Continuum model of vortex oscillations in rotating superfluids*

Mark R. Williams and Alexander L. Fetter
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California 94305
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An elastic-continuum description of the long-wavelength motion of an infinite vortex lattice provides a model for the dynamics of a rotating superfluid. In the classical limit, the resulting differential equations reproduce the ordinary hydrodynamics of a rotating inviscid fluid; in general, however, they include the lowest-order effects of the quantized vortex array. This formalism is used to calculate certain collective modes of rotating superfluids in cylindrical containers.

I. INTRODUCTION

The equilibrium configuration of an unbounded rotating superfluid consists of a two-dimensional triangular array of quantized vortices.[1–5] Such a lattice can support small-amplitude oscillatory collective motion,[3–5] and the calculation of these collective modes is a straightforward matter, at least in principle. For a bounded fluid, however, determining the equilibrium vortex configuration itself is already too difficult for more than a few vortices. Unfortunately, most systems of interest contain many vortices, precluding an exact calculation of the equilibrium configuration and the resulting collective motion.

Elasticity theory suggests an alternative approach to the long-wavelength motion of a dense vortex array. We first consider plane-wave disturbances in the infinite vortex lattice. The equilibrium configuration is known, and the dynamical equations for the plane-wave amplitudes can readily be solved in the limit of wavelengths long compared to the intervortex spacing. The resulting equations for the plane-wave amplitudes can be treated as the Fourier transforms of *local differential equations* that constitute the basis for a continuum description of the vortex array. In this way, we approximate the vortex lattice by an elastic continuum that reproduces the correct long-wavelength dynamics of an infinite system. This elastic description is assumed to remain valid for long-wavelength phenomena in *finite* systems of rectilinear vortices. As a result, solution of the differential field equations subject to the appropriate boundary conditions determines the long-wavelength collective motion of finite rotating superfluid systems.

We begin by considering the dynamics of infinite vortex arrays (Sec. II). Much of this material has appeared previously,[3–7] and only an abbreviated account will be presented. One new feature is our retention of the full three-dimensional character of the vortex dynamics, whereas previous work ignored motion along the vortex axis.[3–6] For infinite systems, this axial motion makes no dynamical contribution and is therefore superfluous. For finite systems, however, the three-dimensional character of the motion becomes crucial.

After considering the dynamics of vortex arrays, we specialize to long-wavelength phenomena (compared to the intervortex spacing), where the continuum limit becomes appropriate (Sec. III). Correspondingly, the equations of motion of a rotating superfluid contain the classical hydrodynamics of ordinary inviscid fluids but with added corrections that reflect the quantization of circulation (Sec. IV). Finally, we consider some simple examples of collective motion in superfluid systems and discuss further applications of this formalism (Secs. V and VI).

II. VORTEX LATTICE DYNAMICS

Consider an infinite incompressible superfluid of mass density ρ containing a system of rectilinear vortices with circulation $\kappa = h/m$ aligned along the z axis. The equilibrium position of the jth vortex is denoted by the two-dimensional vector $\vec{r}_j = x_j\hat{x} + y_j\hat{y}$. Each vortex is assumed to undergo small displacements about its equilibrium position, with the provision that the vortex bend only slightly. We denote this small displacement by the three-dimensional vector $\vec{u}_j(z_j)$. The velocity field induced by the vortices at the point $\vec{R}$ is then given by[8]

$$\vec{v}(\vec{R}) = \sum_j \frac{\kappa}{4\pi} \int_j \frac{d\vec{s}_j \times (\vec{R} - \vec{R}_j)}{|\vec{R} - \vec{R}_j|}, \qquad (1)$$

where $\vec{R}_j = \vec{r}_j + z_j\hat{z} + \vec{u}_j(z_j)$, and the line integral is along the vortex axis. Since each point on the axis of a vortex moves with the local fluid velocity, we can immediately derive the equation of motion for any vortex

$$\frac{\partial \vec{u}_i(z_i,t)}{\partial t} = \vec{v}(\vec{R}_i) = \sum_j{}' \frac{\kappa}{4\pi} \times \int_j \frac{d\vec{s}_j \times (\vec{R}_i - \vec{R}_j)}{|\vec{R}_i - \vec{R}_j|}, \quad (2)$$

where $\sum_j{}'$ omits the term $i=j$ in the sum, ignoring, for the moment, any self-induced motion.

