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Phase transitions and tricritical points: An exactly soluble model for magnetic or distortive systems

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We present and solve exactly a lattice model for magnetic or for structural phase transitions. The model proposed here can be seen as an extension of the spherical model. We obtain the following results: (i) The free energy of the system is calculated rigorously in a general case for lattices of any dimensionality d or structure. (ii) The critical properties are worked out explicitly for a special case of the interaction on a cubic d-dimensional lattice (d may be a fractional number). If $d > 2$, second-order phase transitions may occur. The critical exponents are those of the sperical model. (iii) For a special choice of the interaction and if $d \geq 3$, the existence of a line of tricritical points can be demonstrated. The tricritical exponents are computed explicitly; they are identical to the exponents of the "Gaussian" model. (iv) Finally, first-order phase transitions are shown to exist in one and two dimensions.

I. INTRODUCTION

In this paper we present and solve exactly a continuous-spin model. The model may be considered as an extension of the spherical model of Berlin and Kac,[1] but was first thought as a modification of the well-known continuous-spin Ising model, which describes magnetic[2] as well as structural phase transitions.[3,4]

Although it may be far removed from the physical reality, it seems nevertheless worthwhile examining the model proposed here. In fact, we shall show that the critical properties of such a model system are quite interesting and that tricritical phenomena may be treated exactly. Moreover, the predictions of the renormalization-group theory can be checked.

The results obtained in this paper are the following: (i) The free energy of the system is calculated rigorously in a general case, for lattices of any dimensionality d or structure. (ii) The one-component model is shown to be equivalent, in the thermodynamic limit, to an n-component continuous-spin model proposed recently[5] when $n \to \infty$. (iii) The critical properties are worked out explicitly for a special case of the interaction and for cubic (hypercubic) d-dimensional lattices (d may be a fractional number). The critical exponents are those of the spherical model.[1,2,6] (iv) In this special case, for a convenient choice of the parameters in the Hamiltonian, the existence of a line of tricritical points is demonstrated. The tricritical exponents are those of the Gaussian model.[1,2,7] (v) Finally, the existence of an unconventional phase transition is established in one and two dimensions.

In Sec. II we define the exactly soluble (ES) model and compute the free energy in the thermodynamic limit. The equivalence to the n-component model is then established. In Sec. III we examine the critical properties of the free energy and derive the thermodynamic functions. A special case of the model is solved explicitly in Sec. IV. The one- and two-dimensional case is treated in Sec. V.

II. ES MODEL

Consider a d-dimensional (d is an integer) finite lattice with N lattice points denoted by $\{1, \ldots, j, \ldots, N\}$, and let $\vec{x}$ be a function defined for each lattice point j and assuming the values x_j, with $-\infty \leq x_j \leq \infty$. We first consider the following Hamiltonian:

$$H_N(\vec{x}) = \sum_{j=1}^{N} V(x_j^2) - \sum_{i,j}^{N} J_{ij} x_i x_j - h \sum_{j=1}^{N} x_j , \quad (2.1)$$

where

$$V(z) \geq -V_0 > -\infty \text{ for all } z , \quad \lim_{z\to\infty} z^{-2} V(z) > 0 ,$$
$$J_{ij} = J_{ji} = J(\vec{i} - \vec{j}) , \quad J(\vec{0}) = 0 , \quad \sum_j |J_{ij}| < \infty , \quad (2.2)$$

and h is a real number. Equation (2.1) is the Hamiltonian for the well-known continuous-spin Ising model.[2] A special case of (2.1) was chosen by Wilson[8] as a starting point of the renormalization theory. Moreover, interpreting the x_j's as one-component displacements from a reference lattice position, (2.1) may also be regarded as a model for structural phase transitions.[3,4] Finally, field theoreticians proved that (2.1) is the so-called "lattice approximation" for the $P(\phi)_2$ Euclidean quantum field theory.[9] Nevertheless, the model defined by (2.1) has not yet been solved.

Therefore, we propose a modification of (2.1) which leads to an exactly soluble model. In (2.1) make the change

$$\sum_{j=1}^{N} V(x_j^2) \to NV\left(N^{-1} \sum_{j=1}^{N} x_j^2\right), \tag{2.3}$$

so that the new Hamiltonian, defining the ES model, is given by

$$H_N^{\mathrm{ES}}(\vec{x}) = NV\left(N^{-1} \sum_{j=1}^{N} x_j^2\right) - \sum_{i,j}^{N} J_{ij} x_i x_j - h \sum_{j=1}^{N} x_j . \tag{2.4}$$

As a consequence, the local interaction $\sum V(x_j^2)$ in (2.1) is replaced by a uniform long-range interaction between all sites of the lattice. Note, however, that if $V(z)$ is a polynomial the intensity of the long-range interaction decreases at least as N^{-1}, when $N \to \infty$. It must be emphasized that the symmetry of the problem is modified by the transformation (2.3): $\sum V(x_j^2)$ has a *discrete* hypercubic symmetry in R^N, while $V(N^{-1} \sum x_j^2)$ has a *continuous* spherical symmetry in R^N. Still, (2.1) and (2.4) have a common property: the ground states of both models coincide provided the J_{ij} are all non-negative.[10]

Now, we are interested in computing the free energy corresponding to (2.4) in the thermodynamic limit:

$$-\beta\psi(\beta, h) = \lim_{N\to\infty} [-\beta\psi_N(\beta, h)], \tag{2.5}$$

where

$$-\beta\psi_N(\beta, h) = N^{-1} \ln Z_N(\beta, h) \tag{2.6}$$

and

$$Z_N(\beta, h) = \int_{\mathsf{R}^N} d^N x \exp[-\beta H_N^{\mathrm{ES}}(\vec{x})]. \tag{2.7}$$

In a previous paper[11] we have already indicated a method for computing the free energy (2.5). Let us here make the derivation more precise. Using (2.4) and the spherical symmetry, we may rewrite (2.7) as

$$Z_N(\beta, h) = \int_0^\infty dr \exp[-\beta N V(N^{-1} r^2)] \times \int_{\sum x_j^2 = r^2} d\sigma_r \exp\left(\beta \sum J_{ij} x_i x_j + \beta h \sum x_j\right), \tag{2.8}$$

where $d\sigma_r$ is the surface element of the sphere of radius r in R^N. Making the change

$$r \to N^{+1/2} r, \quad x_j \to r x_j \text{ for all } j, \tag{2.9}$$

(2.8) becomes

$$\begin{aligned} Z_N(\beta, h) &= N^{1/2} \int_0^\infty dr\, r^{N-1} \exp[-\beta N V(r^2)] \int_{\sum x_i^2 = N} \exp\left(\beta r^2 \sum J_{ij} x_i x_j + \beta r h \sum x_j\right) d\sigma_r \\ &= N^{1/2} \int_0^\infty dr\, r^{N-1} \exp[-\beta N V(r^2)] Q_N(r^2) \equiv N^{1/2} \int_0^\infty dr\, r^1 \exp\{N[-\beta V(r^2) + \ln r - \beta F_N(r^2)]\}. \end{aligned} \tag{2.10}$$

