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Motion of negative ions at supercritical drift velocities in liquid ^{4}He at low temperatures

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The motion of negative ions at supercritical drift velocities in pressurized liquid helium is governed at low temperatures by the spontaneous emission of rotons. Assuming a constant matrix element the transition probability for two-roton emission processes is calculated using perturbation theory. The effect of recoil on the ionic motion is studied qualitatively by means of a simple kinetic approach and quantitative calculations are performed using the Boltzmann equation. The calculated dependence of drift velocity on electric field E shows that an $E^{1/3}$ variation occurs over a substantial range of fields as found experimentally. Comparison with experimental data for fields $2 < E < 10^3$ kV m^{-1} is made and deviations from the $E^{1/3}$ law at high fields are accurately accounted for. The role of vortex nucleation is discussed and further experiments are suggested.

INTRODUCTION

Some years ago, Rayfield[1] observed that the maximum velocity to which negative ions in liquid helium II can be accelerated by an electric field is a function of pressure. Below 14 bar an ion reaches the critical velocity v_v for creation of vortex rings and undergoes a transition to a charged vortex state. The velocity of the ion is then reduced to that of the vortex-ring–ion complex. As the electric field is increased the detected signal due to bare ions decreases dramatically and the largest velocity for which a bare-ion pulse is received gives an estimate of the vortex nucleation velocity v_v whose variation with pressure is shown in Fig. 1.

But for pressures above 15 bar the maximum velocity is close to the Landau velocity v_L. The behavior of the ions is then qualitatively different. The Landau critical velocity is that for which the onset of dissipation via creation of elementary excitations (rotons) should occur in the flow of superfluid helium through a channel or in the motion of a heavy rigid object through the stationary fluid. An ion can hence lose energy through emission of rotons and, under the action of an accelerating electric field, moves through the helium with an average velocity close to the Landau velocity. The size of the bare-ion pulse then remains sensibly constant as the electric field increases. This behavior is consistent with the situation shown in Fig. 1 in which the vortex nucleation velocity v_v is greater than the Landau critical velocity v_L at high pressures. In this region there is only a small probability that an ion may reach the velocity v_v and become trapped on a vortex ring.

In Rayfield's experiments[1] the temperature was sufficiently high that the drift velocity of the ions was limited by scattering of thermal excitations in addition to excitation-creation processes. More recent experiments[2,3] carried out at lower temperatures and at a pressure of 25 bar, showed that in strong electric fields negative ions can travel at velocities in excess of the Landau velocity. These results have provided a unique test of Landau's criterion for the breakdown of superfluidity through the roton-creation mechanism, but interpretation of the dependence of the ionic drift velocity on electric field requires a dynamical theory of the mechanism of the supercritical dissipation and a statistical theory of the ionic motion.

We have previously calculated the probability of roton emission[4,5] by a moving ion using perturbation theory with the assumption of a constant matrix element for either one-quantum or two-quantum processes. Together with a simple kinetic approach to the recoil motion of the ion, excellent agreement with experimental data[4,5] was found provided the supercritical drag is dominated by two-roton emission processes. The apparent

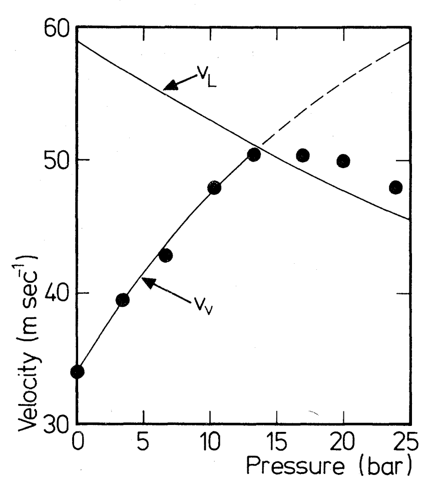

FIG. 1. Pressure variation of Landau critical velocity v_L and vortex nucleation velocity v_v. The data points refer to the maximum velocity achieved by a bare-ion pulse in the experiments of Rayfield (Ref. 1). Above 15 bar, v_v is obtained extrapolation.

complete absence of the one-quantum process has not been accounted for theoretically since the magnitudes of the matrix elements are not known.

Subsequently measurements have been extended to higher electric fields,[6] with correspondingly greater drift velocities, which require a more precise theoretical treatment. The kinetic description does not take full account of the distribution of ionic velocities, particularly at higher fields. To do this it is necessary to introduce a velocity distribution function which is calculated by means of the Boltzmann transport equation. However, in contrast to our previous kinetic theory, a complete solution may now be obtained only by numerical means. In addition, at high velocities, certain simplifications made previously in the calculation of the two-roton emission probability are no longer permissible.

