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Dynamics of classical *XY* spins in one and two dimensions*

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We suggest that low-temperature spin waves in classical spin systems can be understood in terms of a "fixed-length" hydrodynamic theory. A theory is constructed along these lines which is exactly soluble in one and two dimensions for models with an easy-plane anisotropy. The results should apply at low temperatures to one-dimensional ferromagnets such as $CsNiF_3$, and agree with a microscopic truncated-spin-wave theory proposed by Villain. In two dimensions, we expect the calculations to be valid in a band of temperatures for *XY* magnets with an underlying hexagonal symmetry. The calculations should describe in addition the long-wavelength, low-frequency dynamics of third-sound propagation in films of ^4He and ^3He-^4He mixtures. We also show that the critical exponent ν is $\nu \approx 1/2\sqrt{\epsilon}$ for *XY* models in $2+\epsilon$ dimensions. Some results for dynamics in three dimensions are presented as well.

I. INTRODUCTION

There has been considerable progress recently in understanding the low-temperature static critical properties of n-component fixed-length spins near two dimensions.[1] One implication of this work is that the usual long-wavelength or "hydrodynamic" picture of magnetism[2] changes drastically for $d \leq 2$. Specifically, the Landau-Ginzburg expansion of the free-energy functional in powers of a local course-grained n-component magnetization $\vec{M}(\vec{r})$, namely,

$$\bar{\mathcal{H}} \equiv -\frac{H}{k_B T} = -\int d\vec{r}\,[\tfrac{1}{2}(\vec{\nabla}\vec{M})^2 + \tfrac{1}{2}r|\vec{M}|^2 + u|\vec{M}|^4 + \cdots] \tag{1.1}$$

must be replaced by

$$\bar{\mathcal{H}} = -\tfrac{1}{2}K\int d\vec{r}\,(\vec{\nabla}\vec{M})^2, \tag{1.2a}$$

$$|\vec{M}(\vec{r})|^2 = 1 \tag{1.2b}$$

in one and two dimensions. Indeed, this "fixed-length" hydrodynamic description is implicit in the work of Ref. 1, and has been shown to be stable against small perturbations by Amit and Ma.[3] The coupling K in Eq. (2a) is expected[1] to be inversely proportional to temperature. Although (1.2) can be regarded as the continuum limit of a microscopic model of fixed-length spins on a lattice,[1,4] it is rather striking that the fixed-length character is preserved even in a long-wavelength description. In contrast, the "hydrodynamics" of microscopically fixed-length spins in *three* dimensions is almost certainly described by (1.1).

For *XY* models ($n=2$), in the continuum limit the static theory summarized by (1.2) neglects phase-slip singularities in one dimension and vortices in $d=2$. Although it is easy to show that phase slips give only exponentially small corrections in $d=1$, vortices are known to be of critical importance in two dimensions above a critical temperature T_c.[5-7] Below T_c, however, Kosterlitz[5] has shown that vortices are irrelevant variables, and can hence be neglected in the long-wavelength limit. Consequently, we expect that Eq. (1.2) pertains to two-dimensional *XY* models below T_c, in a "phase" describable by a line of critical points with continuously variable critical exponents.[5-7] The calculations described in Refs. 5 and 6 suggest that the two-dimensional *XY* model will also have a conventional high-temperature phase, characterized by correlations which fall off exponentially at large distances. Order parameter correlations fall off as power laws below the T_c of this model. We shall not have anything to say about dynamics in the high-temperature phase.

In this paper, we build on these results for the *static* low-temperature properties by constructing a phenomenological model of *XY* spin *dynamics* using methods developed in studies of dynamic critical phenomena in $4-\epsilon$ and $6-\epsilon$ dimensions.[8,9] According to the prescriptions described and developed in Refs. 8 and 9, a long-wavelength dynamic theory can be constructed by first determining the Landau-Ginzburg free-energy functional appropriate to the problem at hand. One then constructs dynamical equations from this functional which are consistent with the conservation laws, contain all allowable dissipative terms and display the nondissipative couplings implied by the "Poisson-bracket" relations.[8,9] One looks for a stable dynamical fixed point, and discards those couplings which are irrelevant with respect to it. Usually, the resulting equations are nonlinear and difficult to solve. In contrast, the model we have obtained in this way for three-component spins with an easy-plane anisotropy turns out to be exactly soluble. Small deviations from the exactly soluble model are irrelevant variables at low temperatures in one dimension, which allows us to produce the universal limiting (small q and ω) Fourier-transformed spin-spin correlation function $S(q,\omega)$. This structure function exhibits spin-wave peaks for $q\xi > 1$ and relaxational behavior for $q\xi < 1$, in qualitative agreement with experiments on one-dimensional ferromagnets.[10,11] For small q and ω, our results coincide with a microscopic truncated-spin-wave theory proposed by Villain.[12] This theory was found to be in qualitative accord with inelastic-neutron-scattering measurements on $CsNiF_3$ at low temperatures.[13,14] Our analysis provides a "hydrodynamic" interpretation of Villain's results, and suggests that they should apply quite generally in the low-temperature limit.

One might wonder about the relationship between our work and the body of exact results for the quantum spin-$\frac{1}{2}$ XY model in one dimension (these are reviewed in Ref. 11). We believe that no comparison is possible because exact calculations have not yet been done with the quantum analogue of the M_z self-coupling displayed in Eq. (2.1a). Indeed, the dynamics for classical spins is totally different if we suppress this coupling by, say, setting $g=0$ in Eqs. (2.5a) and (2.5b). For small values of g, we observe a rapid crossover to the limiting behavior described in the preceding paragraph.

Deviations from the exactly soluble model are also irrelevant in *two* dimensions. We find that the limiting form of $S(q,\omega)$ in this case displays temperature-dependent power-law divergences at the spin-wave frequencies, signalling a breakdown of the usual hydrodynamic picture.[15] Other correlation functions have the standard hydrodynamic form, however. We emphasize that these results apply throughout the low-temperature phase of isotropic two-dimensional XY systems, up to and including the critical temperature. Villain has proposed a microscopic spin model[12] in two dimensions which is equivalent to ours in the long-wavelength limit, but was only able to obtain an order-of-magnitude estimate of the order-parameter correlation function. Blank *et al.*[16] obtained results very similar to ours for the XY model in $d=2$ simply by postulating that the phase of the order parameter obeys a wave equation. It is interesting that the hydrodynamic approach presented here ultimately gives rise to an identical dynamical theory.

Although isotropic XY critical behavior may be rather rare in nature,[5] it has been suggested[6,7,17] recently that this behavior should persist over a band of temperatures in two dimensional magnetic systems with an underlying hexagonal symmetry. The expected phase diagram[6] is shown in Fig. 1, as a function of hexagonal-crystal-field strength h_6. The dynamic theory presented here should describe the long-wavelength long-time behavior in the band of temperatures bounded by $T_1(h_6)$ and $T_2(h_6)$. Because of strong dimensionality crossover effect,[7] it may be rather difficult to actually produce the phase diagram shown in Fig. 1 in real, quasi-two-dimensional, magnetic crystals.