For simple vortex arrays, Eq. (2) predicts that the equilibrium array will rotate uniformly about the z axis.[3–6] Since we are interested in small displacements with respect to this equilibrium configuration, we transform to a reference frame rotating uniformly about the z axis with angular frequency Ω, redefining $\vec{u}_i(z_i,t)$ to be the small displacement measured with respect to an equilibrium configuration at rest in the rotating frame and linearizing in these small displacements. Upon carrying out this procedure, we obtain[6]

$$\frac{\partial \vec{u}_i}{\partial t} = -\Omega \hat{z} \times \vec{u}_i + \frac{\kappa}{4\pi} \sum_j{}' \int_{-\infty}^{\infty} \left[\hat{z} \times \left(\frac{\vec{r}_{ij}}{|\vec{R}_{ij}^0|^3} - \frac{4\pi}{\kappa}\Omega \vec{r}_i \right) + \hat{z} \times \left(\frac{(\vec{u}_i - \vec{u}_j) - (z_i - z_j)\vec{u}'_j}{|\vec{R}_{ij}^0|^3} \right) - \hat{z} \times \left(\frac{3\vec{r}_{ij}[\vec{R}_{ij}^0 \cdot (\vec{u}_i - \vec{u}_j)]}{|\vec{R}_{ij}^0|^5} \right) + \frac{\vec{u}'_j \times \vec{r}_{ij}}{|\vec{R}_{ij}^0|^3} \right] dz_j, \quad (3)$$

where $\vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j$, $\vec{R}_{ij}^0 \equiv \vec{r}_{ij} + (z_i - z_j)\hat{z}$, and $\vec{u}'_i \equiv \partial \vec{u}_i(z_i)/\partial z$.

The first term in the integrand is independent of $\vec{u}$, and it must vanish in order that the equilibrium configuration be at rest in the rotating frame

$$\sum_j{}' \int_{-\infty}^{\infty} \hat{z} \times \left(\frac{\vec{r}_{ij}}{|\vec{R}_{ij}^0|^3} - \frac{4\pi\Omega}{\kappa} \vec{r}_i \right) dz_j = 0. \quad (4)$$

This condition determines Ω self-consistently; for a triangular lattice, Eq. (4) is satisfied if $\Omega = \frac{1}{2}\kappa n$, where n is the number density of vortices in the xy plane.[3,6] The equation of motion then becomes

$$\frac{\partial \vec{u}_i(z_i)}{\partial t} = -\Omega(\hat{z} \times \vec{u}_i) + \frac{\kappa}{4\pi} \sum_j{}' \int_{-\infty}^{\infty} \left[\hat{z} \times \left(\frac{(\vec{u}_i - \vec{u}_j) - (z_i - z_j)\vec{u}'_j}{|\vec{R}_{ij}^0|^3} - \frac{3\vec{r}_{ij}[\vec{R}_{ij}^0 \cdot (\vec{u}_i - \vec{u}_j)]}{|\vec{R}_{ij}^0|^5} \right) + \frac{\vec{u}'_j \times \vec{r}_{ij}}{|\vec{R}_{ij}^0|^3} \right] dz_j. \quad (5)$$

Within the linearized theory, Eq. (5) is exact. To proceed, we shall assume that the equilibrium configuration of a rotating superfluid is a triangular lattice of N vortices of length L occupying an area A in the xy plane. We impose periodic boundary conditions and use the continuous translational invariance in the z direction and the discrete translational invariance in the xy plane to expand $\vec{u}_i(z_i)$ in plane waves

$$\vec{u}_i(z_i) = (NL)^{-1/2} \sum_k \sum_{\vec{l}} e^{-i\vec{l}\cdot\vec{r}_i} e^{-ikz_i} \vec{u}_{k\vec{l}}, \quad (6)$$

where $k = 2\pi s/L$ $(s = 0, \pm 1, \ldots)$ and $\vec{l}$ is a vector in reciprocal-lattice space lying within the first Brillouin zone. Substituting (6) into (5), and using translational invariance to shift the origins of the sums and integrals, we obtain

$$\frac{\partial \vec{u}_{k\vec{l}}}{\partial t} = \frac{\kappa}{4\pi} \sum_j{}' \int_{-\infty}^{\infty} \left[(\hat{z} \times \vec{u}_{k\vec{l}}) \left(\frac{1 - e^{ikz} e^{i\vec{l}\cdot\vec{r}_j}}{(r_j^2 + z^2)^{3/2}} + \frac{ikz e^{ikz} e^{i\vec{l}\cdot\vec{r}_j}}{(r_j^2 + z^2)^{3/2}} \right) - 3(\hat{z} \times \vec{r}_j)(\vec{R}_j^0 \cdot \vec{u}_{k\vec{l}}) \left(\frac{1 - e^{ikz} e^{i\vec{l}\cdot\vec{r}_j}}{(r_j^2 + z^2)^{5/2}} \right) - \vec{u}_{k\vec{l}} \times \vec{r}_j \left(\frac{ik e^{ikz} e^{i\vec{l}\cdot\vec{r}_j}}{(r_j^2 + z^2)^{3/2}} \right) \right] dz - \Omega(\hat{z} \times \vec{u}_{k\vec{l}}). \quad (7)$$