Here $Q_N(r^2)$ is the partition function of the spherical model (apart from a renormalization factor) with interaction $r^2 J_{ij}$ and applied field rh. The corresponding free energy is $F_N(r^2)$. Noting that the limit

$$\lim_{N\to\infty} F_N(r^2) = F(r^2) \tag{2.11}$$

exists for all r^2, it is easy to show[10] that the convergence of the sequence $\{F_N\}$ is uniform in r^2 on every bounded real interval. Therefore, we get from (2.10)

$$\left| \beta\psi_N + N^{-1} \ln\sqrt{N} \times \int_0^\infty dr\, r^{-1} \exp\{N[-\beta V(r^2) + \ln r - \beta F(r^2)]\} \right| < \epsilon, \tag{2.12}$$

for all finite r^2. Now, for a function $G(r^2)$ which decreases sufficiently fast for $r^2 \to \infty$, it is true[10] that

$$\lim_{N\to\infty} N^{-1} \ln \int_0^\infty dr\, r^{-1} \exp N G(r^2) = \max_{0 \le r^2 \le \infty} G(r^2). \tag{2.13}$$

Using (2.12) and (2.13), where

$$G(r^2) = -\beta V(r^2) + \ln r - \beta F(r^2), \tag{2.14}$$

we finally get

$$-\beta\psi(\beta, h) = -\min_{0 \le r^2 \le \infty} [\beta V(r^2) - \ln r + \beta F(r^2)]. \tag{2.15}$$

The free energy of the spherical model is well known for lattices of any dimensionality d or structure.[1,6] Here, without loss of generality, we shall only consider ferromagnetic interactions ($J_{ij} \geq 0$) and d-dimensional cubic (hypercubic) lattices. In that case, the free energy of the spherical model takes the form[1,6]

$$-\beta F(r^2) = \tfrac{1}{2}\ln\pi\beta^{-1} - \ln r + \beta r^2 t - \tfrac{1}{2}f_d(t) + \beta h^2/4[t - \hat{J}(\vec{0})]\,, \qquad (2.16)$$

where $t = t(r^2)$ is the solution of the saddle-point equation

$$\beta r^2 = \tfrac{1}{2} f_{d,t}(t) + \beta h^2/4[t - \hat{J}(\vec{0})]^2 \qquad (2.17)$$

and

$$f_d(t) = (2\pi)^{-d}\int_0^{2\pi} d^d\omega \ln[t - \hat{J}(\vec{\omega})]\,, \qquad (2.18)$$

$$f_{d,t}(t) = \frac{\partial}{\partial t} f_d(t)\,,$$

$$\hat{J}(\vec{\omega}) = \sum_{\vec{l}} J(\vec{l})\cos(\vec{\omega}\cdot\vec{l})\,. \qquad (2.19)$$

Introducing (2.16) in (2.15) yields

$$\beta\psi(\beta,h) = \min_{0 \leq r^2 \leq \infty}\left(\beta V(r^2) - \beta r^2 t + \tfrac{1}{2}f_d(t) - \frac{\beta h^2}{4[t - \hat{J}(\vec{0})]}\right) - \tfrac{1}{2}\ln\pi\beta^{-1}\,. \qquad (2.20)$$

The minimum in (2.20) can be determined by equating to zero the derivative of the expression in large parentheses with respect to r^2; we get

$$\beta V'(r^2) - \beta t - \left(\beta r^2 - \tfrac{1}{2} f_{d,t}(t) - \frac{\beta h^2}{4[t - \hat{J}(\vec{0})]^2}\right)\frac{\partial t}{\partial(r^2)} = 0\,. \qquad (2.21)$$

According to (2.17) the term in brackets on the left-hand side of (2.21) vanishes so that the necessary condition for a minimum in (2.20) can be simply written

$$V'(r^2) = t(r^2)\,. \qquad (2.22)$$

This condition is sufficient, when, in addition, the second derivative with respect to r^2 is positive, that is, when

$$V''(r^2) > t'(r^2)\,. \qquad (2.23)$$

Thus the free energy of the ES model is given by the set of relations

$$\psi(\beta,h) = V(r^2) - r^2 t + \tfrac{1}{2}\beta^{-1} f_d(t) - h^2/4[t - \hat{J}(\vec{0})] - \tfrac{1}{2}\beta^{-1}\ln\pi\beta^{-1}\,, \qquad (2.24)$$

$$t(r^2) = V'(r^2)\,, \qquad (2.25)$$

$$\beta r^2 = \tfrac{1}{2} f_{d,t}(t) + \beta h^2/4[t - \hat{J}(\vec{0})]^2\,. \qquad (2.26)$$

Before concluding this section, let us make some comments:

(i) In some particular cases where $V(r^2)$ is a polynomial, the results (2.24)–(2.26) may also be obtained with the aid of integral transformations and of the method of steepest descent.[12] However, the derivation of the free energy in this way is more complicated and difficult to make rigorous.

(ii) In a previous paper[11] we emphasized that the free energy of the ES model in the thermodynamic limit is identical to the free energy of an n-component continuous-spin model proposed by Emery[5] and defined by the Hamiltonian

$$H_N^{(n)}(\{\vec{S}_j\}) = \sum_{j=1}^{N} nV(n^{-1}\vec{S}_j^2) - \sum_{i,j}^{N} J_{ij}\vec{S}_i\cdot\vec{S}_j - \sum_{j=1}^{N}\vec{h}\cdot\vec{S}_j\,, \qquad (2.27)$$

when $n\to\infty$. Here $\vec{S}_j$ is an n-component vector and $\vec{h} = h(1,\ldots,1)$. Thus, if the ES model is considered as an extension of the spherical model as defined by Berlin and Kac,[1] the Hamiltonian (2.27) may be regarded as the corresponding extension of the n-component model defined by Stanley. We have then extended the result of Stanley that the n-component vector model is equivalent to the spherical model.[13,14]

III. CRITICAL PROPERTIES OF THE FREE ENERGY AND THERMODYNAMIC FUNCTIONS

The solutions of Eq. (2.25) determine the behavior of the system. Note first that $t(r^2)$ is obtained by solving Eq. (2.26) and that the properties of $t(r^2)$ are connected with those of the function $f_{d,t}(t)$. This last function is well known from the theory of the spherical model.[1,6,15,16] Let us emphasize that $t(r^2)$ is a continuous monotonically decreasing function of r^2 provided that, if $h=0$, one defines[10]

$$t(r^2) = \hat{J}(\vec{0}) \quad \text{if } \tfrac{1}{2}\beta^{-1} f_{d,t}(\hat{J}(\vec{0})) < r^2 < \infty \qquad (3.1)$$

(see Fig. 1). For short-range interactions J_{ij} the following results are true[1,6]

$$f_{d,t}(\hat{J}(\vec{0})) < \infty \quad \text{if } d > 2\,, \qquad (3.2)$$

$$f_{d,t}(\hat{J}(\vec{0})) \text{ diverges if } d \leq 2\,. \qquad (3.3)$$

Now, we have to distinguish between two cases:

(i) Equation (2.25) has only one solution; in this case the minimum in Eq. (2.20) is uniquely determined.