Before formulating the Boltzmann equation for the problem we first summarize briefly the simple approach in which the trajectory of an individual ion is studied.[4] This illustrates most clearly the kinetics of the ionic motion and gives some insight into the form of the velocity distribution to be derived in the solution of the transport equation. We then describe the modifications which must be made to our previous treatment in order to extend the results to the region of high electric fields studied in the most recent experiments.[6]

KINETIC APPROACH

A central feature of the problem is the recoil motion of the ion upon emission of rotons. For two-roton processes there is a threshold velocity[7] $v' = v_L + p_0/m$, where p_0 is the roton momentum at the minimum of the liquid-helium dispersion curve and m is the effective mass of the ion. In an electric field E the motion of the ion may be pictured as in Fig. 2. If the initial velocity $v_0 < v'$, the ion is accelereated past the threshold up to a velocity v_e at which emission of a pair of rotons of total momentum $2p_0$ occurs. After recoil the ion returns to its initial velocity v_0 and the process is repeated.

The average time τ for which the ion exceeds the threshold velocity is found from the probability of roton emission during this period. As we have previously shown[4,5] an approximate formula for the transition rate for two-roton emission by an ion moving with velocity $v > v'$ is

$$R(v) = \alpha(v - v')^2 . \quad (1)$$

Thus after having passed the threshold velocity v', the probability $P(t)$ that an ion survives for a time t before emitting a roton pair, is given by

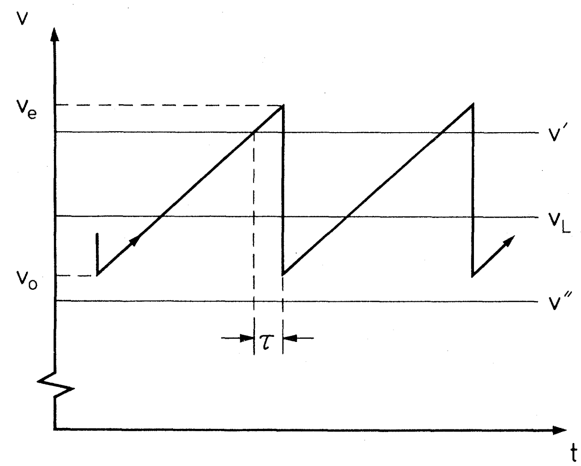

FIG. 2. Velocity trajectory of an ion showing successive periods of acceleration by the electric field followed by instantaneous recoil after roton emission.

$$\frac{dP}{dt} = -R(v)P(t) . \quad (2)$$

Since, between emission events the motion of the ion is determined by the accelerating field, $v = v' + eEt/m$, and thus Eq. (2) can be integrated to give

$$P(t) = \exp\left[-\frac{\alpha m}{3eE}\left(\frac{eEt}{m}\right)^3\right] . \quad (3)$$

We take as a measure of the average time before emission occurs, the time τ for which $P(\tau) \cong e^{-1}$. Thus we obtain

$$\frac{eE\tau}{m} \cong \left(\frac{3eE}{\alpha m}\right)^{1/3} . \quad (4)$$

But we see from the sawtooth velocity trajectory of Fig. 2 that the average velocity $\bar{v} = \frac{1}{2}(v_0 + v_e) = v_e - p_0/m$, since the recoil momentum $2p_0 = m(v_e - v_0)$. Moreover, the velocity at which emission occurs is $v_e = v' + eE\tau/m$, and thus $\bar{v} = v_L + eE\tau/m$. Hence, from Eq. (4), the form of the drift velocity $\bar{v}$ vs electric field E curve is

$$\bar{v} - v_L \cong (3eE/\alpha m)^{1/3} \quad (\bar{v} - v_L < 2p_0/m) . \quad (5)$$

Note that this expression is only valid when the velocity immediately after recoil $v_0 < v'$ which is equivalent to the condition $\bar{v} - v_L < 2p_0/m$ given above.

This type of dependence was observed in experiments[3,5] in electric fields up to 3×10^5 V m^{-1}. However, later experiments[6] in fields up to 10^6 V m^{-1} have revealed deviations from this behavior. Such deviations are expected for drift velocities sufficiently large that $\bar{v} - v_L > 2p_0/m$. This is the situation when the entire sawtooth trajectory lies above the threshold velocity for roton emission, i.e., $v_0 > v'$. In that case a similar calculation[4] shows that the drift velocity is given by

$$(\bar{v} - v_L)^3 - (\bar{v} - v_L - 2p_0/m)^3 \cong 3eE/m \qquad (\bar{v} - v_L > 2p_0/m)\,. \quad (6)$$

This gives a more rapid field dependence of $\bar{v} - v_L$ than $E^{1/3}$, tending to $E^{1/2}$ in the limit of very large E. However the theory is inapplicable in this limit which is also experimentally inaccessible.