The z integrals can be done be recalling the integral representation of the Bessel function[9]

$$K_\nu(kr) = \frac{\Gamma(\nu + \frac{1}{2})(2r)^\nu}{k^\nu \Gamma(\frac{1}{2})} \int_0^\infty \frac{\cos kz \, dz}{(z^2 + r^2)^{\nu + 1/2}}. \quad (8)$$

Using this result, translational invariance, and the symmetry of the triangular lattice, we find that many of the lattice sums in Eq. (7) vanish. Then, the Fourier transformed equations of motion in Cartesian coordinates become

$$\left(\frac{\partial \vec{u}_{k\vec{l}}}{\partial t} \right)_x = (\Omega'_g - \eta + \xi)(\vec{u}_{k\vec{l}})_y + \alpha(\vec{u}_{k\vec{l}})_x, \quad (9)$$

$$-\left(\frac{\partial \vec{u}_{k\vec{l}}}{\partial t} \right)_y = (\Omega'_g - \eta - \xi)(\vec{u}_{k\vec{l}})_x + \alpha(\vec{u}_{k\vec{l}})_y, \quad (10)$$

$$\left(\frac{\partial \vec{u}_{k\vec{l}}}{\partial t} \right)_z = 2i[\nu_y(\vec{u}_{k\vec{l}})_x - \nu_x(\vec{u}_{k\vec{l}})_y], \quad (11)$$

where we have defined the lattice sums[6]

$$\Omega'_g = \Omega + \frac{\kappa k^2}{4\pi} \sum_j{}' K_0(kr_j), \quad (12)$$

$$\eta = \frac{\kappa k^2}{4\pi} \sum_j{}' (1 - e^{i\vec{l}\cdot\vec{r}_j}) K_0(kr_j), \quad (13)$$

$$\xi = \frac{\kappa k^2}{4\pi} \sum_j{}' (1 - e^{i\vec{l}\cdot\vec{r}_j}) \frac{y_j^2 - x_j^2}{r_j^2} K_2(kr_j), \quad (14)$$

$$\alpha = \frac{\kappa k^2}{4\pi} \sum_j{}' (1 - e^{i\vec{l}\cdot\vec{r}_j}) \frac{2x_j y_j}{r_j^2} K_2(kr_j), \quad (15)$$

$$\nu_x = \frac{\kappa k^2}{4\pi} \sum_j{}' (1 - e^{i\vec{l}\cdot\vec{r}_j}) \frac{x_j}{r_j} K_1(kr_j), \quad (16)$$

$$\nu_y = \frac{\kappa k^2}{4\pi} \sum_j{}' (1 - e^{i\vec{l}\cdot\vec{r}_j}) \frac{y_j}{r_j} K_1(kr_j). \quad (17)$$

For an infinite array, it is clear from Eqs. (9), (10), and (11) that the x and y equations of motion determine the entire dynamics. Therefore, previous work has omitted Eq. (11).[3–6] In general, however, the z displacements of the vortex cores are nonzero, and Eq. (11) is essential in maintaining the incompressibility of the fluid. Further, if applications to finite geometries are anticipated, the boundary conditions on the z component of the vortex displacements will affect the xy motion in a nontrivial way.

To this point, we have dealt only with a model consisting of idealized filamentary vortex lines. This model is unphysical due to the divergence of the velocity field near the vortex axis. We must modify our treatment by considering the structure of the vortex core. Since a finite core gives rise to self-induced motion, we expect that our dynamics will be slightly altered. We are primarily interested in long-wavelength phenomena, however, and do not expect the details of the core structure to be critical.

As a simple model, consider a core of radius a containing p ($p \gg 1$) identical elementary vortex filaments with circulation κ/p. Equation (5) may be applied to the system of vortex cores in a triangular lattice by considering the motion of each elementary filament. The collective motion of a given core is then found by averaging the motion of the filamentary vortices over the core. This procedure is straightforward; in the case of a uniform distribution of filamentary vortices within each core, the only modification of Eqs. (9), (10), and (11) is the replacement of Ω'_g by

$$\Omega_g = \Omega'_g + \kappa k^2 (4\pi)^{-1} [\ln(2/ka) - \gamma + \tfrac{1}{4}],$$

where $\gamma = 0.5772\cdots$ is Euler's constant.[6]