(ii) Equation (2.25) has more than one solution; in this case the solution which leads to an absolute minimum in Eq. (2.20) has to be found.

First consider case (i). A very simple necessary and sufficient condition for the existence of one

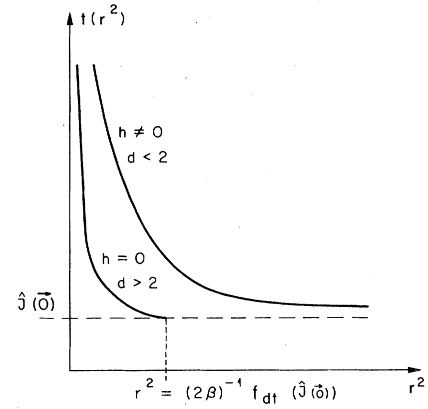

FIG. 1. Behavior of $t(r^2)$, the solution of Eq. (2.26), when: (a) $h=0$ and $d>2$; (b) $h\neq 0$ for all d or any h and $d\leq 2$.

and only one solution of Eq. (2.25) is[10] *(A)* $V(r^2)$ *is a concave function of* r^2 *for all* r^2 *such that* $V'(r^2)\geq \hat{J}(\vec{0})$. Then, since $t(r^2)$ is decreasing, the solution exists and always satisfies condition (2.23) (see Fig. 1). Assume that $V(r^2)$ has the property (A), in which case the appearance of a phase transition in the system is strongly related to the singularities of the free energy. Now the only term in (2.24) which may become singular at a finite temperature is the function $f_d(t)$, an analytic function in the whole complex t plane except along the real interval $[-\infty, \hat{J}(\vec{0})]$.[1,6] At the point $t=\hat{J}(\vec{0})$, f_d has a branch-cut singularity responsible for a second-order phase transition, as is the case in the spherical model.[1,6] However, according to (3.2), Eq. (2.25) can have a solution with $t(r^2)=\hat{J}(\vec{0})$ at finite temperature only if $f_{d,t}(\hat{J}(\vec{0}))<\infty$ and $h=0$. We then have the following result[10]:

Proposition 3.1: Assume that $V(r^2)$ *has the property (A), assume that* $f_{d,t}(\hat{J}(\vec{0}))$ *diverges or that* $h\neq 0$. *Then the free energy is an analytic function of temperature and magnetic field.*

For $d=1$ and $d=2$, $f_{d,t}(\hat{J}(\vec{0}))$ diverges if the interactions J_{ij} are short-range.[6] Applying the above result we conclude that the ES model has no phase transition in one and two dimensions as long as $V(r^2)$ has the property (A).

Now let $d>2$ and $h=0$ and $V(r^2)$ have the property (A). Then $V'(r^2)>\hat{J}(\vec{0})$ as soon as $V'(0)>\hat{J}(0)$, and there exists no solution of Eq. (2.25) such that $t(r^2)=\hat{J}(\vec{0})$. In this case, too, the free energy is an analytic function of β and h. If $V'(0)<\hat{J}(\vec{0})$, however, there exists an $r_0^2>0$ such that $V'(r_0^2)=\hat{J}(\vec{0})$. There exists a critical temperature given by

$$r_0^2=\tfrac{1}{2}\beta_c^{-1} f_{d,t}(\hat{J}(\vec{0})) \tag{3.4}$$

or

$$\beta_c=\tfrac{1}{2} r_0^{-2} f_{d,t}(\hat{J}(\vec{0})) . \tag{3.5}$$

For $\beta\geq\beta_c(T\leq T_c)$, and according to (3.1) and (3.4), r_0^2, the solution of

$$V'(r_0^2)=\hat{J}(\vec{0}) , \tag{3.6}$$

is always the solution of Eq. (2.25) and the free energy is singular. If $\beta<\beta_c(T>T_c)$, r_0^2 is never a solution of Eq. (2.25) and the free energy is regular. We summarize these results as follows[10]:

Proposition 3.2: Assume that $V(r^2)$ *has the property (A). Assume that* $d>2$ *and that the* J_{ij} *are short-range ferromagnetic interactions. Then if* $V'(0)>\hat{J}(\vec{0})$ *the free energy of the ES model is an analytic function of temperature and magnetic field. If* $V'(0)<\hat{J}(\vec{0})$ *and* $h=0$, *however, there exists a critical temperature given by (3.5). Below the critical temperature the free energy always has a branch point singularity.*

We now turn our attention to case (ii) where $V(r^2)$ does not have the property (A). Then Eq. (2.25) may have more than one stable solution at sufficiently small temperatures, and the solution must be determined which corresponds to an absolute minimum in (2.15). Let r_1^2 and r_2^2 be two stable solutions of Eq. (2.25). The interesting case occurs when, with variation of temperature, applied field or model parameters, the absolute minimum first given by r_1^2 is suddenly given by r_2^2. At the values of β and h such that

$$\psi(\beta,h;r_1^2)=\psi(\beta,h;r_2^2) , \quad r_1^2\neq r_2^2 , \tag{3.7}$$

a first-order phase transition will occur. Such transitions are examined in detail in Secs. IV and V.

Let us now derive from the free energy (2.24) the most interesting thermodynamic functions of the ES model. The magnetization as a function of h is given by

$$\begin{aligned} m(h) &= -\frac{\partial\psi}{\partial h} \\ &= -\frac{\partial}{\partial r^2}\left(V(r^2)-r^2 t+\tfrac{1}{2}\beta^{-1} f_d(t)-\frac{h^2}{4[t-\hat{J}(\vec{0})]}\right)\frac{\partial r^2}{\partial h} \\ &\quad +\frac{h}{2[t-\hat{J}(\vec{0})]} . \end{aligned} \tag{3.8}$$

Now, according to (2.21) the first term on the right-hand side of (3.8) vanishes so that

$$m(h)=h/2[t-\hat{J}(\vec{0})] . \tag{3.9}$$

We define the spontaneous magnetization by

$$m_s\equiv\lim_{h\to 0^+} m(h)=\lim_{h\to 0^+}\frac{h}{2[t-\hat{J}(\vec{0})]} . \tag{3.10}$$

The derivative of $m(h)$ with respect to h gives the field-dependent susceptibility

$$\chi(h)=\frac{\partial}{\partial h}\,m(h)=\frac{1}{2[t-\hat{J}(\vec{0})]}\left(1-2m(h)\,\frac{\partial t}{\partial h}\right), \qquad (3.11)$$

and the isothermal susceptibility is defined by

$$\chi_T=\lim_{h\to 0^+}\chi(h)\,. \qquad (3.12)$$

Now the relationship between phase transitions and singularities of the free energy becomes clear. In (3.10) and (3.12) we see that a spontaneous magnetization can only exist and the susceptibility can only diverge if

$$t(r^2)-\hat{J}(\vec{0})\underset{h\to 0}{\sim} h\,. \qquad (3.13)$$

In this case (see the above considerations) the free energy is singular when $h=0$. Note also that χ_T remains divergent as long as the spontaneous magnetization exists.