To the extent that the long-wavelength properties of superfluids are describable by an XY model,[18] our calculations should also apply to third-sound propagation below the transition temperature in ^{4}He films. Third-sound propagation in films made of mixtures of ^{4}He and ^{3}He is also treated briefly. In Appendix B, we show that the finite superfluid density $\rho_s(T)$ at T_c predicted by Kosterlitz's theory is consistent with the Josephson relation evaluated in $2+\epsilon$ dimensions. In particular, we demonstrate that the critical exponent ν,

$$\nu = (1/2\sqrt{\epsilon})[1+O(\sqrt{\epsilon})], \tag{1.3}$$

in $d=2+\epsilon$.

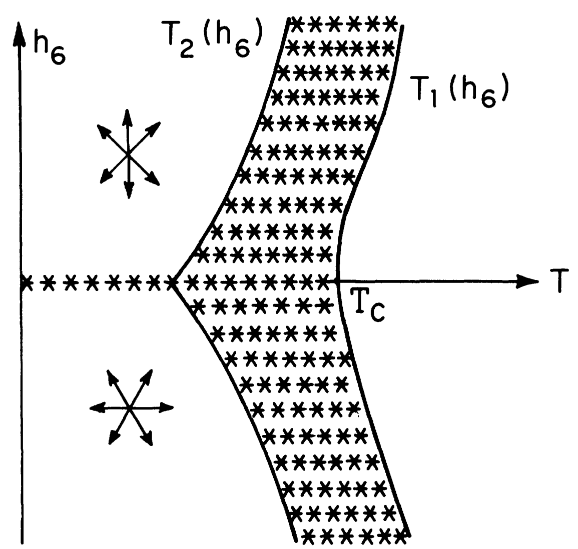

FIG. 1. Schematic phase diagram for a two-dimensional XY model in the presence of a hexagonal-symmetry-breaking field h_6. For $h_6=0$, a line of critical points with continuously variable critical exponents runs from $T=0$ out to $T=T_c$. On lowering the temperature at any finite h_6, one first passes at $T_1(h_6)$ from the disordered phase into a phase with the continuous symmetry and variable exponents of the isotropic XY model. Upon lowering the temperature still further, there is a second phase transition at $T_2(h_6)$ into a phase with the discrete symmetry of the hexagonal crystal field. The dynamical theory constructed here should apply in the region of the phase diagram marked by asterisks.

Although the calculations presented here are limited to XY models, we expect a qualitatively similar picture for $n=3$ antiferromagnets in one dimension.[19] In particular, we expect that spin waves above T_c in such systems are due to the fixed-length nature of the underlying hydrodynamics, and that dynamic scaling results such as (3.4a) hold with $z=1$.

In Sec. II we introduce the basic model, and discuss the relatively simple properties of the defining linear equations. Results for order-parameter correlations are derived in one and two dimensions in Sec. III. Section IV discusses the applicability of the analysis to superfluid ^{4}He films, while the generalization of the model to ^{3}He-^{4}He mixtures is considered briefly in Sec. V. The derivation of the two-dimensional order-parameter correlation function is presented in detail in Appendix A, while the continuation of the static recursion relations of the XY model into $2+\epsilon$ dimensions is relegated to Appendix B. Some results for the three-dimensional case are contained in Appendix C.

II. MODEL

If the course-grained magnetization $M(\vec{r})$ entering (1.2) is taken to have two components, we expect this Hamiltonian to describe the long-wavelength static properties of *three* component spins with an easy plane anisotropy. This has been shown in detail in $2+\epsilon$ dimensions by Pelcovits and Nelson,[20] who treated a fixed-length three-component model with a small quadratic anisotropy term favoring, say, the xy plane. This anisotropy grows[20] under repeated iterations of a renormalization transformation until the x and y components of $\vec{M}(r)$ are effectively fixed length, and the z component $M_z(r)$ can be explicitly integrated out of the problem (this integration was not actually carried out in Ref. 20). We are left with a two-component fixed-length problem.

Because $M_z(\vec{r})$ plays an important role in the spin-wave *dynamics* we shall *not* integrate it out, but consider instead the Hamiltonian

$$\mathcal{H}=-\frac{H}{k_BT}=-\frac{1}{2}K\int d^dr\,[(\vec{\nabla}M_x)^2+(\vec{\nabla}M_y)^2+M_z^2], \tag{2.1a}$$

$$M_x^2(\vec{r})+M_y^2(\vec{r})=1. \tag{2.1b}$$

The analysis of Ref. 20, which shows that this is indeed the long-wavelength form in $d=2+\epsilon$ for spins with an easy-plane anisotropy, is easily extended to include all dimensions $d\lesssim 2$. An additional coupling of the form $(\nabla M_z)^2$ was found to be irrelevant[20] at low temperatures. We have rescaled $M_x(\vec{r})$ and $M_y(\vec{r})$ so that the sum of their squares is precisely unity, and have rescaled $M_z^2(\vec{r})$ (which is effectively unconstrained at low temperatures) so that it appears with coefficient $\frac{1}{2}K$ in (2.1a).

Equation (2.1) simplifies considerably upon making the standard substitution

$$M_x(\vec{r})=\cos\theta(\vec{r}),\quad M_y(\vec{r})=\sin\theta(\vec{r}), \tag{2.2}$$

which gives

$$-\frac{H}{k_BT}=-\frac{1}{2}K\int d\vec{r}\,[(\vec{\nabla}\theta)^2+M_z^2]. \tag{2.3}$$

The partition function and static correlations associated with (2.3) can be obtained by calculating the functional integral of $e^{\mathcal{H}}$ over $\theta(\vec{r})$ and $M_z(\vec{r})$, as has been done by Wegner[21] and Berezinskii.[22] Fluctuations are enormously important at low dimensionality in, for example, M_x-M_x correlations,

$$\begin{aligned}C_{M_xM_x}(\vec{r}-\vec{r}\,')&\equiv\tfrac{1}{2}\langle\cos[\theta(\vec{r})-\theta(\vec{r}\,')]\rangle\\&=\tfrac{1}{2}\exp\{-\tfrac{1}{2}\langle[\theta(\vec{r})-\theta(\vec{r}\,')]^2\rangle\}.\end{aligned} \tag{2.4}$$

Because of strong-fluctuation effects, the correlation function (2.4) falls off exponentially at large distances in $d=1$ and as a power law in $d=2$.