III. LONG-WAVELENGTH APPROXIMATION—CONTINUUM LIMIT

In the case of a dense vortex lattice where the intervortex spacing is small compared to the dimension of the system, it becomes appropriate to study Eqs. (9)–(11) in the long-wavelength limit. If we define $b = (n\pi)^{-1/2}$ to be the mean intervortex spacing, the long-wavelength limit corresponds to taking $kb \ll 1$ and $lb \ll 1$ in the lattice sums (12)–(17). The lattice sums can be evaluated[10] through order b^2:

$$\Omega_g \approx \Omega [2 + (kb)^2 C_\Omega], \quad (18)$$

$$\eta \approx \Omega l^2 (l^2 + k^2)^{-1}, \quad (19)$$

$$\xi \approx \Omega (l_y^2 - l_x^2) [(l^2 + k^2)^{-1} - \tfrac{1}{8} b^2], \quad (20)$$

$$\alpha \approx 2\Omega l_y l_x [(l^2 + k^2)^{-1} - \tfrac{1}{8} b^2], \quad (21)$$

$$\nu_x \approx -i \Omega l_x k [(l^2 + k^2)^{-1} - C_\nu b^2], \quad (22)$$

$$\nu_y \approx -i \Omega l_y k [(l^2 + k^2)^{-1} - C_\nu b^2], \quad (23)$$

where C_Ω and C_ν are constants defined by

$$C_\Omega = \frac{1}{2} \ln\left(\frac{b}{a}\right) - \frac{1}{4}\gamma - \frac{1}{8} + \frac{1}{4} \sum_j{}' [E_1(r_j^2 b^{-2}) + r_j^2 b^{-2} \exp(r_j^2 b^{-2})]$$
$$\approx \frac{1}{2} \ln\left(\frac{b}{a}\right) - 0.244, \quad (24)$$

$$C_\nu = \frac{1}{4} + \frac{1}{2} \sum_j{}' \exp(r_j^2 b^{-2}) \approx 0.330. \quad (25)$$

These quantities give the equations of motion of the Fourier coefficients, correct through order b^2:

$$\frac{\partial (\vec{u}_{k\vec{l}})_x}{\partial t} = 2\Omega \left[\left(\frac{l_y^2 + k^2}{l^2 + k^2} - \frac{b^2}{16}(l_y^2 - l_x^2) + \frac{k^2 b^2}{2} C_\Omega \right) (\vec{u}_{k\vec{l}})_y + \left(\frac{l_x l_y}{l^2 + k^2} - \frac{b^2 l_x l_y}{8} \right) (\vec{u}_{k\vec{l}})_x \right], \quad (26)$$

$$-\frac{\partial (\vec{u}_{k\vec{l}})_y}{\partial t} = 2\Omega \left[\left(\frac{l_x^2 + k^2}{l^2 + k^2} + \frac{b^2}{16}(l_y^2 - l_x^2) + \frac{k^2 b^2}{2} C_\Omega \right) (\vec{u}_{k\vec{l}})_x + \left(\frac{l_x l_y}{l^2 + k^2} - \frac{b^2 l_x l_y}{8} \right) (\vec{u}_{k\vec{l}})_y \right], \quad (27)$$

$$\frac{\partial (\vec{u}_{k\vec{l}})_z}{\partial t} = 2\Omega \left[\left(-\frac{l_x k}{l^2 + k^2} + C_\nu l_x k b^2 \right) (\vec{u}_{k\vec{l}})_y - \left(-\frac{l_y k}{l^2 + k^2} + C_\nu l_y k b^2 \right) (\vec{u}_{k\vec{l}})_x \right]. \quad (28)$$

We can now take the continuum limit by interpreting the displacements as continuous functions of position rather than defined only at discrete lattice positions. This transformation is accomplished by the substitution $\vec{u}_i(z_i) \rightarrow \vec{u}(\vec{r})$. If Eqs. (26)–(28) are multiplied by (l^2+k^2), inversion of the Fourier transform readily yields the differential equation of motion satisfied by the vector field $\vec{u}(\vec{r})$,

$$(\vec{\nabla}\vec{\nabla}\cdot - \vec{\nabla}\times\vec{\nabla}\times)\frac{\partial\vec{u}(\vec{r},t)}{\partial t} = 2\Omega\left\{\vec{\nabla}\times\vec{\nabla}\times(\hat{z}\times\vec{u}) + \frac{b^2}{16}\left[-\vec{\nabla}\times\vec{\nabla}\times\vec{\nabla}\times[\vec{\nabla}\cdot(\vec{u}-\vec{u}\cdot\hat{z}\hat{z})]\hat{z} + \left(\vec{\nabla}-\hat{z}\frac{\partial}{\partial z}\right)\nabla^2(\vec{\nabla}\cdot\hat{z}\times\vec{u}) + 16C_\nu\hat{z}\frac{\partial}{\partial z}\nabla^2(\vec{\nabla}\cdot\hat{z}\times\vec{u}) + 8C_\Omega(\vec{\nabla}\vec{\nabla}\cdot - \vec{\nabla}\times\vec{\nabla}\times)\frac{d^2}{dz^2}\hat{z}\times\vec{u}\right]\right\}. \qquad (29)$$

This equation has several noteworthy features. First, it is first order in time derivatives and thus requires only the initial configuration to determine the subsequent motion of the system. This is because the interaction between vortices affects the vortex velocity and not the acceleration. Second, the most straightforward approach to this problem would have been to attempt a direct expansion of Eq. (5) in powers of b. The form of Eq. (29) shows, however, that $\nabla^2 \partial\vec{u}/\partial t$, not $\partial\vec{u}/\partial t$ itself, has a simple expansion in powers of the intervortex spacing. Third, the terms proportional to b^2 constitute the lowest-order correction due to the existence of the vortex lattice. As shown in Sec. IV, the classical equations of motion for a rotating fluid can be recovered simply by letting $b^2 \rightarrow 0$. This dependence on b^2 is the only remnant of the quantum-mechanical nature of the vortex system.

