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Amplitude universality and confluent corrections to scaling for Ising models

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Using existing series for spin-s Ising models and for continuous-spin Ising models, we have verified the universality of the ratio of the amplitudes associated with the leading confluent corrections to the susceptibility and to the correlation length. We have, thereby, tested the universality of the subcritical part of the correlation function, of which the universality of the amplitude ratio in question is a direct consequence.

Several authors have shown that the small, apparent violations of universality found in naive "one singularity" analysis of Ising and Heisenberg series and of experiments on the critical behavior of the ^{4}He superfluid can be attributed to confluent corrections to the leading singularity.[1-3] Including one confluent correction, the forms of the susceptibility, zeroth correlation function moment μ_0, and of the square of the correlation length become

$$\mu_0 = A_s t^{-\gamma}(1+B_s t^{\Delta_1}), \quad t = 1 - T_c/T , \tag{1}$$

$$\mu_2/\mu_0 = A_\kappa t^{-2\nu}(1+B_\kappa t^{\Delta_1}) . \tag{2}$$

μ_2 is the second correlation function moment. All of the above investigators found values of the correction exponent Δ_1 in the range $0.50 < \Delta_1 < 0.65$, and none claimed to present convincing evidence that their "best value" was correct to within less than 5% or 10%.[1-3] Renormalization group calculations corroborate both the approximate numerical value for Δ_1 and its approximate model independence.[4] This work[1-4] verified, to within rather large confidence limits, the universality predictions that the value of Δ_1 should be the same for all correlation function moments and should be independent of all irrelevant parameters, e.g., spin in the magnetic models and pressure for the ^{4}He superfluid transitions.

We extend this earlier work by investigating, for perhaps the first time,[5] the universality of the subcritical part of the correlation function, from which the universality of B_κ/B_s follows as a direct consequence through use of the following scaling and universality arguments. If we write the correlation function as a power series in the irrelevant field ϕ, we find

$$\Gamma(\vec{r}, t, \phi) = (l/r)^{d-2+\eta}\left[D_0((gt)^\nu r/l) + D_1((gt)^\nu r/l)\,\phi(gt)^{\Delta_1} + \cdots\right] ,$$

where we assume the critical function $D_0(x)$ and the subcritical function $D_1(x)$ are both universal, with all the nonuniversal system-dependence contained in the irrelevant field ϕ and in the scale factors l and g.[7] Integrating this expression with an r^n weight factor to determine the nth correlation function moment, we find

$$\mu_n = l^{d+n}(gt)^{-\gamma-n\nu}\{\alpha_{0,n} + \alpha_{1,n}[\phi(g)^{\Delta_1}]t^{\Delta_1} + \cdots\} ,$$

where $\alpha_{0,n}$ and $\alpha_{1,n}$ are universal integrals depending only on the label n and the form of the functions $D_0(x)$ and $D_1(x)$. From this expression we see that the nonuniversal contribution to the correction amplitude $B_n = \phi g^{\Delta_1}\alpha_{1,n}/\alpha_{0,n}$ is the same for all n, so that any ratio B_n/B_m will be universal. Clearly, $B_s = B_0$ and $B_\kappa = B_2 - B_0$, so that B_κ/B_s will also be universal.[6]

To investigate the universality of the amplitude ratio B_κ/B_s for various Ising models, we use the twelfth-order high-temperature series for the nearest-neighbor spin-s Ising model for the fcc lattice from Ref. 1, and from Ref. 2, the tenth-order high-temperature series for nearest-neighbor continuous-spin Ising models having the single-spin Hamiltonian H_i, $-\beta H_i = -\lambda(\mu_i^4 - 2\mu_i^2)$, in which λ determines the sharpness of the peaking of the spin distribution about $\mu = \pm 1$, the two discrete values of the projection for an $s = \frac{1}{2}$ spin.

