.17}$$

The Hamiltonian H_S is obtained from H_S' by setting $q=1$,

$$H_S=-\sum_i\mathrm{Tr}_\mu\prod_m\left[1+\sum_r\mu^r\lambda_{im}^{q-r}\right]\left[K_S+\frac{h'}{z}\sum_{m'}\sum_{r'}\lambda_{im'}^{r'}\right]-\frac{g}{2}\sum_{im}\sum_r\lambda_{im}^r\ . \tag{2.18}$$

The spin models (2.8) and (2.18) for the bond and site percolation models are similar except that the

spins are placed on the sites in the bond model and on the bonds in the site model. In the bond model only nearest-neighbor pair interactions occur while the site model is more complicated because the interaction involves all the spins on the bonds surrounding a site.

III. MEAN-FIELD THEORY

It is of interest to discuss the mean-field theory of the two models, (2.8) and (2.18).

A. Bond model

We introduce an order parameter $z_r = \langle \lambda^r \rangle$ and from (2.8) the mean-field Hamiltonian is

$$H_{\mathrm{BM}} = -\sum_r \lambda^r \left[z K_B z_{q-1} + zk \left(\sum_{r'} z_{r'} \right) + h + zk \right] . \tag{3.1}$$

From symmetry we may take $z_r = R_B$ for all r and in the limit $q = 1$ the self-consistency condition for R_B is

$$R_B = 1 - e^{-zK_B R_B - h - zk} . \tag{3.2}$$

The generating function $C_B(h,g) = 1 - R_B$ and from (3.2),

$$C_B(h,g) = \sum_{s=1}^{\infty} \frac{(zsK_B)^{s-1}}{s!} e^{-s(zK_B + h + zk)} . \tag{3.3}$$

From (1.1), (1.3), and (3.3), equating powers of e^{-hs}, we find

$$D_B(s,g) = \frac{(zsK_B)^{s-1}}{s!} e^{-sz(K_B + k)} . \tag{3.4}$$

The average size of the clusters diverges at p_c (*for* $g = 0$), and this determines $zK_{Bc} = z \ln[1/(1-p_c)] = 1$. Close to p_c we are interested in large values of s and b (i.e., small g), and the right-hand side of (3.4) can be approximated by

$$D_B(s,g) = P_B(s) e^{-szpg + s\gamma g^2} , \tag{3.5}$$

where

$$P_B(s) = [1/(2\pi s^3)^{1/2}] e^{-sr_0^2/2}$$

is the probability of a site lying in a cluster of s sites, $r_0 = z(p_c - p)/(1 - p_c)$, and $\gamma = \frac{1}{2} zp(1 - p - zp)$. For large b the right-hand side of (1.3) is of the form of a Laplace transform and $P_{s,b}$ is determined by inverting this transform. From (3.5) we find

$$P_{s,b}^{(B)} = P_B(s) \frac{1}{(4\pi s\gamma)^{1/2}} \exp\left[\frac{-(b - zps)^2}{4s\gamma} \right] . \tag{3.6}$$

The average number of bonds in clusters of size s, $\langle b \rangle_s$, and the fluctuations around this average are then given by

$$\langle b \rangle_s = zps , \tag{3.7a}$$

$$\frac{(\langle b^2 \rangle_s - \langle b \rangle_s^2)^{1/2}}{\langle b \rangle_s} = \frac{1}{zp} \left(\frac{2\gamma}{s} \right)^{1/2} . \tag{3.7b}$$

Equation (3.7a) gives some information on the shapes of the clusters near p_c. For large z, $p_c \simeq 1/z$ and $\langle b \rangle_s \simeq s$. For large one-dimensional clusters, we have $b = s - 1$, while for compact globular clusters, we have $b \simeq \frac{1}{2} zs$. Thus (3.7a) shows that the clusters are more extended than compact. Equation (3.7b) shows that the root-mean-square fluctuations are small $\sim s^{-1/2}$ for large clusters. Close to p_c where both s and b are large the distribution (3.6) is essentially a δ function

$$P_{s,b}^{(B)} \simeq P_B(s) \delta(b - zps) . \tag{3.8}$$

The distribution (3.6) is conveniently written in a scaling form (close to p_c)