Although the theory described above shows correctly the trend of the high-field experimental results[6] it is deficient in two respects. First, the concept of a mean pre-emission time τ does not take account of the distribution of times given by the probability $P(t)$. Thus there is an undetermined numerical factor in Eqs. (5) and (6) which may be expected to be weakly field dependent as the velocity distribution of the ions changes from that implicitly assumed in the kinetic method. For an ensemble of ions the sawtooth velocity trajectory corresponds to a distribution of velocities $f(v)$ of rectangular shape, $f(v)$ = constant for $v_0 < v < v_e$, and zero elsewhere. But the probability $P(t)$ in Eq. (3) also gives the probability that an ion reaches a velocity $v = v' + eEt/m$, so that for $v > v'$ the actual velocity distribution function is of the form

$$f(v) \sim \exp\left(-\frac{\alpha m}{3eE}(v - v')^3\right), \quad (v > v')\,.$$

For $v < v'$ the distribution reflects the availability of ions for acceleration past the threshold velocity. Since an ion with velocity $v + 2p_0/m$ recoils to a velocity v, the distribution function below threshold is given approximately by

$$f(v) \sim 1 - f(v + 2p_0/m) \quad (v'' < v < v')\,,$$

where the lower limit $v'' = v' - 2p_0/m$. In the steady state there are no ions with lower velocities. We have compared these distributions in Fig. 3. At relatively low fields the actual distribution is a distorted rectangular shape with a tail to higher velocities. In the kinetic approach this distortion is simulated by a bodily shift of the rectangular distribution to higher velocities. In high fields the distribution becomes much broader and more smoothly varying but must be obtained by solution of the Boltzmann equation.

Secondly, the above theory is based on the two-roton emission rate given in Eq. (1). The quadratic dependence on the velocity above threshold is an approximation and in the next section we derive an expression which is valid for the large drift velocities observed in the high-field experiments. The assumption that the recoil momentum is exactly $2p_0$ is likewise only true near the threshold velocity. At high velocities the roton momenta are not quite parallel to the ionic velocity and a correction is needed to allow for the consequent reduction in recoil velocity. This may also be readily incorporated into the Boltzmann-equation treatment.

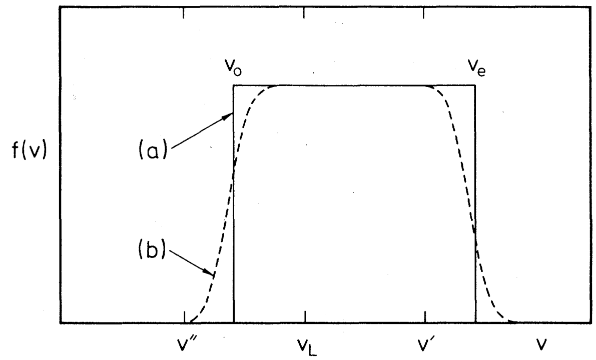

FIG. 3. Distribution function $f(v)$ of ionic velocities. (a) Rectangular distribution assumed in kinetic method. (b) Form of actual velocity distribution for the same average velocity as in (a).

TWO-ROTON EMISSION RATE

We assume that the emission process may be described in perturbation theory and that the transition rate for emission of rotons $\vec{k}$ and $\vec{q}$ by an ion with initial momentum $m\vec{v}$ is given by

$$R_{\vec{k}\vec{q}}(v) = (2\pi/\hbar)\,|V_{\vec{k}\vec{q}}|^2 \delta(E_f - E_i)\,.$$

The transition matrix element $V_{\vec{k}\vec{q}}$ is taken to be a constant. There is no theoretical justification for this save that, at least for small excess drift velocities $\bar{v} - v_L$, the majority of emitted rotons tend to have momenta of magnitudes close to the minimum value $p_0 = \hbar k_0$ and directions almost parallel to the ion velocity $\vec{v}$. Over such a restricted range of available states a weak dependence of the matrix element on the wave vectors $\vec{k}$ and $\vec{q}$ could therefore be inappreciable.

For a constant matrix element, evaluation of the total emission rate

$$R(v) = \sum_{\vec{k},\vec{q}} R_{\vec{k}\vec{q}}(v)\,, \quad (7)$$

reduces to a computation of the volume of available phase space which is limited by the requirements of energy and momentum conservation. Since after recoil the momentum of the ion becomes $m\vec{v} - \hbar(\vec{k} + \vec{q})$, the difference between the final and initial energies is

$$E_f - E_i = \epsilon_k + \epsilon_q + \tfrac{1}{2} m \{\vec{v} - (\hbar/m)(\vec{k} + \vec{q})\}^2 - \tfrac{1}{2} m v^2$$
$$= \bar{\epsilon}_k + \bar{\epsilon}_q - \hbar\vec{k}\cdot\vec{v} - \hbar\vec{q}\cdot\{\vec{v} - (\hbar\vec{k}/m)\}\,, \quad (8)$$

where ϵ_k, ϵ_q are the roton energies and

$$\bar{\epsilon}_k = \epsilon_k + \hbar^2 k^2/2m\,, \quad \bar{\epsilon}_q = \epsilon_q + \hbar^2 q^2/2m\,. \tag{9}$$