IV. CLASSICAL LIMIT

It is interesting to recall the linearized form of Euler's equations in a classical inviscid incompressible rotating fluid.[11] Conservation of momentum yields the classical relation

$$\frac{\partial\vec{v}(\vec{r},t)}{\partial t} + 2\Omega\hat{z}\times\vec{v}(\vec{r},t) = -\vec{\nabla}\rho^{-1}\bar{P}(\vec{r},t), \qquad (30)$$

where $\vec{v}(\vec{r},t)$ is the fluid velocity and $\bar{P}$ is the "reduced pressure," given by $\bar{P}=(P-\frac{1}{2}\rho|\vec{\Omega}\times\vec{r}|^2 - V)$. Here P is the hydrodynamic pressure and V is any external potential that couples to the fluid mass. The continuity equation has the usual form

$$\vec{\nabla}\cdot\vec{v}(\vec{r},t) = 0. \qquad (31)$$

Motivated by Eq. (31), we consider the combination

$$\frac{\partial}{\partial t}[l_x(\vec{u}_{k\vec{l}})_x + l_y(\vec{u}_{k\vec{l}})_y + k(\vec{u}_{k\vec{l}})_z].$$

Using Eqs. (26)–(28), we have

$$\frac{\partial}{\partial t}[l_x(\vec{u}_{k\vec{l}})_x + l_y(\vec{u}_{k\vec{l}})_y + k(\vec{u}_{k\vec{l}})_z] = 2\Omega b^2[\tfrac{1}{16}(l_y^2+l_x^2) + \tfrac{1}{2}(C_\Omega + 2C_\nu)k^2] \times [l_x(\vec{u}_{k\vec{l}})_y - l_y(\vec{u}_{k\vec{l}})_x]. \qquad (32)$$

Inversion of the Fourier transform yields

$$\vec{\nabla}\cdot\frac{\partial}{\partial t}\vec{u} = -2\Omega b^2\left[\frac{1}{16}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}(C_\Omega+2C_\nu)\frac{\partial^2}{\partial z^2}\right](\hat{z}\cdot\vec{\nabla}\times\vec{u}). \qquad (33)$$

This equation simplifies Eq. (29) to the form

$$\vec{\nabla}\times\vec{\nabla}\times\left[\frac{\partial\vec{u}}{\partial t} + 2\Omega\hat{z}\times\vec{u} - \frac{2\Omega b^2}{16}\times\left\{\vec{\nabla}\times[\vec{\nabla}\cdot(\vec{u}-\vec{u}\cdot\hat{z}\hat{z})]\hat{z} + (16C_\nu - 1)\times\frac{\partial}{\partial z}(\vec{\nabla}\cdot\hat{z}\times\vec{u})\hat{z} + 8C_\Omega\frac{\partial^2}{\partial z^2}\hat{z}\times\vec{u}\right\}\right] = 0, \qquad (34)$$

which can be satisfied whenever

$$\frac{\partial\vec{u}}{\partial t} + 2\Omega\hat{z}\times\vec{u} - \frac{2\Omega b^2}{16}\times\left\{\vec{\nabla}\times[\vec{\nabla}\cdot(\vec{u}-\vec{u}\cdot\hat{z}\hat{z})]\hat{z} + (16C_\nu - 1)\times\frac{\partial}{\partial z}(\vec{\nabla}\cdot\hat{z}\times\vec{u})\hat{z} + 8C_\Omega\frac{\partial^2}{\partial z^2}\hat{z}\times\vec{u}\right\} = -\vec{\nabla}F, \qquad (35)$$

with F an arbitrary scalar function. Equations (33) and (35) form a system of four equations in four unknowns (u_x, u_y, u_z, F) which must be satisfied subject to the appropriate boundary and initial conditions.