$$P_{s,b}^{(B)} = r_0^{\nu d + x} f(s r_0^{\Delta}, (b - as) r_0^{x}) . \tag{3.9}$$

with $\nu = \frac{1}{2}$, $\Delta = 2$, $x = 1$, dimensionality $d = 6$, and $a \simeq zp$ is a constant. The new exponent x is connected with the width of the distribution and has been discussed by Leath.[5] Taking f in (3.9) to be even in $(b - as)$ then $\langle b \rangle_s = as$ and $(\langle b^2 \rangle_s - \langle b \rangle_s^2)^{1/2} \sim s^{x/\Delta}$ (assuming it is independent of r_0 as $r_0 \to 0$).

The order parameter has the form

$$R_B(h,g) = r_0^{\nu d - \Delta} f\left(\frac{h + ag}{r_0^{\Delta}}, \frac{g}{r_0^{x}} \right) . \tag{3.10}$$

The field g enters with two exponents Δ and x with $\Delta > x$. Close to p_c the term $(h + ag) r_0^{-\Delta}$ is more important and if the term $g r_0^{-x}$ is omitted the distribution $P_{s,b}$ is proportional to a δ function as in (3.8). We note a difficulty with the mean field theory; the width γ is negative for certain values of p. This difficulty can be resolved by using a more sophisticated form of the mean-field theory, e.g., the Bethe-Peierls approximation which is exact for a Bethe lattice and gives a result of the form (3.8), i.e., $P_{s,b}^{(B)} = P_B(s) \delta(b - s + 1)$.

B. Site model

As in the bond model we introduce an order parameter $\langle \lambda^r \rangle = z_r$ and set $z_r = R_S$ for all r. A mean-field Hamiltonian is obtained from (2.18) by replacing λ_{im}^r by R_S for all spins except that under consideration. This form of the mean-field theory neglects all short-range correlations and gives (omitting constants and setting $q = 1$)

$$H_{\rm SM} = -2K_S[1-(1-R_S)^{z-1}]\sum_r \lambda^r - \frac{2h'}{z}\left[1+(z-1)(1-R_S)^{z-1} + [1-(1-R_S)^{z-1}]\sum_r \lambda^r\right] \times \sum_r \lambda^r - g\sum_r \lambda^r \ . \tag{3.11}$$

The self-consistency condition is (for $q=1$)

$$R_S = 1-\exp\{-2K_S[1-(1-R_S)^{z-1}] -2h'(1-R_S)^{z-1}-g\} \ , \tag{3.12}$$

and the generating function (2.16) is given by

$$C_S(h,g) = \frac{e^{-h}}{1-p+pe^{-h}}(1-R_S)^z \ . \tag{3.13}$$

Equation (3.12) cannot be solved exactly, but near p_c, we expand the exponential in powers of R_S, h', and g which gives

$$r_0 R_S + uR_S^2 = g+2h' \simeq g+2p_c h \ , \tag{3.14}$$

where $r_0 = 1-2(z-1)K_S$ and $u \simeq \frac{1}{2}(z-1)$. The condition $r_0=0$ determines p_c through the equation $-\ln(1-p_c) = 1/2(z-1)$. In the disordered phase ($p<p_c$) from (3.13) and (3.14), the average number of sites and bonds (internal and external) are

$$\langle s\rangle = 2p_c z/r_0, \quad \langle b+b_e\rangle = z/r_0 \ . \tag{3.15}$$

These results give some information about the shapes of the clusters. Thus for large z, $p_c \sim 1/2z$, and from (3.15), $\langle b+b_e\rangle \simeq z\langle s\rangle$. For a large one-dimensional cluster $b+b_e \simeq zs$, which for a compact globular cluster $b+b_e \simeq \frac{1}{2}zs$. Result (3.15) then shows that the clusters near p_c are more extended than compact as in the bond problem.