Differentiating the free energy with respect to $J_{i,i+l}=J(\vec{l})$, one gets the correlation function

$$\langle x_i x_{i+l}\rangle=-\frac{\partial}{\partial J(\vec{l})}\,\psi(\beta,h)$$

$$=\tfrac{1}{2}\beta^{-1}(2\pi)^{-d}\int_0^{2\pi}d^d\omega\,\frac{\cos(\vec{\omega}\cdot\vec{l})}{t-\hat{J}(\vec{\omega})}+m^2(h)\,. \qquad (3.14)$$

In derivating (3.14) we have again used (2.21) to eliminate the partial derivative of ψ with respect to r^2. Now, setting $\vec{l}=0$ yields

$$\langle x_i^2\rangle=\tfrac{1}{2}\beta^{-1}f_{d,t}(t)+m^2(h)=r^2\,. \qquad (3.15)$$

Moreover, in the limit $|\vec{l}|\to\infty$ the integral in (3.14) vanishes[17] (Riemann-Lebesgue lemma) and we get

$$\lim_{|\vec{l}|\to\infty}\langle x_i x_{i+l}\rangle=m^2(h)\,. \qquad (3.16)$$

The internal energy per site u is defined by

$$u=\frac{d}{d\beta}\,(\beta\psi)=V(r^2)-r^2t-\frac{h^2}{4[t-\hat{J}(\vec{0})]}+\tfrac{1}{2}\beta^{-1}\,. \qquad (3.17)$$

Once again we have used Eq. (2.21). The specific heat is then given by

$$c=-k\beta^2\,\frac{\partial}{\partial\beta}u$$

$$=\tfrac{1}{2}k\left(1+2\beta^2[r^2+m^2(h)]\,t'(r^2)\,\frac{\partial r^2}{\partial\beta}\right). \qquad (3.18)$$

IV. CRITICAL AND TRICRITICAL BEHAVIOR

In this section the most interesting application of the ES model will be discussed, namely, the case for which tricritical behavior is found. The interest of this application resides in the fact that exact treatments of tricritical points are quite rare and often fastidious. The ES model is, however, very simple to analyze in the tricritical region. Here we shall reexamine the case where $V(z)$ is a polynomial of degree three in z:

$$V(z)=\tfrac{1}{6}Cz^3+\tfrac{1}{4}Bz^2+\tfrac{1}{2}Az\,,\quad C>0\,,\quad A,B\in\mathrm{R}\,. \qquad (4.1)$$

Moreover, we consider ferromagnetic interactions between nearest neighbors only, so that J_{ij} is defined by

$$J_{ij}=\begin{cases}\tfrac{1}{2}J\,, & i,j \text{ nearest neighbors}\,;\\ 0\,, & \text{otherwise}\,.\end{cases} \qquad (4.2)$$

Then

$$\hat{J}(\vec{\omega})=J\sum_{i=1}^{d}\cos\omega_i\,,\quad \hat{J}(\vec{0})=dJ\,. \qquad (4.3)$$

If we choose C fixed (positive) and define

$$\Delta=A-\hat{J}(\vec{0})=A-2dJ\,, \qquad (4.4)$$

we have to distinguish three regions in the B-Δ plane (see Fig. 2):

$$\begin{aligned}&\text{region I}\quad \Delta\geq 0;\ B\geq 0\,;\\ &\text{region II}\quad \Delta<0;\ B\in\mathrm{R}\,;\\ &\text{region III}\quad \Delta\geq 0;\ B<0\,.\end{aligned} \qquad (4.5)$$

A. Regions I and II: Critical behavior

In region I, it is easy to verify that $V'(0)>\hat{J}(\vec{0})=dJ$ as long as $\Delta>0$, so that, from proposition 3.2, no phase transition can occur in this region of the B-Δ plane. If $\Delta=0$, critical behavior exists at zero temperature. For a discussion of this particular case see Ref. 18.

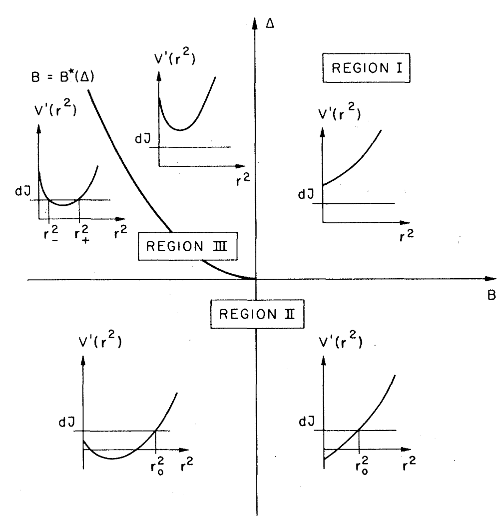

FIG. 2. Three regions in the B-Δ plane as defined by (4.5) and the behavior of $V'(r^2)$. Here $\Delta=2dJ$, r_0^2 is given by (4.7) and $r_\pm^2$ by (4.21). On the curve $B=B^*(\Delta)$, $V'(r^2)$ is tangent to the line dJ and $r_+^2=r_-^2=(\Delta C^{-1})^{1/2}$.

In region II, $V'(r^2)$ always has the property (A), so that Eq. (2.25) has one and only one solution, which is stable. Moreover, $V'(0) > J(\vec{0}) = dJ$, and for $d > 2$ and $h = 0$, there is a phase transition at the critical temperature

$$\beta_c = (2J)^{-1} r_0^{-2} q(d), \tag{4.6}$$

where r_0^2 is the (unique) positive solution of

$$V'(r^2) = \tfrac{1}{2}(Cr^4 + Br^2 + A) = dJ, \tag{4.7}$$

and $q(d)$ is given by[1,6,19]

$$q(d) = (2\pi)^{-d} \int_0^{2\pi} d^d\omega \left(d - \sum_{i=1}^{d} \cos\omega_i\right)^{-1}. \tag{4.8}$$

For $\beta \geq \beta_c$, the solution of Eq. (2.25) sticks in r_0^2 and the free energy always has a branch-cut singularity.