Comparison of Eqs. (30) and (31) with Eqs. (33) and (35) is suggestive. We must, however, relate the displacements $\vec{u}(\vec{r},t)$ to the fluid velocity $\vec{v}(\vec{r},t)$. In essence, our reliance on the displacements $\vec{u}(\vec{r},t)$ constitutes a Lagrangian description of the fluid,[12] where we follow the motion of a given element of the fluid as it moves through space. Hence, the position of the element of fluid initially at $\vec{r}$ is given at a later time by $\vec{r}+\vec{u}(\vec{r},t)$. This view differs from the usual Eulerian description, which focuses on a given element of space. Since each element of fluid moves in space with the local fluid velocity, the relation between the Lagrangian and Eulerian descriptions is given by

$$\vec{v}(\vec{r}+\vec{u}(\vec{r},t),t) = \frac{\partial \vec{u}(\vec{r},t)}{\partial t}.$$

For small $\vec{u}$, we have

$$\vec{v}(\vec{r},t) + [\vec{u}(\vec{r},t)\cdot\vec{\nabla}]\,\vec{v}(\vec{r},t) = \frac{\partial \vec{u}(\vec{r},t)}{\partial t},$$

and, in the linearized regime, we have simply $\partial\vec{u}(\vec{r},t)/\partial t = \vec{v}(\vec{r},t)$.

It is now possible to connect Euler's equations with the equations of motion of the vortex system. First differentiate (35) with respect to time, and then replace $\partial\vec{u}/\partial t$ by $\vec{v}(\vec{r},t)$ in Eqs. (33) and (35) to obtain

$$\vec{\nabla}\cdot\vec{v}(\vec{r},t) = O(b^2), \qquad (36)$$

$$\frac{\partial \vec{v}(\vec{r},t)}{\partial t} + 2\Omega\hat{z}\times\vec{v} + O(b^2) = -\frac{\vec{\nabla}\,\partial F}{\partial t}. \qquad (37)$$

Since $b^2 = \hbar/m\,\Omega$, the classical limit $(\hbar\to 0)$ is obtained by taking $b^2\to 0$ in (36) and (37), and identifying $\partial F/\partial t = \rho^{-1}\bar{P}$. In this way, we exactly reproduce the classical equations of motion.

V. APPLICATIONS

For an unbounded fluid, the equations of motion (26)–(28) for the Fourier coefficients predict a plane-wave dispersion relation[10]

$$\omega^2 = 4\Omega^2[k^2 + \tfrac{1}{16}l^4 b^2 + \tfrac{1}{2}C_\Omega(l^2+2k^2)k^2b^2](l^2+k^2)^{-1}. \qquad (38)$$

In the classical limit $(b\to 0)$, these solutions become the familiar transverse left-handed circularly polarized inertial waves of a rotating fluid[13]; the dispersion relation for these waves is given by $\omega^2 = (2\Omega\hat{n}\cdot\hat{z})^2$, where $\hat{n} = (\vec{l}+k\hat{z})(l^2+k^2)^{-1/2}$ is the direction of wave propagation and $\hat{z}$ is along the axis of rotation. The system is highly dispersive, with the phase velocity $\vec{v}_p = v_0(\hat{n}\cdot\hat{z})\hat{n}$ and the group velocity $\vec{v}_g = v_0[\hat{z}-(\hat{n}\cdot\hat{z})\hat{n}] = v_0\hat{z}-\vec{v}_p$, where $v_0 \equiv 2\Omega(l^2+k^2)^{-1/2}$. Each vortex experiences bending along its axis and undergoes elliptical motion in the xy plane with a given element of fluid executing circular motion in the plane perpendicular to the direction of propagation. For fixed k, the system exhibits the unusual property that the frequency *decreases* with increasing l; equivalently, the group velocity for propagation perpendicular to the rotation axis is *opposite* to l. The dashed curves in Fig. 1 illustrate this feature of the classical system, showing that $\omega\to 0$ as $k/l\to 0$.

It is just in the regime of small k/l, however, where the underlying vortex structure modifies this classical picture. When the wave propagates nearly perpendicular to the rotation axis $(k/l\to 0)$, the vortices remain essentially undeformed, executing elliptical motion in the xy plane. For $k/l = 0$, these solutions are Tkachenko waves[3–6]; they are nondispersive with $v_p = v_g = \frac{1}{2}\Omega b$, and have no analog in a classical fluid, where $b = 0$. Such Tkachenko motion alters the dynamical spectrum, as seen by the solid curves in Fig. 1; the terms proportional to b^2 in Eq. (38) dominate as $k/l\to 0$, and the dispersion relation for fixed nonzero k has a minimum as a function of l, markedly reducing the low-frequency density of modes.[14]

As a simple example of a finite system, we consider a rotating superfluid in a rigid cylinder of length L and radius R. The equations of motion (33) and (35) must be solved subject to the boundary condition that the normal component of $\vec{u}$ vanish at the boundaries of the cylinder.[15] We seek a solution of the form