Equation (3.14) can be solved for R_S,

$$R_S = -\frac{r_0}{2u}\left[1-\left[1+\frac{4u}{r_0^2}(g+2p_c h)\right]^{1/2}\right] \ . \tag{3.16}$$

The distribution function $P_{s,b}$ then follows by substituting R_S in (3.13) [with $(1-R_s)^z \simeq 1-zR_S$] and taking the double inverse Laplace transform

$$P_{s,b} = z\left(\frac{p_c}{2\pi u s^3}\right)^{1/2}\exp\left(-\frac{sr_0^2}{8up_c}\right) \times \delta\left(b+b_e-\frac{s}{2p_c}\right) \ . \tag{3.17}$$

This is of exactly the same form as in the bond problem, Eq. (3.8). As we have only retained terms linear in h and g in (3.14) the width of the distribution is zero.

IV. CRITICAL EXPONENTS IN $1+\epsilon$ AND $6-\epsilon$ DIMENSIONS

A simple renormalization-group analysis of the spin Hamiltonians for the bond and site problems can be carried out near one dimension using the Migdal approximation.[6] In each case in one dimension we can write the Hamiltonian

$$H = -\sum_{nn} V(\lambda_i,\lambda_j) \ , \tag{4.1}$$

and define the transfer matrix

$$[T(\{K_l\})]_{\lambda_i,\lambda_j} = e^{V(\lambda_i,\lambda_j)} \ , \tag{4.2}$$

where $\{K_l\}$ denotes the set of coupling constants in H. The Migdal recursion relation can then be written

$$[T^2(\{K_l\})]_{\lambda_i,\lambda_j} = (\text{const})[T(\{K_l' 2^{1-d}\})]_{\lambda_i,\lambda_j} \ , \tag{4.3}$$

where $K_l' = K_l'(\{K_l\})$ are the coupling constants that occur in the Hamiltonian when the length scale is doubled.

For the bond model, which is described by the Hamiltonian (2.3), Eq. (4.3) leads to the following recursion relations when q is set equal to 1:

$$\begin{aligned} e^{K'2^{1-d}} &= \frac{e^h+e^{2K}-1}{e^h+2e^K-2} \ , \\ e^{(h'/2)2^{1-d}} &= \frac{e^{h/2}(e^{K+k+h}+e^K-1)}{e^h+2e^K-2} \ , \\ e^{(K'+k'+h')2^{1-d}} &= \frac{e^{2(K+k+h)}}{e^h+2e^K-2} \ . \end{aligned} \tag{4.4}$$

In addition to the trivial fixed points $K=0$ and $K=\infty$, these relations possess the fixed point $k=h=0$, $K_c \simeq 1/(d-1)$, which corresponds to the critical concentration[7] $p_c \simeq 1-e^{-1/(d-1)}$.

The linearized recursion relations near this fixed point are

$$\begin{aligned} (K-K_c)' &= 2^{d-1}(K-K_c) \ , \\ k' &= 2^{d-1}k \ , \\ k'+h' &= 2^d(k+h) \ , \end{aligned} \tag{4.5}$$

and give the exponents $\nu = (d-1)^{-1}$, $x=1$, and $\Delta = d/(d-1)$. These results indicate that the order parameter

$$R_B \sim (r_0^{\nu d-\Delta}) f\left(\frac{h+k}{r_0^{\Delta'}}, \frac{k}{r_0^x}\right)$$

and as $k \sim g$ this is of the same form as (3.10). We interpret the exponent x to be connected with the width of the distribution $P_{s,b}$ as in (3.9).

The Migdal approximation for the site problem gives identical results except that in the linearized recursion relations (4.5) the fields g and h are inter-

changed. The order parameter is thus of the form

$$R_S \sim (r_0^{\nu d-\Delta}) f\left(\frac{h+g}{r_0^{\Delta}}, \frac{h}{r_0^{x}}\right)$$

with exactly the same exponents as in the bond problem.

We now briefly discuss the results of the renormalization group analysis in $6-\epsilon$ dimensions. The usual method consists of writing the partition function as an integral over complex order parameters. For the bond model where only pair interactions occur this procedure is straightforward and has been described in Ref. 4. For the site model where we have multiple-spin interactions a method of passing to a continuum model is described in the Appendix. The effective Hamiltonian in both the bond and site problems is of the form (we use the notation of Ref. 4)