The equations we propose to describe the dynamics of the local fluctuating variables $\theta(\vec{r},t)$ and $M_z(\vec{r},t)$ are

$$\frac{\partial\theta}{\partial t}=-\Gamma\frac{\delta H}{\delta\theta}+g\frac{\delta H}{\delta M_z}+\zeta, \tag{2.5a}$$

$$\frac{\partial M_z}{\partial t}=\lambda\nabla^2\frac{\delta H}{\delta M_z}-g\frac{\delta H}{\delta\theta}+\Upsilon, \tag{2.5b}$$

where the Gaussian fluctuating noises $\zeta(\vec{r},t)$ and $\Upsilon(\vec{r},t)$ satisfy

$$\langle\zeta(\vec{r},t)\zeta(\vec{r}\,',t')\rangle=2k_BT\Gamma\delta(\vec{r}-\vec{r}\,')\delta(t-t'), \tag{2.5c}$$

$$\langle\Upsilon(\vec{r},t)\Upsilon(\vec{r}\,',t')\rangle=-2k_BT\lambda\nabla^2\delta(\vec{r}-\vec{r}\,')\delta(t-t'), \tag{2.5d}$$

$$\langle\zeta(\vec{r},t)\Upsilon(\vec{r}\,',t')\rangle=0. \tag{2.5e}$$

According to Eq. (2.5a), $\theta(\vec{r},t)$ relaxes toward equilibrium at a rate Γ, and precesses about the local z component of the magnetization. The conserved variable $M_z(\vec{r},t)$ exhibits a diffusive self-coupling in addition to a Larmor precession term. The correlations between noise sources are consistent with the fluctuation dissipation theorem, and ensure that $\theta(\vec{r},t)$ and $M_z(\vec{r},t)$ relax toward an equilibrium distribution given by e^{-H/k_BT}. Although Eqs. (2.5) were derived for ferromagnetic couplings between nearest-neighbor spins, identical equations should hold for XY antiferromagnets as well, where $\cos\theta(\vec{r},t)$ and $\sin\theta(\vec{r},t)$ now represent the components of a staggered-two-component order parameter. The variable $M_z(\vec{r},t)$ still represents the conserved z component of the uniform magnetization. Our model can be considered a fixed-length version of the model E employed by Halperin *et al.*[23] in $4-\epsilon$ dimensions.

Equations (2.5a) and (2.5b) are linear and, of course, exactly soluble. They display hydrodynamic spin waves very similar to those found *below* T_c in three-dimensional magnets,[18] although here these excitations occur, for example, in magnets *above* T_c for $d=1$. In sharp contrast to the situation near four dimensions,[23] neglected nonlinearities in (2.5) are irrelevant variables, for $T \gtrsim 0$ in one dimension, and along the Kosterlitz-Thouless line of critical points in $d=2$. In fact it is straightforward to check that the dynamic couplings Γ and λ in (2.5) are also irrelevant at small wave numbers and frequencies in these temperature ranges. A consequence is that the Fourier-transformed M_z-M_z and θ-θ correlation functions display sharp peaks at the spin-wave velocities. Using, for example, methods described in the book by Forster,[15] we find, in the limit $\Gamma, \lambda \to 0$,

$$C_{\theta\theta}(q,\omega) \equiv \int d\vec{r}\, e^{i\vec{q}\cdot\vec{r}} \int dt\, e^{-i\omega t} \langle \theta(\vec{r},t)\theta(\vec{0},0)\rangle = \frac{\pi}{Kq^2}[\delta(\omega-cq)+\delta(\omega+cq)], \quad (2.6a)$$

$$C_{M_zM_z}(q,\omega) \equiv \int d\vec{r}\, e^{i\vec{q}\cdot\vec{r}} \int dt\, e^{-i\omega t} \times \langle M_z(\vec{r},t)M_z(\vec{0},0)\rangle = \frac{\pi}{K}[\delta(\omega-cq)+\delta(\omega+cq)]. \quad (2.6b)$$

The spin-wave velocity c which follows from (2.4) is just

$$c = k_B T g K. \quad (2.7)$$

III. ORDER-PARAMETER CORRELATIONS IN ONE AND TWO DIMENSIONS

In addition to $C_{\theta\theta}$ and $C_{M_zM_z}$, we are, of course, interested in the order-parameter correlations $C_{M_xM_x}$ and $C_{M_yM_y}$. It is in these quantities that one sees very strong-fluctuation effects in analogy to the results found by Wegner[21] and Berezinskii[22] for the statics. As in the static case,[21,22] $C_{M_xM_x}$ and $C_{M_yM_y}$ are readily obtained from the θ-θ correlations.

A. $d=1$

In one dimension, we obtain

$$\begin{aligned} S(r,t) &\equiv \langle M_x(\vec{r},t)M_x(\vec{0},t)+M_y(\vec{r},t)M_y(\vec{0},t)\rangle = \langle \cos[\theta(\vec{r},t)-\theta(\vec{0},0)]\rangle \\ &= \exp\{-\tfrac{1}{2}\langle[\theta(\vec{r},t)-\theta(\vec{0},0)]^2\rangle\} = \exp[-(1/4K)(|x-ct|+|x+ct|)], \end{aligned} \quad (3.1)$$

upon setting λ and Γ to zero. According to (3.1), $S(r,t)$ remains fixed at its static $t=0$ value for times $t<x/c$. For $t>x/c$, a spin wave has had time to propagate between the two spins, and $S(r,t)$ begins to decay exponentially in time. It is very easy to Fourier transform (3.1) and obtain the structure factor

$$S(q,\omega) = \frac{8\kappa^2/c}{[\kappa^2+(\omega/c-q)^2][\kappa^2+(\omega/c+q)^2]}, \quad (3.2)$$

where we have defined an inverse correlation length, $\kappa = \frac{1}{2}K^{-1}$. Thus, κ is proportional to temperature as $T\to 0$, in agreement with the exact results of Wegner[21] for an XY model on a lattice.

The structure factor (3.2) has poles at the complex frequencies,

$$\omega_\pm(q) = \pm cq + ic\kappa; \quad (3.3)$$

The imaginary part represents a q-independent fluctuation contribution to the damping. As shown in Fig. 2, $S(q,\omega)$ exhibits fluctuation-broadened spin-wave peaks for $q/\kappa > 1$, but displays dissipative behavior for $q/\kappa < 1$. We note that the frequencies (3.3) [as well as $S(q,\omega)$ itself] satisfy dynamic scaling,[24,25]

$$\omega_\pm(q) = q^z \Omega_\pm(q/\kappa), \quad (3.4a)$$

where

$$z=1, \quad \Omega_\pm(x) = c(\pm 1 + i/x). \quad (3.4b)$$

In fact, it is not difficult to show that the dynamic exponent z, which is expected to be $z=\frac{1}{2}d$ for $d>2$,[8,9] locks into the value unity for $d\leq 2$. Accepting the result $z=1$, the contribution of the damping proportional to κ follows from the dynamic scaling assumption (3.4a) and the requirement that $\omega_\pm(q) \sim$ const. as $q\to 0$ (M_x and M_y are not conserved quantities).