Converting the sums in Eq. (7) to integrals, we perform the angular integrations using polar coordinates for $\vec{k}$ relative to $\vec{v}$ as polar axis and for $\vec{q}$ relative to $\vec{w} = \vec{v} - (\hbar\vec{k}/m)$. Taking a constant matrix element V_0 this gives

$$R(v) = \frac{\Omega^2 |V_0|^2}{(2\pi)^3 \hbar} \int k^2 dk \int q^2 dq \int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' \delta(\bar{\epsilon}_k + \bar{\epsilon}_q - \hbar k v \mu - \hbar q w \mu')\,,$$

where Ω is the volume of the system, $\mu = \vec{k}\cdot\vec{v}/kv$ is the cosine of the angle between $\vec{k}$ and $\vec{v}$ and similarly $\mu' = \vec{q}\cdot\vec{w}/qw$. The integral over μ' will be nonzero only if the value defined by the δ function lies within the range of integration, which in effect means that this value is less than unity, i.e.,

$$\mu' = (\bar{\epsilon}_k + \bar{\epsilon}_q - \hbar k v \mu)/\hbar q w < 1\,. \tag{10}$$

This inequality restricts the subsequent integration over μ to the range $\mu_m < \mu < 1$. Now the experimental data give $|\hbar\vec{k}/m| \sim p_0/m \cong 5$ m sec^{-1} whereas $v > 50$ m sec^{-1} so that the approximation

$$w = |\vec{v} - \hbar\vec{k}/m| \cong v - \hbar k \mu/m$$

is accurate to better than 1%. The inequality (10) then gives

$$\mu_m = \frac{\bar{\epsilon}_k + \bar{\epsilon}_q - \hbar q v}{\hbar k v \{1 - (\hbar q/mv)\}}\,. \tag{11}$$

To estimate the maximum angle of emission θ_m we make the approximations $\hbar k = \hbar q \cong p_0$, $\epsilon_k = \epsilon_q \cong \Delta$ the minimum roton energy and, noting that $v_L \cong \Delta/p_0$ to within an accuracy[8] of 1%, find

$$\mu_m = \cos\theta_m \cong 1 - 2[(v - ')/v_1]\,,$$

where $v_1 = v - (p_0/m)$ and v' is the emission threshold velocity. Thus θ_m increases from zero at the emission threshold velocity v', to of the order 60° at the large velocities ~70 m sec^{-1} observed experimentally. This shows clearly that at high drift velocities the average recoil momentum may be substantially less than $2p_0$.

The result of performing the integrations over μ and μ' is

$$\int_{\mu_m}^{1} \frac{d\mu}{\hbar q w} = \frac{m}{\hbar q} \int_{\mu_m}^{1} \frac{d\mu}{mv - \hbar k \mu}$$

$$= \frac{m}{\hbar^2 k q} \ln\left(\frac{mv - \hbar k \mu_m}{mv - \hbar k}\right) \theta(1 - \mu_m)\,, \tag{12}$$

where $\theta(x)$ is the unit step function. Using (11) the argument of the logarithm simplifies to

$$\left(\frac{m^2 v^2 - m(\bar{\epsilon}_k + \bar{\epsilon}_q)}{(mv - \hbar k)(mv - \hbar q)}\right), \tag{13}$$

and the integrations over the wave numbers k, q are now to be performed. At this stage it is essential to introduce a simplification in view of the complicated dependence of the integrand on k and q and the implicit restriction $\mu_m < 1$ given by the step function. We can estimate the maximum energy of the emitted rotons and hence, from the known dispersion curve[9] at 25 bar, the corresponding range of wave numbers. At the higher drift velocities we find $|k - k_0| hk_0 \lesssim 0.13$. Since the deviation $k - k_0$ can be positive or negative for a given roton energy, we assume that the wave numbers k and q in the integrand can be replaced by the average value k_0 but we take account of the variation in roton energy which originates from the basic conservation law (8).