$$H_E = \frac{1}{2}\sum_k \sum_r (r_0 + k^2) z_{rk}^* z_{rk} - \tilde{h} N^{1/2} \sum_r z_{rk=0} - \frac{u}{6N^{1/2}} \sum_{(k)} \sum_{(r)} z_{r_1 k_1} z_{r_2 k_2} z_{r_3 k_3} \Delta(r_1 + r_2 + r_3)\delta_{k_1+k_2+k_3,0} + O(z^4) \quad . \tag{4.6}$$

where z_{rk} is the Fourier transform of the order parameter z_{ri} (bond model) and z_{rim} (site model). In the third term, $\Delta(r_1+r_2+r_3)$ is 1 if $r_1+r_2+r_3$ is an integral multiple of q and zero otherwise, $r_0 \sim p_c - p$, and $\tilde{h} \sim h + ag$, where a is a constant.

Provided (4.6) is a realistic continuum generalization of the spin models representing the bond and site models the two models will have the same critical exponents near six dimensions. Also to leading order the fields h and g enter in the combination $h+ag$ [terms of order gz^2, g^2z, etc., have been omitted in (3.6)]. The order parameters will then have the form $(|r_0|^{\beta_p}) f((h+ag)/r_0^{\Delta})$ and the distribution $P_{s,b}$ will be proportional to a δ function $\delta(b-as)$ close to p_c. The distribution P_s has been discussed in greater detail in Ref. 4.

V. CONCLUSIONS

The bond and site percolation models are formulated as the limit of spin models and can thus be described by Hamiltonians. Near one dimension the Migdal procedure gives identical exponents for both models. The order parameter in each case provides a generating function for the site and bond distributions $P_{s,b}$ in the finite clusters. Close to p_c, where large clusters occur, this distribution is well represented by a δ function $\delta(b-as)$ so that the fluctuations are small. In order to obtain exponents near six dimensions it is necessary to pass to continuum forms of the spin Hamiltonians. It is argued that the effective continuum Hamiltonians are the same for the bond and site percolation models. Then, provided the effective Hamiltonian is representative of the underlying spin models, the bond and site models will have identical exponents and behavior in $6-\epsilon$ dimensions.

ACKNOWLEDGMENT

We would like to thank P. L. Leath for several helpful discussions.

APPENDIX

The Hamiltonian for the site model (for simplicity we omit the fields) is given in (2.18):

$$H_{S0}(\{\lambda_{im}^r\}) = -K_S \sum_i \mathrm{Tr}_\mu \prod_m \left[1 + \sum_r \mu^r \lambda_{im}^{q-r}\right] \quad . \tag{A1}$$

We have indicated explicitly that it depends on all the spin variables λ_{im}^r. This Hamiltonian is more complicated than the bond one (2.8) because it contains interactions between all the spins around a site whereas the bond Hamiltonian only contains pair interactions. To derive a continuum form for the site Hamiltonian we proceed as follows,

$$Z_{S0} = \mathrm{Tr}_\lambda \exp[-H_{S0}(\{\lambda_{im}^r\})] \tag{A2}$$

$$= C^{-1}\,\mathrm{Tr}_\lambda \int (dz) \exp[H_{S0}(\{\lambda_{im}^r + z_{im,r}\}) - H_{S0}(\{\lambda_{im}^r\})] \quad , \tag{A3}$$

where

$$C = \int (dz)\, e^{H_{S0}(\{z_{im,r}\})} \quad . \tag{A4}$$

The $z_{im,r} = z_{im,q-r}^*$, $r = 1, \ldots, q-1$ are complex variables defined at each spin site and the integration is over all z variables, and for each is over the entire complex plane. The equality of (A2) and (A3) follows by introducing a new integration variable $z_{im,r}' = z_{im,r} + \lambda_{im}^r$ in (A3) and using (A4). The effective Hamiltonian is then defined by

$$\exp[-H_E(\{z_{im,r}\})] = (1/q^{Nz/2})\,\mathrm{Tr}_\lambda \exp[H_{S0}(\{\lambda_{im}^r + z_{im,r}\}) - H_{S0}(\{\lambda_{im}^r\})] \quad . \tag{A5}$$

This effective Hamiltonian may be calculated as a power series in z_r by expanding both sides of (A5). The field terms may also be included in the right-hand side of (A5). A straightforward but lengthy calculation then leads to Eq. (4.6).

*Supported in part by the NSF under Grant No. DMR-75-13350.

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