As mentioned previously, our results agree with Villain's analysis[12] of a microscopic truncated-spin-wave theory in the long-wavelength, low-frequency limit. The approach taken here makes it clear that (3.2) should be the *universal* form of $S(q,\omega)$ at low temperatures for small q and ω in one-dimensional XY magnetic systems. This universality, which is associated with a zero-temperature fixed point, applies only to *classical* spins. At sufficiently low temperatures, quantum effects will, of course, invalidate the results given here.

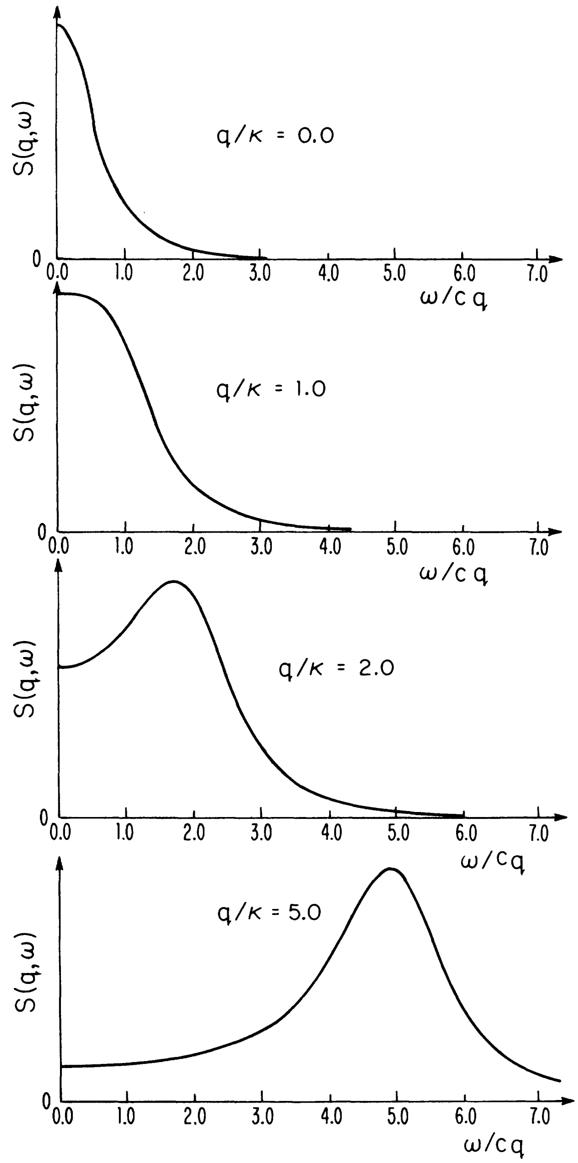

FIG. 2. Universal structure factor $S(q,\omega)$ (Fourier-transformed order-parameter correlation function) for one-dimensional XY spin systems, plotted in arbitrary units. The structure factor exhibits spin wave peaks for $q/\kappa > 1$, but purely relaxational behavior for $q/\kappa < 1$.

B. $d=2$

The hydrodynamic theory should apply in a range of temperatures below T_c for two dimensional XY magnets. The result for the static-order-parameter correlation function in this regime is[21,22]

$$C_{M_xM_x}(r) \sim 1/r^{\eta(K)}, \quad r\to\infty, \tag{3.5}$$

where r is measured in units of the lattice spacing.

The exponent $\eta(K)$, which describes the decay, is proportional to temperature

$$\eta(K)=1/2\pi K. \tag{3.6}$$

Because power-law decay of correlations is associated with systems at a critical point, the "ordered phase" of an XY model in $d=2$ should actually correspond to a *line* of critical points with continuously variable, temperature dependent exponents such as $\eta(K)$. As shown by Kosterlitz,[5] the only effect of vortices is to renormalize K slightly. Here, we assume these renormalizations have already been incorporated into K. Vortices do eventually cause the line of critical points to terminate,[5] at a "temperature" K_c^{-1} such that

$$\eta(K_c)=\tfrac{1}{4}. \tag{3.7}$$

The calculation of the structure factor proceeds exactly as in the one-dimensional case. Although θ-θ and M_z-M_z correlations again have the standard form (2.6), the order-parameter correlations are now given by

$$\begin{aligned} S(r,t) &= \exp\{-\langle[\theta(\vec{r},t)-\theta(\vec{0},0)]^2\rangle\} \\ &= \exp[-I(\vec{r},t)], \end{aligned} \tag{3.8}$$

where

$$I(r,t)=\frac{1}{4\pi^2K}\int d^2q\,\frac{1}{q^2}[1-e^{i\vec{q}\cdot\vec{r}}\cos(cqt)]. \tag{3.9}$$

As was the case in $d=1$, we have set the irrelevant variables λ and Γ to zero.

In one dimension, we could ignore effects due to a finite lattice spacing and convergently extend the integrals over q space to infinity. In $d=2$, however, an ultraviolet cutoff is necessary to ensure a finite result. It is convenient to impose this cutoff after performing the angular integral in (3.9), by replacing $I(r,t)$ by

$$\tilde{I}(r,t)=\frac{1}{2\pi K}\int_0^\infty \frac{dq}{q}[1-J_0(qr)\cos(cqt)]e^{-aq}. \tag{3.10}$$

Here, r and ct are measured in units of the lattice spacing, while a is a dimensionless inverse cutoff of order unity. Of course, our results should not depend on the precise form of the cutoff in the limit $r,\ ct \gg a$.

As explained in Appendix B, Eq. (3.10) can be evaluated analytically, with the result

$$\tilde{I}(r,t)=\frac{1}{4\pi K}\ln\tfrac{1}{4}\{[a+A_+(r,t)]^2+[ct+A_-(r,t)]^2\}, \tag{3.11a}$$

where

$$A_{\pm}(r,t)=\{\tfrac{1}{2}[(a^2-c^2t^2+r^2)^2+4a^2c^2t^2]^{1/2} \pm \tfrac{1}{2}(a^2-c^2t^2+r^2)\}^{1/2}. \tag{3.11b}$$

This expression simplifies considerably in the limit $r,\ ct >> a$,

$$\tilde{I}(r,t)=\begin{cases}(1/2\pi K)\ln r, & r>ct,\\ (1/2\pi K)\ln[ct+(c^2t^2-r^2)^{1/2}], & r<ct.\end{cases} \tag{3.12}$$

It follows that $S(r,t)$ can be written in scaling form in this limit

$$S(r,t)=e^{-\tilde{I}(r,t)}=\frac{1}{r^{\eta(K)}}\Phi_\eta\left[\frac{ct}{r}\right], \tag{3.13a}$$

with

$$\Phi_\eta(y)=\begin{cases}1, & y<1,\\ [y+(y^2-1)^{1/2}]^{-\eta(K)}, & y>1\ .\end{cases} \tag{3.13b}$$

For any finite value of a, (3.13) will be a good approximation to (3.11), provided $|c^2t^2-r^2| >> a^2$. This condition will, in general, be satisfied for large ct and r unless $ct \approx r$. Thus, the result (3.13b) for the scaling function $\Phi_\eta(y)$ will always be inaccurate near $y=1$, but in a region about this value which vanishes in the scaling limit. We have normalized $S(r,t)$ such that $\Phi_\eta(0)=1$.