To calculate the spontaneous magnetization we have to determine the behavior of $t - dJ$, when $|h| \to 0$. We find[10]

$$t - dJ \underset{|h|\to 0}{\sim} \begin{cases} \tau > 0, & \beta < \beta_c; \\ |h|^{2/s}, & \beta = \beta_c; \\ (2r_0)^{-1} |h| (1 - \beta_c/\beta)^{-1/2}, & \beta > \beta_c, \end{cases} \tag{4.9}$$

where

$$s = \begin{cases} \tfrac{1}{2}(d+2), & 2 < d < 4; \\ 3, & d \geq 4. \end{cases} \tag{4.10}$$

Therefore, from (3.10), it follows that

$$m_s = \begin{cases} 0, & \beta \leq \beta_c; \\ r_0(1 - \beta_c/\beta)^{1/2}, & \beta > \beta_c, \end{cases} \tag{4.11}$$

and $\tilde{\beta}$, the critical exponent[20] of the magnetization, is then given by

$$\tilde{\beta} = \tfrac{1}{2}. \tag{4.12}$$

At $\beta = \beta_c$ it follows from (4.9) that

$$m(h) \underset{|h|\to 0}{\sim} |h|^{(s-2)/s} \equiv |h|^{1/\delta}, \tag{4.13}$$

so that the critical exponent[20] δ is given by

$$\delta = s/(s-2). \tag{4.14}$$

The other critical exponents[20] can be evaluated in the same way. We get

$$\alpha = (s-3)/(s-2); \quad \beta = \tfrac{1}{2}; \quad \gamma = 2\nu = (s-2)^{-1};$$
$$\delta = s/(s-2), \quad \eta = 0. \tag{4.15}$$

These are the critical exponents of the spherical model.[1,2,6]

An interesting property of the ES model is also the behavior of $\langle x_i^2 \rangle$. From (3.15) we find that at $h = 0$ one has

$$\langle x_i^2 \rangle = \begin{cases} r_0^2, & \beta \geq \beta_c; \\ (2\beta)^{-1} f_{d,t}(t), & \beta < \beta_c. \end{cases} \tag{4.16}$$

Moreover,

$$\langle x_i^2 \rangle_{\beta < \beta_c} - \langle x_i^2 \rangle_{\beta = \beta_c} \underset{\beta \to \beta_c}{\sim} \left(-1 + \frac{\beta_c}{\beta}\right)^{1/(s-2)} \tag{4.17}$$

and

$$\langle x_i^2 \rangle \underset{\beta \to 0}{\sim} \beta^{-1/3}. \tag{4.18}$$

B. Region III: Tricritical behavior

We first consider the case $d = 3$ and $h = 0$. In this region $V'(0) > dJ = 3J$; nevertheless V does not have property (A), and proposition 3.2 does not apply. Moreover, Eq. (2.25) may have more than one solution and the equation

$$V'(r^2) = \tfrac{1}{2}\,(Cr^4 + Br^2 + A) = 3J \tag{4.19}$$

may have two positive solutions. If $C > 0$ and $\Delta \geq 0$ have fixed values, an important quantity is

$$B^*(\Delta) = -2(\Delta C)^{1/2}. \tag{4.20}$$

For $0 > B > B^*(\Delta)$, Eq. (4.19) has no real solution. For $B < B^*(\Delta)$, Eq. (4.19) has two real solutions

$$r_\pm^2 = (2C)^{-1}[-B \pm (B^2 - 4\Delta C)^{1/2}]. \tag{4.21}$$

If $B = B^*(\Delta)$,

$$r_+^2 = r_-^2 = r_0^2 = (\Delta C^{-1})^{1/2}. \tag{4.22}$$

Therefore, if $0 > B > B^*(\Delta)$, Eq. (2.25) has no solution r^2 with

$$t(r^2) = 3J. \tag{4.23}$$

Nevertheless, if the curvature of $t(r^2)$ is large enough for small temperatures, Eq. (2.25) may have three solutions $r_3^2 \geq r_2^2 \leq r_1^2$. The stability condition (2.23) is never satisfied for r_2^2. However, r_3^2 and r_1^2 are stable solutions and the solution which gives an absolute minimum of (2.15) must be found. Therefore, we have two essential tasks to perform: on the one hand, to compute the curvature of $t(r^2)$ in order to enumerate the solutions of (2.25); on the other hand, to determine the solution which minimizes (2.15). These two points are treated in detail in Ref. 10. In what follows we describe the results obtained there.

The curvature of $t(r^2)$ depends on the values of Δ, J, and $q(3)$, and one has to distinguish between two cases:

$$\Delta \leq J\eta = J \times 4\pi^2 q^2(3) \tag{4.24}$$

and

$$\Delta > J\eta. \tag{4.25}$$

1. $\Delta \leqslant J\eta$

In this case another important value of B must be defined:

$$B^{**}(\Delta) = -(4/\sqrt{3})(\Delta C)^{1/2} \leq B^{*}(\Delta)\,. \tag{4.26}$$

Note that $B^{**}(\Delta)$ is the value at which the order parameter jumps for $T=0$ and $h=0$ (Ref. 10). If $B > B^{*}(\Delta)$, Eq. (2.25) has a unique solution for all temperatures. Nevertheless, this solution never satisfies (4.19), so that the free energy is always regular. In this domain of region III there is no phase transition.

When $B^{*}(\Delta) \geq B \geq B^{**}(\Delta)$, Eq. (2.25) has two stable solutions $r_1^2 \leq r_3^2$. It can be shown that the solution r_M^2 which minimizes (2.15) is given by

$$r_M^2 = \begin{cases} r_3^2(\beta) \geq r_+^2\,, & \beta_c \geq \beta \geq 0\,, \\ r_+^2\,, & \beta_1 \geq \beta > \beta_c\,, \\ r_1^2(\beta) \leq r_-^2\,, & \infty \geq \beta > \beta_1\,, \end{cases} \tag{4.27}$$

where

$$r_1^2(\beta = \infty) = 0\,; \quad r_3^2(\beta_c) = r_+^2 \tag{4.28}$$

and

$$\beta_c = (2J)^{-1} q(3) r_+^{-2}. \tag{4.29}$$

The new critical temperature β_1 cannot be computed explicitly, but the following relations hold:

$$\begin{aligned} &\beta_1 \geq \beta_c\,, \\ &\beta_1 = \infty \quad \text{for } B = B^{**}(\Delta)\,, \\ &\beta_1 = \beta_c = \beta_t \quad \text{for } B = B^{*}(\Delta)\,. \end{aligned} \tag{4.30}$$

Finally, if $B < B^{**}(\Delta)$, Eq. (2.25) still has two stable solutions $r_1^2 \leq r_3^2$, but the best solution is r_3^2, so that r_M^2 is simply given by

$$r_M^2 = \begin{cases} r_3^2(\beta) \geq r_+^2\,, & \beta_c \geq \beta \geq 0\,, \\ r_+^2\,, & \infty \geq \beta > \beta_c\,, \end{cases} \tag{4.31}$$

where r_3^2 has to satisfy (4.28). In this case there is only one critical temperature.