Previous experience testing the universality of amplitude ratios has shown that, not unreasonably, the values used for the indices in determining the amplitudes should be constrained to universal values, because usually a small change in the value of the critical temperature or a critical index will effect a larger change in the value of a resulting amplitude.[7] Therefore, allowing nonuniversal variations in the indices, which are used as input in the amplitude determination, forces a nonuniversal variation of the amplitudes. In this investigation, the values of the indices γ, ν, and Δ_1 are constrained to a set of universal values in determining B_κ, B_s, and B_κ/B_s for various values of the spin-s and of the peaking parameter λ. For the Ising models, the values of γ and ν and therefore T_c have quite small confidence limits; however, the value of Δ_1 has large confidence limits,

TABLE I. Values of the amplitudes B_s and B_κ and the ratio B_κ/B_s for various Ising models. These values for the amplitudes were determined by extrapolating Saul-Wortis sequences, which were constructed using the values $\gamma=1.25$, $2\nu=1.275$, the values of K_c^{-1} shown, and the two values of Δ_1 shown. The ratio was determined by extrapolating the sequence formed by taking the ratio of corresponding terms in the amplitude sequences.

			$\Delta_1=0.575$			$\Delta_2=0.600$	
Model	K_c^{-1}	B_s	B_κ	B_κ/B_s	B_s	B_κ	B_κ/B_s
$s=1$	10.229	0.142 ± 0.003	0.196 ± 0.005	1.39 ± 0.02	0.150 ± 0.002	0.212 ± 0.004	1.40 ± 0.03
$s=\frac{3}{2}$	10.362	0.207 ± 0.003	0.294 ± 0.002	1.41 ± 0.01	0.222 ± 0.002	0.316 ± 0.010	1.41 ± 0.02
$s=2$	10.421	0.241 ± 0.002	0.340 ± 0.004	1.41 ± 0.02	0.259 ± 0.004	0.367 ± 0.006	1.41 ± 0.03
$s=\infty$	10.522	0.320 ± 0.006	0.456 ± 0.010	1.41 ± 0.02	0.345 ± 0.008	0.489 ± 0.016	1.41 ± 0.03
$\lambda=\frac{1}{4}$	10.613	0.392 ± 0.003	0.545 ± 0.015	1.405 ± 0.03	0.417 ± 0.009	0.583 ± 0.020	1.406 ± 0.04
$\lambda=\frac{1}{2}$	10.479	0.289 ± 0.003	0.410 ± 0.004	1.42 ± 0.03	0.307 ± 0.003	0.433 ± 0.010	1.42 ± 0.03
$\lambda=1$	10.303	0.188 ± 0.003	0.265 ± 0.003	1.41 ± 0.02	0.198 ± 0.002	0.280 ± 0.001	1.41 ± 0.02
$\lambda=\frac{3}{2}$	10.186	0.139 ± 0.005	0.190 ± 0.004	1.37 ± 0.04	0.147 ± 0.002	0.199 ± 0.004	1.36 ± 0.04
$\lambda=2$	10.104	0.101 ± 0.005	0.136 ± 0.006	1.30 ± 0.06	0.113 ± 0.005	0.145 ± 0.007	1.30 ± 0.06

lying in the range $0.50<\Delta_1<0.65$. Even though the most reliable series evidence from the Ising models seems to narrow the range to $0.565<\Delta_1<0.610$, this still represents a large uncertainty in Δ_1. For this reason, we determine the amplitude ratios for several values of Δ_1 in the above range.

Using Saul-Wortis four-fit methods,[1,8,9] modified to fix all parameters in Eqs. (1) and (2) excepting the amplitudes,[10] and using $\gamma=1.250$, $2\nu=1.275$, and the best values of $K_c=J/kT_c$ from Refs. 1 and 2 given in Table I, we have determined the values of the amplitudes B_s and B_κ and their ratio B_κ/B_s, all of which are presented in Table I for two values of Δ_1. The uncertainties quoted in Table I reflect only the scatter in the Saul-Wortis sequence, and do not reflect uncertainties in γ, ν, Δ_1, or K_c; as an example, consider the sequence for B_κ/B_s for the $\lambda=\frac{1}{4}$ continuous-spin model, using $\Delta_1=0.575$, which follows 1.292, 1.281, 1.304, 1.320, 1.330, 1.339, 1.347, 1.354, and 1.360, a Neville-table extrapolation of which yields 1.405 ± 0.030.