As in the one-dimensional case, $S(r,t)$ remains locked at its static value for times $ct<r$. It decays in time after a spin wave has had time to propagate between the two points, corresponding to a "light cone singularity" in $\Phi_\eta(y)$ at $y=1$. Although $S(r,t)$ ultimately decays as a power law in time, we note that there is a square-root cusp in $\Phi(y)$ for $y=1^+$. For any finite a, this cusp will be rounded off. It is only present in the scaling limit discussed in the previous paragraph. Similar remarks apply to the one-dimensional problem in the presence of a cutoff.

We have also computed the Fourier transform of $S(r,t)$, which is, of course, directly relevant to neutron-scattering experiments. This can be written in the form

$$S(q,\omega)=\frac{4\pi}{c}\int_0^\infty dr\, r^{2-\eta(K)}J_0(qr) \times\int_0^\infty dy\cos\left[\frac{r\omega y}{c}\right]\Phi_\eta(y). \tag{3.14}$$

In Appendix B, we show that the integrals which enter (3.14) can be evaluated in terms of an infinite series of hypergeometric functions. This allows the singularities in $S(q,\omega)$ to be determined analytically, and a convenient numerical evaluation of the function for various values of $\eta(K)$. The structure factor can be written in a scaled form

$$S(q,\omega)=\frac{1}{q^{3-\eta(K)}}\Psi_\eta\left[\frac{\omega}{cq}\right], \tag{3.15}$$

where the scaling function $\Psi_\eta(y)$ depends on η. The function $\Psi_\eta(y)$ for $\eta=\frac{1}{4}$ (corresponding to $T=T_c$) and $\eta=\frac{1}{9}$ are shown in Fig. 3. The plots for other values of η are very similar. The function $\Psi_\eta(y)$ diverges at the spin-wave frequencies according to an η-dependent power law

$$\Psi_\eta(y)\sim 1/|1-y^2|^{1-\eta(K)}, \tag{3.16}$$

as $y\to\pm1$. The large-y behavior is

$$\Psi_\eta(y)\sim 1/|y|^{3-\eta(K)},\quad |y|\to\infty, \tag{3.17}$$

while $\Psi_\eta(y)$ is analytic in y^2 about $y=0$. Evidently, $S(q,\omega)$ exhibits spin wave "peaks" at $\omega=\pm cq$ for all temperatures below T_c. These peaks are very sharp, are *not* describable by a simple hydrodynamic pole of $S(q,\omega)$, and exhibit an $\eta(K)$-dependent fluctuation induced broadening. The divergence of $S(q,\omega)$ at the spin-wave frequencies can be traced back to the square-root cusp in $\Phi_\eta(y)$. Effects due to finite values

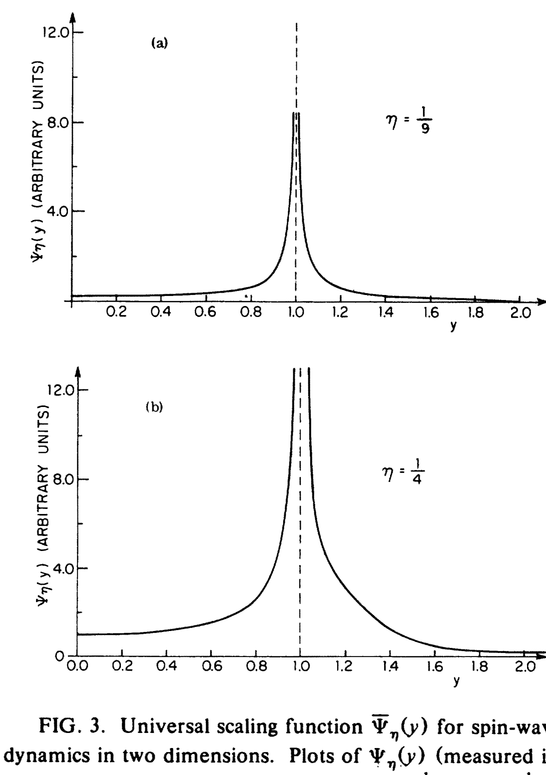

FIG. 3. Universal scaling function $\overline{\Psi}_\eta(y)$ for spin-wave dynamics in two dimensions. Plots of $\Psi_\eta(y)$ (measured in the same arbitrary units) are shown for $\eta=\frac{1}{9}$ and $\eta=\frac{1}{4}$. Plots for other values of η are very similar. If the nonconservation of M_z can be neglected, the function $\Psi_{1/4}(y)$ should describe the dynamics along the line $T_1(h_6)$ in Fig. 1, while $\Psi_{1/9}(y)$ will control the dynamics along the line $T_2(h_6)$ in this figure.

of λ and Γ and a finite lattice spacing which smooth out this cusp will round off the divergence in $S(q,\omega)$. The height of the resulting peaks will, however, diverge as q and ω tend to zero.

C. Hexagonal symmetry breaking

As T varies from zero to T_c in an isotropic XY system, results such as (3.13) and (3.15) should hold with $\eta(K)$ varying smoothly from 0 to $\frac{1}{4}$. (The precise relationship between η and temperature is nonuniversal, however.) As was mentioned in Sec. I, isotropic XY *static* critical behavior should appear in two-dimensional magnets with a hexagonal-symmetry-breaking field in a band of temperatures (see Fig. 1). However, a strong hexagonal coupling changes the *dynamics* drastically, because this interaction breaks the conservation of M_z. If M_z is no longer conserved, we expect the order parameter dynamics to be dominated by Eq. (2.5a) with $g=0$ and finite Γ. The purely diffusive dynamics which arises in this case has been treated by de Gennes[26] (who was interested in nematic liquid crystals), and should describe the long-wavelength behavior in the region between $T_2(h_6)$ and $T_1(h_6)$.

Provided the nonconservation of M_z can be neglected, the spin wave results of Sec. III B may apply in an *intermediate* range of wave numbers and frequencies even for finite h_6. Results such as (3.14) would then hold in the shaded region of Fig. 1, with $\eta(K)$ varying smoothly from $n=\frac{1}{9}$ at $T_2(h_6)$ to $\eta=\frac{1}{4}$ at $T_1(h_6)$. As k and ω tend to zero, however, this spin-wave description must eventually break down, and reduce to the diffusive dynamical theory associated with a nonconserved M_z.