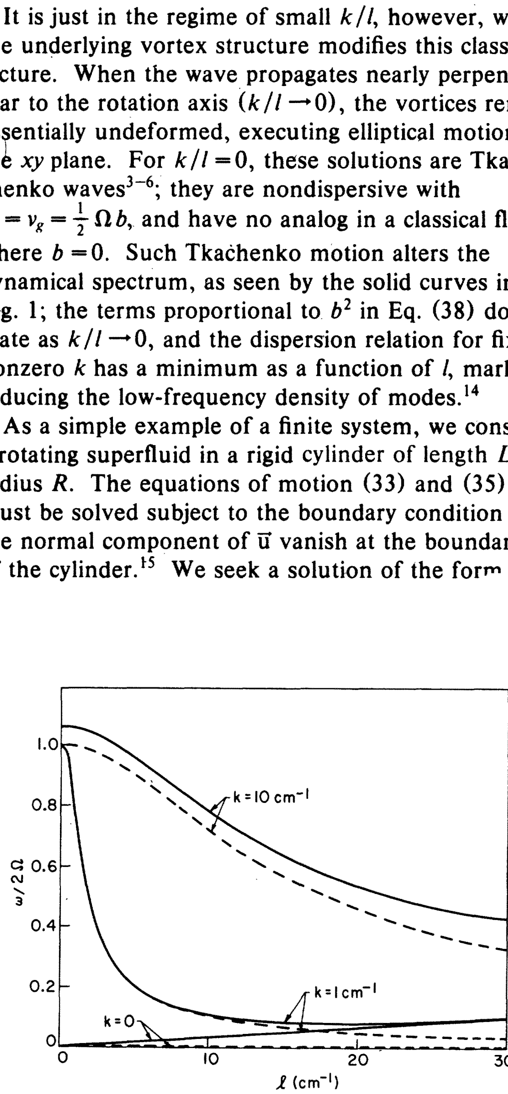

FIG. 1. Dispersion relation for a plane wave in an infinite rotating fluid. The curves show the dimensionless frequency $\omega/2\Omega$ for wave number l perpendicular to the axis of rotation; they are labeled with the fixed wave number k parallel to the axis of rotation. The dashed curves represent the classical limit $(b\to 0)$; the solid curves include the lowest-order modification due to the quantization of circulation in a typical sample of He II ($\Omega \approx 1$ sec^{-1}, $\kappa \approx 10^{-3}$ cm^2 sec^{-1}, $b \approx 1.3\times 10^{-2}$ cm, $a \approx 1\times 10^{-8}$ cm).

$$\vec{u}(\vec{r},t)=[u_r(r)\cos(kz)\hat{r}+u_\phi(r)\cos(kz)\hat{\phi}+u_z(r)\sin(kz)\hat{z}]\times\exp(im\phi-i\omega t),$$

where $k=p\pi/L$, $p=0,1,2,\ldots$, and $m=0,\pm1,\pm2,\ldots$. The resulting system of coupled differential equations is difficult to solve in general, and we only consider the two special cases of axisymmetric modes and Tkachenko modes.

Axisymmetric modes ($m=0$; $p=0,1,2,\ldots$). If $\vec{u}(\vec{r})$ is independent of ϕ, then the equations of motion (33) and (35) lead to a fourth-order differential equation for $u_r(r)$. This differential equation can be factored into two commuting second-order differential operators acting on $u_r(r)$. It is then straightforward to solve this equation subject to the boundary conditions that $u_r(R)$ vanish and $u_r(0)$ be finite. A rather lengthy calculation yields the eigenfunctions

$$\vec{u}_{n,0,p}(\vec{r},t)=\left[\left[\left(\hat{r}-\frac{2i\Omega}{\omega_{n,0,p}}\left\{1+\frac{b^2}{16}\left[8C_\Omega\left(\frac{p\pi}{L}\right)^2-\left(\frac{j_{1,n}}{R}\right)^2\right]\right\}\hat{\phi}\right)J_1\left(j_{1,n}\frac{r}{R}\right)\cos\left(\frac{p\pi z}{L}\right)-\hat{z}\left(\frac{j_{1,n}L}{p\pi R}\right)\right.\right.$$
$$\left.\times\left\{1-\frac{b^2}{16}\left(\frac{2\Omega}{\omega_{n,0,p}}\right)^2\left[\left(\frac{j_{1,n}}{R}\right)^2+(8C_\Omega+16C_\nu)\left(\frac{p\pi}{L}\right)^2\right]\right\}J_0\left(j_{1,n}\frac{r}{R}\right)\sin\frac{p\pi z}{L}\right]e^{-i\omega_{n,0,p}t}. \tag{39}$$

Here, the eigenfrequencies are given by

$$\omega_{n,0,p}=2\Omega\left\{1+\left(\frac{b\pi p}{L}\right)^2\left[C_\Omega+\frac{C_\Omega}{2}\left(\frac{j_{1,n}L}{p\pi R}\right)^2+\frac{1}{16}\left(\frac{j_{1,n}L}{p\pi R}\right)^4\right]\right\}^{1/2}\left[1+\left(\frac{j_{1,n}L}{p\pi R}\right)^2\right]^{-1/2}, \tag{40}$$

where $n=1,2,\ldots$, $p=0,1,2,\ldots$, and $j_{m,n}$ is the nth zero of the mth Bessel function.