Using Eqs. (9) and (11), the condition $\mu_m < 1$ can be written

$$\epsilon_k + \epsilon_q + \frac{\hbar^2 (k+q)^2}{2m} - \hbar(k+q)v < 0\,.$$

Defining $E_1 = \epsilon_k - \Delta$, $E_2 = \epsilon_q - \Delta$ and making the approximation $k = q \cong k_0$, shows that the total roton energy, measured from 2Δ, is limited by the condition

$$0 < E_1 + E_2 < E_m = 2\hbar k_0 (v - v')\,. \tag{14}$$

The same approximation is used to simplify the expression (13) for the argument of the logarithm in Eq. (12). To rewrite the integrals over k and q in terms of the energy variables E_1 and E_2, we use the roton dispersion relation of standard form

$$\epsilon_k - \Delta = \hbar^2 (k - k_0)^2 / 2m_r\,, \tag{15}$$

where m_r is the roton effective mass. This gives the emission rate

$$R(v) = \frac{\Omega^2 m |V_0|^2}{(2\pi\hbar)^3} \left(\frac{2m_r k_0^2}{\hbar^2}\right) \int_0^\infty \int_0^\infty \frac{dE_1 dE_2}{(E_1 E_2)^{1/2}} \ln\left(1 + \frac{E_m - E_1 - E_2}{m v_1^2}\right) \theta(E_m - E_1 - E_2)\,.$$

For ion velocities $v \sim 70$ m sec^{-1}, $E_m/2\Delta \sim 0.45$ which corresponds to the 13% range of roton wave numbers quoted above. Since $E_m/mv_1^2 \sim 0.05$ we may safely expand the logarithm and hence obtain the emission rate as a power series in $E_m/mv_1^2 \propto v - v'$.

The integrals are most easily performed by changing to variables $\xi = E_1 + E_2$, $\eta = E_1 - E_2$ giving

$$\int_0^{E_m} d\xi \int_{-\xi}^{\xi} \frac{d\eta}{(\xi^2 - \eta^2)^{1/2}} \ln\left(1 + \frac{E_m - \xi}{mv_1^2}\right) = \pi \int_0^{E_m} d\xi \left[\frac{E_m - \xi}{mv_1^2} - \frac{1}{2}\left(\frac{E_m - \xi}{mv_1^2}\right)^2 + \cdots\right] = \frac{\pi E_m^2}{2mv_1^2}\left(1 - \frac{E_m}{3mv_1^2} + \cdots\right), \quad (16)$$

provided $E_m > 0$, i.e., $v > v'$ and is zero otherwise. Since the second term only gives a correction ~1.5% it may be ignored. Hence the two-roton emission rate is, for $v > v'$,

$$R(v) = K\left(\frac{v - v'}{v - (p_0/m)}\right)^2, \quad (17)$$

where

$$K = \Omega^2 m_r k_0^4 |V_0|^2 / 2\pi^2\hbar^3 .$$

For $v \sim v'$ this reduces to the form (1) given previously provided $\alpha = K/v_L^2$.

Within the framework of the above model we expect our calculation to be accurate to within a few percent for ion velocities $v \sim 70$ m sec^{-1}. This also corresponds to the accuracy of the recent experimental results[6] which have random errors ~2% and there is a similar systematic error in absolute values of the velocities owing to uncertainties in the length of the drift space. Unfortunately the model omits certain features whose effects are difficult to estimate quantitatively. The roton dispersion curve deviates from the parabolic form (15) for energies well above the minimum. By including a quartic term in $k - k_0$ we can show that the resultant correction to the emission rate gives a factor $\sim 1 + 0.05(E_m/E_0)$, where E_0 is the roton energy above Δ at which the quartic term contributes ~10%. From the published dispersion curves[9] we can make a rough estimate $E_0 \sim \frac{1}{2}\Delta$, whereas $E_m \lesssim \Delta$, so that this correction could increase our calculated rate by at most 10% at drift velocities $\bar{v} \sim 65$ m sec^{-1}.

A possible complication in very strong electric fields is the quantum uncertainty $\hbar/\tau_1$ in the energy of the ion, where τ_1 is the mean free time between emission events. But $\tau_1 = 2p_0/eE$ since it is the time required by the electric field to increase the momentum of the ion by $2p_0$. We only expect this "collision broadening" effect to become significant when $\hbar/\tau_1 = eE/2k_0$ is greater than the energy of the emitted rotons $E_m \sim 2\Delta$. This condition reduces to $E > 4k_0\Delta/e \sim 5 \times 10^7$ V m^{-1}, which is well outside the range of fields studied experimentally.

We note that for velocities $v \gg v'$, the emission rate (17) tends to a constant value which implies that there is a maximum field for which it is possible to maintain steady-state ionic motion. However, this field also exceeds 10^7 V m^{-1} and our formula (17) is not applicable at such high velocities. It is also possible that other emission processes become important for these very high fields.