IV. APPLICATION TO TWO-DIMENSIONAL SUPERFLUIDS

The analogy between superfluidity and XY magnetism goes back to work by Matsubara and Matsuda,[27] and has been exploited and reviewed in Ref. 18. As in the discussion of model E by Halperin *et al.*,[23] we expect (2.3) and (2.5) to represent a long-wavelength description of superfluidity, where M_x and M_y are the components of the superfluid order parameter. In three dimensions, M_z represents an appropriate linear combination of the superfluid mass and energy densities.[23] On comparing Eq. (2.5) with the Atkins hydrodynamic treatment of propagating excitations in films,[28] we are lead to identify M_z with a linear combination of deviation of the film height from equilibrium, and the mass and energy densities. With this identification, spin-wave excitations in the magnet correspond to third sound[28] in ^{4}He films.

According to the results of Sec. III B, we expect a linear dispersion relation for third sound as $k\to 0$, with a velocity that remains finite even as $T\to T_c$ from below. The corresponding prediction in the Kosterlitz-Thouless description of the *static* critical properties is that the superfluid density $\rho_s(T)$ should remain finite as $T\to T_c^-$. Indeed, a particularly beautiful consequence of this theory is that the ratio $\rho_s(T)/T$ should approach a universal constant as T goes to T_c from below.[29] In Appendix B, we show how the finite value of $\rho_s(T)$ at T_c is also suggested by the Josephson relation evaluated in $2+\epsilon$ dimensions.

V. ^{3}He-^{4}He MIXTURES

It is possible to generalize the model described in Sec. II so that it provides a long-wavelength description of dynamics in films of ^{3}He-^{4}He mixtures. We shall follow closely the treatment of dynamics in ^{3}He-^{4}He mixtures near four dimensions by Siggia and Nelson.[30] For small concentrations of ^{3}He, we expect the *static* features of the superfluid transition in the film to remain unchanged. The analysis presented below ignores the possibility of a tricritical or first order transition, which could occur for sufficiently large concentrations.

The new feature associated with ^{3}He-^{4}He mixtures is a conserved concentration variable $c(\vec{r},t)$ in addition to combination of the concentration and the conserved densities present in the pure system, which we call $q(\vec{r},t)$. We will build a dynamic theory from a "Hamiltonian" functional which depends on $q(\vec{r},t)$, $c(\vec{r},t)$ and a phase variable $\theta(\vec{r},t)$, namely,

$$-\frac{H}{k_BT}\equiv-\frac{1}{2k_BT}\times\int d^2\vec{r}\,[k_BTK(\vec{\nabla}\theta)^2+q^2+c^2]. \tag{5.1}$$

A convenient normalization has been chosen for q and c. The dynamic equations we propose are

$$\frac{\partial\theta}{\partial t}=-\Gamma\frac{\delta H}{\delta\theta}+g_1\frac{\delta H}{\delta c}+g_2\frac{\delta H}{\delta q}+\zeta, \tag{5.2a}$$

$$\frac{\partial c}{\partial t}=\lambda\nabla^2\frac{\delta H}{\delta c}+L\nabla^2\frac{\delta H}{\delta q}-g_1\frac{\delta H}{\delta\theta}+\Upsilon, \tag{5.2b}$$

$$\frac{\partial q}{\partial t}=K\nabla^2\frac{\delta H}{\delta q}+L\nabla^2\frac{\delta H}{\delta c}-g_2\frac{\delta H}{\delta\theta}+\phi. \tag{5.2c}$$

The fluctuating noise sources $\zeta(\vec{r},t)$, $\Upsilon(\vec{r},t)$, and $\phi(r,t)$ obey

$$\langle\zeta(r,t)\zeta(r',t')\rangle=2k_BT\Gamma\delta(\vec{r}-\vec{r}')\delta(t-t'), \tag{5.3a}$$

$$\langle\Upsilon(\vec{r},t)\Upsilon(\vec{r}',t')\rangle=-2k_BT\lambda\nabla^2\delta(\vec{r}-\vec{r}')\delta(t-t'), \tag{5.3b}$$

$$\langle\phi(\vec{r},t)\phi(\vec{r}',t')\rangle = -2k_BTK\nabla^2\delta(\vec{r}-\vec{r}')\delta(t-t'), \tag{5.3c}$$

$$\langle\Upsilon(\vec{r},t)\phi(\vec{r}',t')\rangle = -2k_BTL\nabla^2\delta(\vec{r}-\vec{r}')\delta(t-t'), \tag{5.3d}$$

while all other noise correlations vanish.

The justification of Eqs. (5.2) and (5.3) for a mixture in $d=2$ is very similar to the analysis for bulk mixtures given in Ref. 30. The principle difference is that the propagating modes which arise in (5.2) (see below) now correspond to third sound,[28] rather than the second-sound excitations present in bulk mixtures.

As before, more complicated dynamical couplings which could, in principle, appear in (5.2) are irrelevant variables, and the model as it stands is exactly soluble. Dissipative couplings such as Γ, λ, L, and K are also irrelevant and will again be set to zero. Using, for example, the methods of Kadanoff and Martin,[31] it is straightforward to show that the c-c and q-q correlations have the standard hydrodynamic form in the dissipationless limit

$$C_{cc}(k,\omega) = \pi\left[\frac{g_1^2K}{c^2}\delta(\omega+ck) + \frac{g_1^2K}{c^2}\delta(\omega-ck) + \left(1-\frac{2g_1^2K}{c^2}\right)\delta(\omega)\right], \tag{5.4a}$$

$$C_{qq}(k,\omega) = \pi\left[\frac{g_2^2K}{c^2}\delta(\omega+ck) + \frac{g_2^2K}{c^2}\delta(\omega-ck) + \left(1-\frac{2g_2^2K}{c^2}\right)\delta(\omega)\right], \tag{5.4b}$$

where

$$c^2 = k_BTK(g_1^2+g_2^2). \tag{5.5}$$

There is a sharp peak at $\omega=0$ in addition to peaks at the third sound velocity c given by (5.5). This central peak is absent, however, in phase correlations in the dissipationless limit

$$C_{\theta\theta}(k,\omega) = (\pi/Kk^2)[\delta(\omega+ck)+\delta(\omega-ck)]. \tag{5.6}$$

Consequently, the results presented in Sec. III B can be applied directly to the order-parameter correlation function, $S(k,\omega)$ in the mixtures. One need only insert the third-sound velocity (5.5) into (3.14) to obtain the long-wavelength form of the structure factor.