The spectrum is obviously discrete, but the modes are similar in other respects to the plane-wave modes in the unbounded fluid. In particular, as b tends to zero, the modes approach the classical inertial modes in a rotating cylinder,[16] and as p/n approaches zero, the modes approach the axisymmetric Tkachenko modes appropriate for this geometry. The motion in the xy plane is again approximately elliptical, the ϕ-motion being exactly out of phase with the r motion. Further, while the axisymmetric modes of the classical system ($b=0$) become dense near $\omega=0$ as $p/n\to0$, when $b\neq0$, the Tkachenko motion of the lattice dominates the spectrum as $p/n\to0$ resulting in a discrete isolated lowest-frequency axisymmetric mode ($n=1, p=0$).

General Tkachenko modes ($m=0,\pm1,\pm2,\ldots;p=0$). If we consider motion with no z dependence, we generate Tkachenko modes. For the axisymmetric case the eigenfunctions and eigenfrequencies can be found from Eqs. (39) and (40),

$$\vec{u}_{n,0,0}(r)=\left\{\hat{r}-\frac{2i\Omega}{\omega_{n,0,0}}\left[1-\frac{b^2}{16}\times\left(\frac{j_{1,n}}{R}\right)^2\right]\hat{\phi}\right\}J_1\left(j_{1,n}\frac{r}{R}\right), \tag{41}$$

$$\omega_{n,0,0}=\Omega b j_{1,n}/2R. \tag{42}$$

For $m\neq0$, we must return to the differential equations, where we find

$$\vec{u}_{n,m,0}(\vec{r},t)=\left(\left[\frac{1}{r}J_m\left(j_{m,n}\frac{r}{R}\right)\right]\hat{r}-i\left\{\left[\frac{1}{r}+\frac{\omega_{n,m,0}}{2\Omega}\left(\frac{j_{m,n}}{mR}\right)^2r\right]\times J_m\left(j_{m,n}\frac{r}{R}\right)-\frac{j_{m,n}}{mR}\times J_{m-1}\left(j_{m,n}\frac{r}{R}\right)\right\}\hat{\phi}\right)\times\exp[i(m\phi-\omega_{n,m,0}t)], \tag{43}$$

$$\omega_{n,m,0}=\Omega b j_{m,n}/2R. \tag{44}$$

It is interesting to note that the Tkachenko modes for $m=0$ and $m=1$ are degenerate in this geometry, even though they correspond to physically distinct solutions.

For laboratory conditions in ^{4}He ($\kappa\approx10^{-3}$ cm^2/sec, $R\approx1$ cm, $\Omega\approx1$ sec^{-1}), the lowest Tkachenko modes have periods on the order of a few hundred seconds. We are unaware of any direct laboratory observations of these modes, although they should, in principle, be observable in photographic experiments that detect the position of quantized vortices in rotating superfluids. Alternatively, Ruderman[17] has proposed that Tkachenko modes within the core of a rotating neutron star might provide an explanation for the long-period phenomenon in pulsars observed after a glitch. For the unphysical, but tractable cylindrical neutron star considered here, the predicted period is on the order of months, in rough agreement with apparent time scales.

VI. DISCUSSION

We have constructed a model of the dynamic behavior of a rotating superfluid based on the continuum limit of an infinite dense array of quantized vortices. The model not only reproduces the usual results of classical hydrodynamics but also predicts the leading corrections due to the underlying vortex lattice. This description is useful in studying the collective motion of finite rotating quantum systems since it avoids a detailed calculation of the equilibrium vortex configuration. For cylindrical geometries, it is possible to obtain explicit normal-mode solutions where the Tkachenko motion of the vortex lattice modifies the classical results.

It would be attractive to apply this model to more complicated geometries. For example, a spherical configuration would provide a better model of a rotating neutron star. Since the cylindrical model was derived by assuming small bending of *rectilinear* vortices, however, caution is necessary for geometries like a sphere that *require* vortex bending to match the boundary conditions. Such a system might be treated with a modification of the present formalism, but the procedure is likely to be complicated; this subject remains for further investigation.

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[15]Rather than requiring only the normal component of $\vec{u}$ to vanish on the boundary, we might want to consider pinned vortices, demanding that all components of $\vec{u}$ vanish on the boundaries. By reducing the order of the differential equations in going from Eq. (34) to (35), however, the freedom to specify the boundary values of all the components of $\vec{u}$ has been lost; this situation is similar to that in going from the Navier-Stokes equation to Euler's equation in classical fluid mechanics. Hence, by choosing to solve the more tractable problem posed by Eq. (35), we cannot consider situations where the vortices are pinned to the walls.

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