Let Δ be fixed, positive, and satisfy (4.24) and consider the phase diagram in the B-β^{-1} plane (see Fig. 3). In this plane there is a line of critical temperatures defined for $B < B^{*}(\Delta)$ and given by

$$\beta_c(B;\Delta) = (2J)^{-1} q(3) r_+^{-2}(B;\Delta)\,. \tag{4.32}$$

The line ends at the point TCP: $(B_t;\beta_t^{-1})$ with

$$\beta_t = (2J)^{-1} q(3)(\Delta^{-1}C)^{1/2}\,, \quad B_t = B^{*}(\Delta)\,. \tag{4.33}$$

$t(r^2) = 3J$ at the temperature β_c, so that the free energy becomes singular with a branch-point singularity; there is a phase transition of the type found in region II, that is, a second-order phase transition. Along the line $\beta_c(B;\Delta)$, the critical exponents are those of the spherical model[1,2,6] and are given by (4.15). All thermodynamic functions behave as in region II.

If $B^{**}(\Delta) \leq B \leq B^{*}(\Delta)$, there is, in addition, another line of critical temperatures $\beta_1(B;\Delta) \geq \beta_c(B;\Delta)$, beginning at the point Q:$(B_Q;\beta_Q^{-1})$, with

$$\beta_Q^{-1} = 0; \quad B_Q = B^{**}(\Delta)\,, \tag{4.34}$$

and terminating at the point TCP defined by (4.31). $t(r^2) = t(r_+^2) = 3J$ between the two temperatures β_c and β_1, and the free energy has a branch-point singularity. At β_1, however, r^2 jumps from r_+^2 to $r_1^2(\beta)$, with $t(r_1^2) > 3J$, so that the singularity suddenly disappears and the thermodynamic functions are discontinuous. Then, at β_1 two phases coexist and $\beta_1(B;\Delta)$ is a line of coexistence or a line of first-order phase transitions. At the point TCP the two lines of critical temperatures $\beta_c(B;\Delta)$ and $\beta_1(B;\Delta)$ meet so that TCP is a tricritical point.[21]

The thermodynamic functions are now easy to compute. The spontaneous magnetization always vanishes for $B > B^{*}(\Delta)$. If $B^{*}(\Delta) \geq B \geq B^{**}(\Delta)$ one has

$$m_s = \begin{cases} 0\,, & (\beta_c \geq \beta \geq 0)\,; \\ r_+(1 - \beta_c/\beta)^{1/2}\,, & (\beta_1 \geq \beta \geq \beta_c)\,; \\ 0\,, & (\infty \geq \beta \geq \beta_1)\,. \end{cases} \tag{4.35}$$

Therefore, m_s has a discontinuity at β_1. For $B < B^{**}(\Delta)$, m_s is simply given by

$$m_s = \begin{cases} 0\,, & \beta_c \geq \beta \geq 0\,; \\ r_+(1 - \beta_c/\beta)^{1/2}\,, & \infty \geq \beta > \beta_c\,. \end{cases} \tag{4.36}$$

The susceptibility diverges when $t = 3J$, that is, for $\beta_c < \beta \leq \beta_1$, if $B^{*}(\Delta) \geq B \geq B^{**}(\Delta)$, and for $\beta_c \leq \beta \leq \infty$, if $B < B^{**}(\Delta)$. Nevertheless,

$$\chi_T \underset{\beta \to \beta_c^-}{\sim} \left(-1 + \frac{\beta_c}{\beta}\right)^{-\gamma} = \left(-1 + \frac{\beta_c}{\beta}\right)^{-2} \quad \text{for } B < B^{*}(\Delta)\,,$$

$$\chi_T \xrightarrow[\beta \to \beta_1^+]{} \text{const} < \infty\,. \tag{4.37}$$

$\langle x_i^2 \rangle$ is given by (4.27) and (4.31). Note that $\langle x_i^2 \rangle$ is discontinuous at β_1 and that

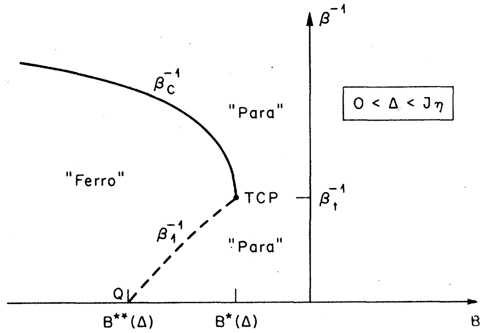

FIG. 3. Phase diagram in the B-β^{-1} plane for $\Delta < J\eta$. The full line β_c^{-1} is the line of second-order phase transition (spherical-model type of critical behavior). The dashed line β_1^{-1} is a coexistence line (first-order phase transition).

$$\langle x_i^2\rangle_{\beta<\beta_c} - \langle x_i^2\rangle_{\beta_c} \underset{\epsilon\to 0^+}{\sim} \left(-1+\frac{\beta_c}{\beta}\right)^2, \quad B<B^*(\Delta), \qquad (4.38)$$

while

$$\langle x_i^2\rangle_{\beta<\beta_t} - \langle x_i^2\rangle_{\beta_t} \underset{\epsilon\to 0^+}{\sim} -1+\frac{\beta_c}{\beta}, \quad B=B^*(\Delta). \qquad (4.39)$$

The standard critical exponents remain those of the spherical model even if the TCP is approached at constant $B=B^*(\Delta)$. However, when the TCP is approached in such a way that $B \downarrow B^*(\Delta)$ and $\beta=\beta_t$, different, so-called tricritical exponents[2,7,22,23] describe the behavior of thermodynamic quantities as a function of $B-B^*(\Delta)$, namely,

$$\tilde{\beta}_t=\tfrac{1}{4}, \quad \alpha_t=\tfrac{1}{2}, \quad \gamma_t=1, \quad \delta_t=5, \quad \phi_t=\tfrac{1}{2}. \qquad (4.40)$$

These are Gaussian exponents.[2,7,24] ϕ_t is the so-called crossover exponent[22,23] and describes the behavior of β_c close to $B=B^*(\Delta)$. It must be emphasized that renormalization-group calculations made with model (2.1) with $V(z)$ defined as in (4.1) give the same result (4.40) for the tricritical exponents.[25-27] However, these authors obtained logarithmic corrections for the tricritical exponents; in the ES model there are no such logarithmic corrections. A recent paper[28] gives an explanation for this fact: the logarithmic corrections are of order $1/n$ and vanish in the ES model which is equivalent to an n-component vector model in the limit $n\to\infty$.

2. $\Delta > J\eta$

In this case there still exists a line of second-order phase transitions $\beta_c(B;\Delta)$ and coexistence line $\beta_1(B,\Delta)$ in the B-β^{-1} plane (see Fig. 4). $\beta_c(B;\Delta)$ is given by (4.32) and terminates at the TCP defined by (4.33). The coexistence line $\beta_1(B;\Delta)$ appears at the point Q defined by (4.34). However, β_1 intersects the second-order line at a point

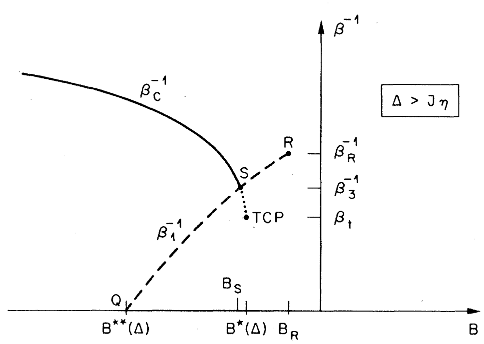

FIG. 4. Phase diagram in the B-β^{-1} plane for $\Delta>J\eta$. Note that the coexistence line (dashed) crosses the second-order line (full) at a point S different from TCP. The segment (S, TCP) of β_c^{-1} (pointed) has no physical meaning.