AVERAGE RECOIL VELOCITY

When a pair of rotons $\vec{k}, \vec{q}$ is emitted the momentum of the ion is reduced by $\hbar(\vec{k} + \vec{q})$. By symmetry the average momentum loss must be in the direction of the ion velocity $\vec{v}$ so that the average rate of loss of momentum is

$$\sum_{\vec{k}, \vec{q}} \hbar(\vec{k} + \vec{q}) \cdot (\vec{v}/v) R_{\vec{k}\vec{q}} ,$$

It follows that the average recoil velocity, which is just the average decrease in ionic velocity per emission event, may be written

$$\sum_{\vec{k}, \vec{q}} \frac{\hbar(\vec{k} + \vec{q}) \cdot \vec{v}}{mv} R_{\vec{k}\vec{q}} \Big/ \sum_{\vec{k}, \vec{q}} R_{\vec{k}\vec{q}} .$$

Now the conservation of energy condition (8) gives

$$\hbar(\vec{k} + \vec{q}) \cdot \vec{v} = \epsilon_k + \epsilon_q + (\hbar^2/2m)(\vec{k} + \vec{q})^2 .$$

Since we are interested in the deviation of the recoil velocity from the value $2p_0/m$ arising from the finite angular spread of emitted rotons, we only evaluate the above expression to lowest order in $v - v'$. Accordingly we may make the approximation $(\vec{k} + \vec{q})^2 \cong 4k_0^2$, which gives

$$\hbar(\vec{k} + \vec{q}) \cdot \vec{v} = E_1 + E_2 + 2\hbar k_0 v' .$$

In terms of the total roton-energy variable $\xi = E_1 + E_2$ and maximum emitted energy $E_m = 2p_0(v - v')$ introduced previously, we have

$$\frac{\hbar(\vec{k} + \vec{q}) \cdot \vec{v}}{mv} = \frac{2p_0}{m} - \frac{E_m - \xi}{mv} .$$

The average over all $\vec{k}$ and $\vec{q}$ can now be reduced to a simple average over ξ with the weighting function $E_m - \xi$ as in Eq. (16). Thus, the average recoil velocity is

$$\frac{2p_0}{m} - \frac{2E_m}{3mv} = \frac{2p_0}{m}\left(1 - \frac{2(v - v')}{3v}\right). \quad (18)$$

The recoil velocity $2p_0/m$ at threshold is therefore reduced by about 20% at high ion velocities $v \sim 70$ m sec^{-1}.

FORMULATION OF BOLTZMANN EQUATION

In a typical experiment a pulse of negative ions is injected into the region between two grids and the time of flight measured as a function of the applied potential difference V. For a drift space of length $L \sim 1$ cm the time of flight $T = L/\bar{v} \sim 2 \times 10^{-4}$ sec. During this time the number of roton emission events for a single ion is $N_e = T/\tau_1$, where $\tau_1 = 2p_0/eE$ is the mean time between successive events. Since $\bar{v}$ is of the same order of magnitude as $v_L = \Delta/p_0$ we have $N_e \sim eV/2\Delta$ which corresponds to 10^6 events for a potential difference of 10^3 V. We can reasonably assume that a steady-state distribution of ionic velocities will be established after relatively few events and will therefore persist for almost the entire transit time T. It is this steady-state distribution which we shall calculate using the Boltzmann equation.

In its most general form the transport equation describing the ion-roton system is very complicated since an ion can both emit and absorb rotons and take part in inelastic roton scattering processes. If we think of scattering processes as giving rise to a viscous drag on the ion then the net effect is to produce a reduction in the effective electric field. At 0.4 K the experimental results[3,5,6] indicate that scattering processes[10] are only important for low fields $E < 3 \times 10^3$ V m^{-1}. At these low temperatures the thermal roton density is so small that we may also ignore absorption and stimulated emission processes in comparison to the spontaneous emission rate calculated above. In addition the density of ions in a typical pulse is sufficiently small that the thermal roton density is not appreciably increased as a result of the pair creation processes.

Even with the restriction to roton emission processes the situation is still complicated by the geometry of an emission event. Let us consider an ion just above the emission threshold when the rotons are emitted parallel to the ionic velocity whose direction remains unchanged by the recoil. Between emission events the ion is accelerated parallel to the applied electric field. Thus one can see that an ion initially moving at some angle to the electric field will after several emission events be essentially travelling parallel to the field. This is illustrated in Fig. 4 on the simplifying assumption that emission occurs immediately when the ion reaches the threshold speed v'. Thus the ion velocity vector diffuses towards the direction of the electric field via the path $ABC\ldots$ shown in Fig. 4. In the steady state the problem becomes