This analysis shows that the amplitude ratio is universal with the value $B_\kappa/B_s=1.41\pm0.05$. Note that although the amplitudes B_κ and B_s are quite sensitive to changes in the value of Δ_1, the ratio of these amplitudes is relatively insensitive to such changes. Indeed, further testing the effect of such changes in Δ_1 on the value of the ratio shows that a 1% change in the value of Δ_1 effects no more than a 0.07% change in the value of B_κ/B_s throughout the range $0.50<\Delta_1<0.65$. Also, it can be seen from Table I that the largest deviations from the universal value quoted above, also the largest uncertainties, occur for those models for which the confluent singularity is least important, small B_κ and B_s, for $s=1$ and $\lambda=1.5$, 2.0 making the determination of the parameters characterizing the confluent singularity more susceptible to irregular scatter in the series.

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[1]The Ising model susceptibility was investigated by D. M. Saul, Michael Wortis, and David Jasnow [Phys. Rev. B 11, 2571 (1975)]; and by William J. Camp and J. P. Van Dyke [Phys. Rev. B 11, 2579 (1975)]. The Ising-model second moment was investigated by William J. Camp, D. M. Saul, J. P. Van Dyke, and Michael Wortis [Phys. Rev. B 14, 3990 (1976)]. The Heisenberg-model susceptibility was investigated by William J. Camp and J. P. Van Dyke [J. Phys. A 9, 721 (1976)].

[2]J. P. Van Dyke and W. J. Camp, Phys. Rev. Lett. 35, 323 (1975); M. Ferer and R. Macy, AIP Conf. Proc. 34, 385 (1976).

[3]D. S. Greywall and G. Ahlers, Phys. Rev. Lett. 28, 1251 (1972).

[4]F. J. Wegner, Phys. Rev. B 5, 4529 (1972); ϵ expansions: first order—F. J. Wegner, Phys. Rev. B 6, 1891 (1972); second order—A. Aharony, Phys. Rev. B 8, 3349 4270 (1973); third order—E. Brezin *et al.*, Phys. Rev. B 8, 5330 (1973). Coupling constant expansion: G. A. Baker, Jr., B. G. Nickel, M. S. Green, and D. I. Meiron, Phys. Rev. Lett. 36, 1351 (1976).

[5]To our knowledge, the only previous quantitative investigation of this "extended" universality was for the $d=2$, $s=\frac{1}{2}$ Ising models and for the spherical models [D. S. Ritchie and D. D. Betts, Phys. Rev. B 11, 2559 (1975)]; it should be noted that for both sets of models, $\Delta_1=1$, suggesting that these models are exceptional

in that the amplitudes of the general correction-to-scaling terms are zero.

[6]The ratio B_K/B_S can be determined from the values given in Refs. 1 and 2; however, the resulting ratios exhibit misleading, nonuniversal variations due to differences between the value of Δ_1 used in determining B_0 and the value used in determining B_2.

[7]M. Ferer and Michael Wortis, Phys. Rev. B 6, 3426 (1972).

[8]D. Saul, Ph.D. thesis (University of Illinois, 1974) (unpublished).

[9]Baker-Hunter methods of analysis [G. A. Baker, Jr. and and D. L. Hunter, Phys. Rev. B 7, 3377 (1973)] give consistent results with rather more scatter.

[10]Unbiased methods reflect a sizeable scatter in the values of T_C and the indices, which scatter can introduce unnecessary scatter in the sequence for the, amplitude, or even apparent but spurious convergence of this sequence.