$S:(B_S;\beta_S^{-1})$ such that

$$B^{**}(\Delta)\le B_S<B, \quad \beta_S^{-1}=\beta_c^{-1}(B;\Delta), \qquad (4.41)$$

which is *not* the TCP, and continues until *another* point $R(B_R;\beta_R^{-1})$ is reached, where

$$B_R>B^*(\Delta); \quad \beta_R^{-1}>\beta_t^{-1}. \qquad (4.42)$$

The consequences of this behavior are the following (Fig. 4):

(i) The segment of the line β_c between S and TCP is no longer a line of critical points since it does not correspond to a stable solution of Eq. (2.25). The second-order line, therefore, terminates in S. However, since the coexistence line continues until R is reached, S is not the tricritical point. There is no tricritical point when $\Delta>J\eta$.

(ii) If $B_S<B\le B_R$, and the temperature is decreased, the spontaneous magnetization always remains zero, even if β_1 is crossed; in fact r^2 is never such that $t(r^2)=3J$. Nevertheless, $\langle x_i^2\rangle$ and χ_T are discontinuous at β_1 since

$$\langle x_i^2\rangle = r_M^2 = \begin{cases} r_3^2(\beta), & \beta_1\ge\beta\ge 0; \\ r_1^2(\beta), & \infty\ge\beta\ge\beta_1, \end{cases} \qquad (4.43)$$

and

$$\lim_{\beta\to\beta_1^-} r_M^2(\beta) - \lim_{\beta\to\beta_1^+} r_M^2(\beta) > 0 \quad (B<B_R). \qquad (4.44)$$

When $\Delta>J_\eta$ the critical exponents are always those of the spherical model.

It must be emphasized that in the B-β^{-1}-Δ space, there exists a line of tricritical points when $\Delta\le J\eta$. This line appears at the origin of the B-β^{-1}-Δ space and terminates at the point $P:(B^*;\beta_c^{-1};\Delta=J\eta)$. The line of tricritical points is given by

$$\beta=\beta_t, \quad B=B^*(\Delta), \quad \Delta\le J_\eta. \qquad (4.45)$$

Note also that all results derived here are in perfect agreement with those which can be obtained at zero temperature.

In region III, phase transitions can still occur when h is real and nonvanishing. In fact, if h has a fixed, sufficiently small positive value, Eq. (4.25) still has three solutions $r_3^2(h)\ge r_2^2(h)\ge r_1^2(h)$ at sufficiently low temperatures.[10] If $B<B^{**}(h;\Delta)<B^{**}(h^*;\Delta)$, where

$$B^{**}(0;\Delta)=B^{**}(\Delta),$$
$$B^{**}(h^*;\Delta)=-\tfrac{2}{3}(5)^{1/2}(\Delta C)^{1/2}\ge B^*(\Delta), \qquad (4.46)$$
$$h^*=\tfrac{8}{3}5^{-5/4}\Delta^{5/4}C^{-1/4},$$

as before, $r_3^2(h)$ is the solution which minimizes the free energy, and there is no phase transition. If $B\ge B^{**}(h;\Delta)$, there is a critical temperature $\beta_1(h,B;\Delta)$, such that if $\beta>\beta_1(h,B;\Delta)$, $r_1^2(h)$ minim-

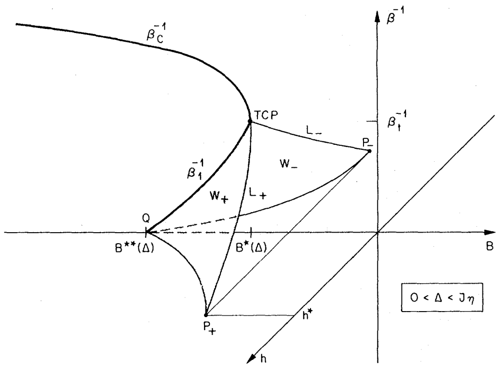

FIG. 5. Phase diagram in the h-B-β^{-1} space for $\Delta < J\eta$ and $d=3$.

izes the free energy, while if $\beta<\beta_1(h,B;\Delta)$, it is done by $r_3^2(h)$. Of course,

$$\beta_1(h=0,B;\Delta)=\beta_1(B;\Delta)\,. \qquad (4.47)$$

For $\Delta\le J\eta$, this critical temperature exists for values of B such that $B^{**}(h;\Delta)\le B\le B^*(h;\Delta)$, where $B^*(h;\Delta)$ cannot be calculated explicitly, but is such that

$$B^*(0;\Delta)=B^*(\Delta)\,,\quad B^*(h^*;\Delta)=B^{**}(h^*;\Delta)\,. \qquad (4.48)$$

When h is varied between zero and h^*, the lines $\beta_1(h,B;\Delta)$ generate a surface W_+ in the $h-B-\beta^{-1}$ space (see Fig. 5). Due to the symmetry with respect to h, another symmetric surface W_- is generated between $-h^*$ and zero. These two surfaces are the "wings" associated with the tricritical point[21,23,24] and are coexistence surfaces. The two lines of points defined by

$$L_\pm:\ B=B^*(\pm h;\Delta)\,,\quad B^{-1}=\beta_1^{-1}(\pm h,B^*;\Delta)\,, \qquad (4.49)$$

are two lines of continuous (second-order) phase transitions. They begin at the points P_+ or P_- and meet at the TCP. At the TCP, therefore, three lines of critical points meet (see Fig. 5).

When $\Delta>J\eta$, the critical temperature $\beta_1(h,B;\Delta)$ exists if $B^{**}(h;\Delta)\le B\le B_R(h;\Delta)$, where

$$B_R(h=0;\Delta)=B_R\,,\quad B_R(h^*;\Delta)=B^{**}(h^*;\Delta)\,, \qquad (4.50)$$

and B_R is given by (4.42). There are still two symmetrical "wings" and two lines of second-order phase transitions:

$$L_\pm:\ B=B_R(\pm h;\Delta)\,;\quad B^{-1}=\beta_1^{-1}(\pm h,B_R;\Delta)\,. \qquad (4.51)$$

However, in this case L_+, L_-, and β_c do not meet at the same point (see Fig. 6).