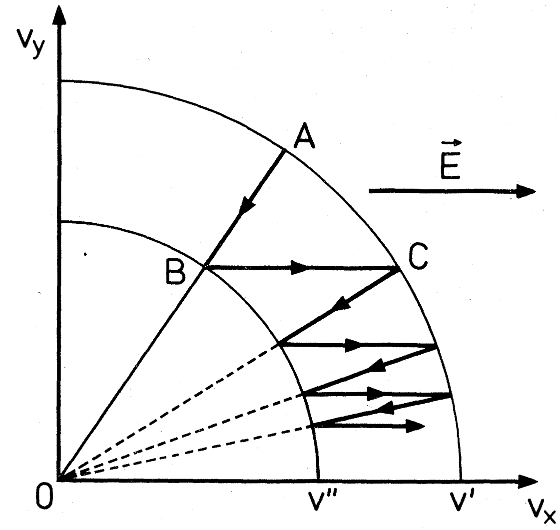

FIG. 4. Transverse diffusion of ion velocity vector. Assuming roton emission occurs exactly at the threshold speed v' the velocity vector $\overrightarrow{OA}$ becomes $\overrightarrow{OB}$ after recoil, $\overrightarrow{OC}$ after acceleration by the field $\vec{E}$ and so diffuses towards the field direction along the x axis.

strictly one-dimensional as considered in our kinetic approach. This is no longer true at high fields, when the ion can be accelerated to rather larger velocities before emission. The rotons are then emitted with an angular spread and upon recoil the ion can be deflected from its original course. This weakens the angular diffusion of the ion velocity towards the field direction. Although ions moving at large angles to the field still tend to diffuse towards the field direction, those already moving parallel to the field may be deflected from this direction. In the steady state there is thus a distribution of transverse velocities. A rough estimate based on modifying the picture of Fig. 4 by allowing the recoil velocity to lie within a cone, suggests that this transverse velocity distribution has a width $\sim 2p_0/m \sim 10$ m sec^{-1} for large drift velocities $\bar{v} \sim 65$ m sec^{-1}. Such a spread in transverse velocities is not so large that a one-dimensional approximation to the complete Boltzmann equation should be seriously in error.

We introduce a distribution function $f(v)$ of ionic velocities, where v is the component of velocity in the field direction. In so doing we are implicity averaging over transverse velocity components. The fraction of ions $f(v)dv$ with velocities in the range dv can be changed by the effect of the accelerating field and by roton emission processes. The rate of change due to the field is

$$-\frac{eE}{m}\frac{df}{dv}dv\,.$$

The rate at which ions are lost from the interval

dv by roton emission is just $R(v)f(v)dv$. But ions within a velocity range du will after emission arrive in the velocity range dv provided the initial velocity u is greater than the final velocity v by the average recoil velocity appropriate to an ion moving with speed u. This is given by Eq. (18) with v replaced by u. The rate of gain of ions due to the recoil of higher velocity ions is therefore $R(u)f(u)du$, giving the net rate of change due to emission processes

$$-R(v)f(v)dv+R(u)f(u)du\,.$$

Since in the steady state the total rate of change is zero we have the Boltzmann equation

$$\frac{eE}{m}\frac{df}{dv}=-R(v)f(v)+R(u)f(u)\frac{du}{dv}\,, \tag{19}$$

where the average recoil velocity $u-v$ is given by

$$u-v=\frac{2p_0}{m}\left(1-\frac{2(u-v')}{3u}\right). \tag{20}$$

The averaging over transverse velocities implies that the rates $R(u)$ and $R(v)$ in Eq. (19) have been similarly averaged. However a transverse velocity component ~10 m sec^{-1} only changes the magnitude of a velocity $v\sim 70$ m sec^{-1} by 1%. For this reason we expect the one-dimensional Boltzmann equation to be an excellent approximation throughout the experimental range of drift velocities.

APPROXIMATE ANALYTIC SOLUTION

Since a complete solution of the Boltzmann equation can only be obtained by numerical means, we first derive an analytic solution applicable to the region of small excess drift velocities $\bar{v}-v_L \ll 2p_0/m$. This illustrates some of the features which will be encountered in the general solution and relates the kinetic method to the present approach employing the transport equation.

In this low-velocity region we can take the recoil velocity to be exactly $2p_0/m$, hence $u=v+(2p_0/m)$ and Eq. (19) becomes

$$\frac{eE}{m}\frac{df}{dv}=-R(v)f(v)+R\left(v+\frac{2p_0}{m}\right)f\left(\frac{2p_0}{m}\right). \tag{21}$$