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APPENDIX A: EVALUATION OF INTEGRALS IN TWO DIMENSIONS

To evaluate the integral

$$\bar{I}(r,t) \equiv 2\pi K\tilde{I}(r,t) = \int_0^\infty \frac{dq}{q} \times [1-J_0(qr)\cos(qct)]e^{-aq}. \tag{A1}$$

It is convenient to first add and subtract the quantity

$$\int_0^\infty \frac{dq}{q}e^{-aq}\cos(qct). \tag{A2}$$

The equation for $\bar{I}$ can then be written

$$\bar{I}(r,t) = \frac{1}{2}\int_0^\infty \frac{dq}{q}e^{-(a+ict)q}[1-J_0(qr)] + \frac{1}{2}\int_0^\infty \frac{dq}{q}e^{-(a-ict)q}[1-J_0(qr)] + \int_0^\infty \frac{dq}{q}e^{-aq}[1-\cos(qct)]. \tag{A3}$$

The first two integrals in Eq. (A3) may be evaluated by first differentiating with respect to $a \pm ict$ and using the standard result[32]

$$\int_0^\infty e^{-\alpha x}J_0(\beta x)\,dx = \frac{1}{(\alpha^2+\beta^2)^{1/2}}. \tag{A4}$$

The third integral can be evaluated trivially after differentiation with respect to a. It is then tedious but straightforward to derive the expressions (3.11) for $\tilde{I}(r,t)$ by integrating these results with respect to $a \pm ict$ or a.

To calculate

$$S(q,\omega) = \frac{4\pi}{c}\int_0^\infty dr\, r^{2-\eta(K)}J_0(qr) \times \int_0^\infty dy\cos\left(\frac{\omega r}{c}y\right)\Phi_\eta(y), \tag{3.14}$$

it is convenient to write the y integral as

$$\begin{aligned}\int_0^\infty dy\cos(\alpha y)\Phi_\eta(y) &= \int_0^1 dy\cos(\alpha y) + \int_1^\infty dy\cos(\alpha y)\frac{1}{[y+(y^2-1)^{1/2}]^\eta}\\ &= \frac{\eta}{\alpha}\int_0^\infty dx\sin(\alpha\cosh x)e^{-\eta x}\\ &\equiv \frac{\eta}{\alpha}G_\eta(\alpha),\end{aligned} \tag{A5}$$

where $\alpha = \omega r/c$, $G_\eta(\alpha)$ is defined by the last equality, and we have made the substitution $y = \cosh x$ and integrated by parts. We first note that $G_\eta(\alpha)$ satisfies the differential equation

$$L_\alpha G_\eta(\alpha) = \eta^2 G_\eta(\alpha) - \eta \sin\alpha, \tag{A6}$$

where L_α is Bessel's operator

$$L_\alpha = \alpha^2 \left(\frac{d}{d\alpha} \right)^2 + \alpha \frac{d}{d\alpha} + \alpha^2. \tag{A7}$$

We can now expand $\sin\alpha$ as a Neumann expansion in integer order Bessel functions and use the small α form of the integrals in (A5) as a boundary condition to solve Eq. (A6):

$$G_\eta(\alpha) = \frac{\Gamma(1+\eta)\Gamma(1-\eta)\sin\frac{1}{2}\pi\eta}{\eta} J_\eta(\alpha) + 2\eta \sum_{n=0}^{\infty} \frac{(-1)^{n+1} J_{2n+1}(\alpha)}{(2n+1)^2 - \eta^2}. \tag{A8}$$

The remaining integrals over r in (3.14) are tablulated in Ref. 32, and lead to the result

$$S(q,\omega) \equiv (q^{\eta-3}/c)\Psi_\eta(y), \tag{A9}$$

where

$$\Psi_\eta(y) = \begin{cases} 2^{4-\eta}\cos\left(\frac{\pi}{2}\eta\right)\eta^2 \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n+1)^2-\eta^2} \frac{\Gamma(n+\frac{3}{2}-\frac{1}{2}\eta)}{(2n+1)!} \\ \quad \times F\left(n+\frac{3}{2}-\frac{1}{2}\eta, n+\frac{3}{2}-\frac{1}{2}\eta, 2n+2; y^2\right), \quad y<1, \\ 2^{3-\eta}\pi^2\eta 2 \frac{\sin(\frac{1}{2}\pi\eta)}{\Gamma(1+\eta)\sin(\pi\eta)} \frac{(y^2-1)^{\eta-1}}{y^{1+\eta}} + \frac{2^{2-\eta}\pi\eta^2}{y^{3-\eta}} \\ \quad \times \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\Gamma(n+\frac{1}{2}-\frac{1}{2}\eta)}{\Gamma(n+\frac{3}{2}+\frac{1}{2}\eta)} F(n+\frac{3}{2}-\frac{1}{2}\eta, -n+\frac{1}{2}-\frac{1}{2}\eta, 1; 1/y^2), \quad y>1 . \end{cases} \tag{A10}$$

This function has a singularity at $y^2=1$ of the form

$$\Psi_\eta(y) \approx 2^{2-\eta}\pi\eta\sin(\tfrac{1}{2}\pi\eta)\Gamma(1-\eta)|y^2-1|^{\eta-1}. \tag{A11}$$

There is also a lower-order singularity proportional to

$$|y^2-1|^{\eta-1/2}.$$

Both of these singularities and their amplitudes can be determined quickly by analyzing the large α effects of the square-root cusp in the y integral in Eq. (3.14), and by identifying singularities in $\Psi_\eta(y)$ with the large-r behavior of the second integral.

For small y, $\Psi_\eta(y)$ is given by

$$\psi_\eta(0) \approx 2^{4-\eta}\cos\left(\frac{\pi}{2}\eta\right)\eta^2 \frac{[\Gamma(\frac{3}{2}-\frac{1}{2}\eta)]^2}{1-\eta^2} \rightarrow 4\pi\eta^2 \text{ as } \eta \rightarrow 0 , \tag{A12}$$

while for large y,

$$\psi_\eta(y) \rightarrow \frac{1}{y^{3-\eta}} \left[2^{3-\eta}\pi\Gamma(1-\eta)\eta\sin\left(\frac{\pi}{2}\eta\right) - 2^{1-\eta}\eta^2\pi^{3/2}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\eta)}{\Gamma(1+\frac{1}{2}\eta)} \right] \rightarrow \frac{1}{y^{3-\eta}} 2\pi^2\eta^2 \text{ as } \eta \rightarrow 0 . \tag{A13}$$

The formidable-looking expressions in (A10) are easily evaluated numerically: We have used a Taylor series in y^2 for $y<1$ and have made use of Gauss's recursion relations for hypergeometric functions for $y>1$. Six or seven terms of the second series yields about 1% accuracy for $\eta = \frac{1}{4}$.

APPENDIX B: XY CRITICAL EXPONENTS IN $2+\epsilon$ DIMENSIONS

It is interesting to consider the finite value of $\rho_s(T)$ at T_c in the context of the Josephson scaling relation,[33] which asserts that

$$\rho_s(T) \sim |T-T_c|^{(d-2)\nu} \tag{B1}$$

as $T \rightarrow T_c$ from below. As shown by Kosterlitz,[5] the exponent ν is infinite in precisely two dimensions, so (4.1) is indeterminate in $d=2$. The Josephson relation becomes useful, however, if we first continue the theory into $2+\epsilon$ dimensions and take the limit $\epsilon \rightarrow 0$.