If $d>3$ and fractional values of d with $3<d\le 4$ are admissible, the qualitative behavior of the model is the same as for $d=3$ in the case $\Delta\le J\eta$. That is, a tricritical point exists for all values of Δ if $d>3$. The phase diagram is again given by Figs. 3 and 5. Along the second-order line $\beta_c(B;\Delta)$ defined by

$$\beta_c(B;\Delta)=(2J)^{-1}q(d)r_+^{-2}(B;\Delta)\,, \qquad (4.52)$$

the critical exponents are those of the spherical d-dimensional model,[2,6] i.e., given by (4.15), as long as $B<B^*(\Delta)$. When $B=B^*(\Delta)$, however, and the temperature β_t is approached, the critical exponents are always those of the *three-dimensional* spherical model, namely

$$\tilde\beta=\tfrac12\,,\quad \gamma=2\,,\quad \alpha=-1\,,\quad \delta=5 \qquad (4.53)$$

for *all* $d>3$. When the TCP is approached in such a way that $B\downarrow B^*(\Delta)$ and $\beta=\beta_t$, the exponents (4.53) are renormalized by the crossover exponent ϕ_t and given by the Gaussian tricritical values

$$\tilde\beta_t=\tfrac14\,,\quad \alpha_t=\tfrac12\,,\quad \gamma_t=1\,,\quad \delta_t=5\,,\quad \phi_t=\tfrac12\,, \qquad (4.54)$$

for all $d>3$.

It must be emphasized that in the neighborhood of the TCP, the dimension $d=3$ plays the same role as $d=4$ at an ordinary critical point,[2] namely, $d=3$ is the border line above which classical exponents are obtained for all $d\ge 3$.

When $2<d<3$, the qualitative behavior of the model is always the same as for $d=3$ in the case $\Delta>J\eta$. Therefore, no tricritical point exists for $d<3$. Nevertheless, for $h=0$, there still exists a line of second-order phase transitions $\beta_c(B;\Delta)$ defined by (4.52) when $B\le B_S$, and a coexistence line $\beta_1(B;\Delta)$ defined for $B^{**}(\Delta)\le B<B_R$. The phase diagram in this case is that drawn in Figs. 4 and 6.

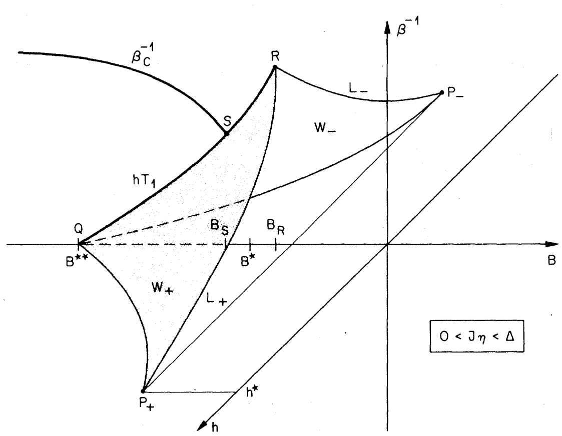

FIG. 6. Phase diagram in the h-B-β^{-1} space for $\Delta > J\eta$ and $d=3$.

V. PHASE TRANSITIONS IN ONE AND TWO DIMENSIONS

A surprising feature of the ES model is the existence of a phase transition in one and two dimensions provided that $V(z)$ has not the property (A). In fact, for $d=1$ or $d=2$, $f_{d,t}(\hat J(\vec 0))$ diverges and $t(r^2)$

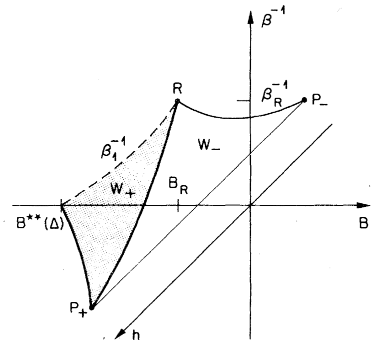

FIG. 7. Qualitative phase diagram in the h-β-β^{-1} space for $d=1$ and $d=2$ in the particular case where $V(z)$ is given by (4.1) and $\Delta>0$.

$\geq \hat{J}(\vec{0})$ at all finite temperatures. The function f_d in the free energy (2.24) is, therefore, analytic in the physical region. Then, even if $V(z)$ has not the property (A), there is no second-order phase transition in one or two dimensions.

However, at sufficiently low temperature, Eq. (2.25) may still have more than one stable solution. Consider once again the particular case where $V(z)$ is given by (4.1), and J_{ij} by (4.2). Then, in region III [Eq. (4.5)], Eq. (2.25) has two stable solutions at low temperatures, and the coexistence line $\beta_1(B;\Delta)$ given by (4.30) still exists for $h=0$ and $B^{**}(\Delta)<B<B_R$. Along the line $\beta_1(B;\Delta)$, two phases coexist, with the same magnetization $\langle x_i\rangle$ but with different values of $\langle x_i^2\rangle$. The quantity

$$S=\langle x_i^2\rangle_{\beta=\beta_1+0^+}-\langle x_i^2\rangle_{\beta=\beta_1+0^-} \qquad (5.1)$$

has to be regarded as a spontaneous order parameter. At the point R, S vanishes and $\partial\psi/\partial S$ diverges. R is then a critical point.

Moreover, if $h\neq 0$, the behavior of the model is still that described in Sec. IV. In particular, the two symmetrical wings $W_\pm$ and the two second-order lines $L_\pm$ do exist. The phase diagram in region III for $d=1$ and $d=2$ is given in Fig. 7.

The occurrence of phase transitions in one and two dimensions can be understood if we keep in mind the presence in the ES model of the long-range uniform interaction. This long-range interaction must yield a mean-field-like critical behavior[29] and we expect classical critical exponents[20] at the critical point R.

VI. CONCLUSION

In this work the critical properties of a model describing structural or magnetic phase transitions have been investigated. Particularly, a modification of the original Hamiltonian (2.1) has been proposed, which leads to an exactly soluble partition function. The corresponding free energy was shown to be identical with the free energy of the n-component extension of (2.1) when $n\rightarrow\infty$.

The ES model enables all interesting thermodynamic functions to be calculated, and exhibits a nontrivial critical behavior. In fact, the critical behavior has been shown to be that of the spherical model. Moreover, tricritical properties may also be examined. The existence of a tricritical point has been demonstrated and the tricritical exponents calculated explicitly. Apart from logarithmic corrections, the tricritical exponents have been found to be identical with those determined by renormalization-group calculations for the original model (2.1).

Finally, it must be emphasized that the critical dynamics of the ES model can also be solved.[12,30] The main feature of the solution is that the self-consistent approximation is exact.

Obviously, the ES model is far from physical reality. The uniform long-range interaction (2.3) has been introduced only for mathematical convenience. Nevertheless, it seems interesting to ask whether the qualitative behavior of the original model is well described by the behavior of the ES model. What are the features of the ES model which are only direct consequences of the presence of the long-range interaction? Is it possible to prove that the original model has a phase transition when the ES model has one? These questions have not yet been clarified. Nevertheless, one fact is certain: the ES model gives an idea of the complexity and the richness of the original model.

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