Kosterlitz derived differential recursion relations for K and a parameter y, which measures the probability of exciting a vortex pair.[5] Initially,

$$y \approx e^{-\pi^2 K/2}, \tag{B2}$$

but under the action of a renormalization group transformation K and y evolve according to[5,6]

$$\frac{dK^{-1}(l)}{dl} = 4\pi^3 y^2(l) + O(y^4(l)) \ , \tag{B3a}$$

$$\frac{dy(l)}{dl} = [2 - \pi K(l)]y(l) + O(y^3(l)) . \tag{B3b}$$

These equations display a line of fixed points (corresponding to a line of critical points) at $y = 0$. Although vortices are irrelevant for $K \gtrsim 2/\pi$, the fixed line becomes inaccessible due to relevant vortex perturbations when $K \lesssim 2/\pi$.

The nonlinear terms in (B3) represent fluctuation integrals evaluated in precisely two dimensions.[5,6] To a leading approximation, these terms can be retained as they stand in $d = 2 + \epsilon$, provided one takes into account that K is no longer dimensionless in (B3a). It is not necessary to account for the corresponding change in the dimension of y in (B3b), because the fixed point value of y will turn out to be of order $\sqrt{\epsilon}$. Thus, the only modification of (B3) required in $d = 2 + \epsilon$ is the replacement of (B3a) by

$$\frac{dK^{-1}(l)}{dl} = -\epsilon K^{-1}(l) + 4\pi^3 y^2(l) , \tag{B4}$$

while (B3b) is unchanged.

Hamiltonian flows generated by (B4) and (B3b) are shown in Fig. 4. There is a nontrivial critical fixed point at

$$K^* = 2/\pi + O(\epsilon), \quad y^* = (\epsilon/8\pi^2)^{1/2} + O(\epsilon) , \tag{B5}$$

with eigenvalues

$$\lambda_\pm = \pm 2\sqrt{\epsilon} + O(\epsilon) . \tag{B6}$$

It follows[33] that the correlation-length exponent is

$$\nu = (1/2\sqrt{\epsilon})[1 + O(\sqrt{\epsilon})] \tag{B7}$$

for XY systems in $2 + \epsilon$ dimensions. Insertion of this results into [35] (B1) suggests that $\rho_s(T) \rightarrow$const as

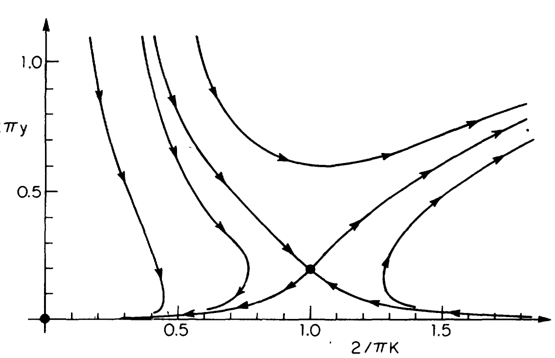

FIG. 4. Hamiltonian flows for an XY model in 2.08 dimensions. Both a zero temperature ($K = \infty$) fixed point and a nontrivial fixed point at $2/\pi K = 1.0$, $2\pi y = 0.2$ are shown. The static critical properties are controlled by the nontrivial fixed point.

$T \rightarrow T_c^-$ in precisely two dimensions. Corrections to the Kosterlitz result[5] $\eta = \frac{1}{4}$ at T_c in $d = 2$ are of order ϵ; the values of other critical exponents in $d = 2 + \epsilon$ follow from η and ν by use of the standard[36] scaling relations.

APPENDIX C: DYNAMICS IN THREE DIMENSIONS

Although the bulk of this paper is concerned with the dynamics of XY models in the one and two dimensions, it is interesting to tabulate the predictions of the model defined by Eqs. (2.5) for $d = 3$ as well. Because fluctuations are not particularly important at low temperatures above two dimensions, the results are very similar to the standard hydrodynamical treatment proposed by Halperin and Hohenberg.[18] We shall see, however, that the model does display interesting coexistence curve singularities.[37]

The predictions for $C_{\theta\theta}(q,\omega)$ and $C_{M_z M_z}(q,\omega)$ are, of course, independent of dimensionality and simply given by (2.6). For order parameter correlations, we find an equation analogous to (3.8),

$$S(r,t) = M_0^2 \exp Q(r,t) \tag{C1}$$

where

$$\begin{aligned} M_0^2 &= \exp\left[\left(-\frac{1}{2\pi^2 K}\right)\int_0^\infty e^{-aq}\,dq\right] \\ &= \exp\left(-\frac{1}{2\pi^2 Ka}\right) \end{aligned} \tag{C2}$$

and

$$Q(r,t) = \frac{1}{2\pi^2 K}\int_0^\infty dq\left(\frac{\sin(qr)\cos(cqt)}{qr}\right)e^{-aq} . \tag{C3}$$

An exponential cutoff has again been imposed. It is easy to evaluate the integral in (C2):

$$\begin{aligned} Q(r,t) = (1/4\pi^2 Kr)[&\arctan((r+ct)/a) \\ &+ \arctan((r-ct)/a)] \ . \end{aligned} \tag{C4}$$

At very large values of r, $S(r,t)$ decays to a nonzero value which we identify with the square of a nonzero spontaneous magnetization.

It is interesting to examine the longitudinal and transverse correlation functions, given by

$$\begin{aligned} S_T(r,t) &= \langle \sin\theta(\vec{r},t)\sin\theta(\vec{0},0)\rangle \\ &= M_0^2 \sinh Q(r,t) \ , \end{aligned} \tag{C5}$$

and

$$\begin{aligned} S_L(r,t) &= \langle \cos\theta(r,t)\cos\theta(\vec{0},0)\rangle - M_0^2 \\ &= M_0^2[\cosh Q(r,t) - 1] \ . \end{aligned} \tag{C6}$$

It follows from (C5) and (C6) that $S_T(r,0)$ and $S_L(r,0)$ fall off as power laws at large r,

$$S_T(r,0) \sim 1/r, \quad S_L(r,0) \sim 1/r^2 \ . \tag{C7}$$

This slow decay is related to the coexistence curve singularities discussed in Ref. 37. As $t \to \infty$ with r fixed we find

$$S_T(r,t) \sim 1/t, \quad S_L(r,t) \sim 1/t^2 \ . \tag{C8}$$

This power-law falloff in time is very similar to the long-time tail phenomena associated with the Navier-Stokes equations in three dimensions.[38] In both cases, the slow decay of correlations in time signals the breakdown of conventional hydrodynamics which will occur in two dimensions and below.

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