We may also use the approximate form for the emission rate

$$R(v)=\alpha(v-v')^2\theta(v-v')\,,$$

where the step function $\theta(v-v')$ indicates explicitly that the rate is zero for $v<v'$. For $v>v'$ both loss and gain terms on the right-hand side of Eq. (19) are nonzero. However, the number of ions reaching the higher speed $v+2p_0/m$ is very much less than the number which attain the lower speed v. As a first approximation we therefore neglect the second term. The resulting equation is then easily solved to give

$$f(v)=A\exp\left(-\frac{\alpha m}{3eE}(v-v')^3\right)\quad (v>v')\,. \tag{22}$$

This approximation will be valid provided the exponential decay of $f(v)$ is very rapid, i.e., $(\alpha m/3eE)^{1/3}(2p_0/m)\gg 1$, which is in fact quite equivalent to the initial assumption of low drift velocities. For $v<v'$ the loss term in Eq. (19) vanishes and we approximate the gain term by substituting for $f(v+(2p_0/m))$ using the expression just found. Solving the resulting equation gives

$$f(v)=B-A\exp\left(-\frac{\alpha m}{3eE}(v-v'')^3\right)\quad (v''<v<v')\,, \tag{23}$$

where $v''=v'-(2p_0/m)=v_L-(p_0/m)$. Both gain and loss terms vanish for $v<v''$ and the only solution then is $f=0$. Physically this arises because, in the absence of any scattering processes, the electric field can accelerate all ions without dissipation up to the threshold velocity v'. In the steady state there is then no mechanism other than recoil by which ions can be reduced to subthreshold velocities. Clearly this implies that there is a lower limit $v'-2p_0/m=v''$ to the range of velocities for which $f(v)$ is nonzero.

The constants A and B are determined from the continuity of the velocity distribution at $v=v'$ and the normalization condition

$$\int_{v''}^{\infty}f(v)dv=1\,.$$

Subject to the above approximation this gives $A=B=(2p_0/m)^{-1}$. It can be seen that the velocity distribution (22) and (23) is in fact identical to that which we deduced from our earlier kinetic considerations. Moreover, calculating the average velocity gives

$$\bar{v}=\int_{v''}^{\infty}vf(v)dv\cong v_L+\Gamma\left(\frac{3eE}{\alpha m}\right)^{1/3},$$

which is also the same as found from the kinetic approach apart from a numerical factor

$$\Gamma=\int_0^{\infty}\exp(-x^3)dx=0.893\,.$$

This factor depends on the form of the velocity distribution function which changes with electric field. It follows that the same correction factor cannot be used to correct the result (6) of the kinetic approach for large drift velocities $\bar{v}-v_L>2p_0/m$. We therefore proceed to the numerical solution of the Boltzmann equation using the more accurate form (17) for the emission rate which we have derived in this paper, and the expression (20) for the average recoil velocity.

NUMERICAL SOLUTION OF BOLTZMANN EQUATION

Written out more fully the Boltzmann equation becomes

$$\frac{eE}{Km}\frac{df}{dv}=-\left(\frac{v-v'}{v-p_0/m}\right)^2\theta(v-v')f(v)+\left(\frac{u-v'}{u-p_0/m}\right)^2\theta(u-v')f(u)\frac{du}{dv}.$$

It is convenient to introduce dimensionless variables

$$x=\frac{v-v'}{2p_0/m},\quad y=\frac{u-v'}{2p_0/m},\quad a=2p_0/mv_L,$$

$$b=(eE/Kmv_L)^{1/3},$$

whence

$$\left(\frac{b}{a}\right)^3\frac{df}{dx}=-\left(\frac{x}{1+ax}\right)^2\theta(x)f(x)+\left(\frac{y}{1+ay}\right)^2\theta(y)f(y)\frac{dy}{dx}. \tag{24}$$

We note that x represents the velocity excess above threshold measured in units of the recoil velocity. Thus $x=0$ corresponds to the threshold velocity v', $x=-1$ corresponds to the lower limit v'', whilst the mid-point $x=-\frac{1}{2}$ refers to the Landau velocity v_L. The variable y refers to the velocity u of those ions which recoil into the velocity state v. For values of u just above the threshold v' we have the linear relation $u=v+2p_0/m$ which becomes $y=x+1$, but in general we must use the nonlinear relation (20) for the recoil velocity. In reduced variables this is

$$y=x+1-\frac{\frac{2}{3}ay}{1+\frac{1}{2}a+ay}, \tag{25}$$

and may be solved explicitly to give the functional form $y(x)$ for y as a function of x.

As in the derivation of the analytic solution there are two distinct velocity ranges, $x>0$ and $-1<x<0$, within each of which Eq. (19) must be solved and the two solutions matched at $x=0$. In the region $x>0$ above threshold, both gain and loss terms on the right-hand side of Eq. (19) are nonzero, though at least at low fields the loss term dominates. We can write the solution corresponding to the case when only the loss term (first term) is present on the right-hand side, in the form $e^{-\phi(x)}$, where

$$\frac{d\phi}{dx}=\left(\frac{a}{b}\right)^3\left(\frac{x}{1+ax}\right)^2.$$

A particular solution for ϕ is

$$\phi(x)=b^{-3}[(1+ax)-(1+ax)^{-1}-2\ln(1+ax)],$$

which for small x, reduces to $\phi\cong(ax)^3/3b^3$ corresponding to the cubic dependence on $(v-v')$ displayed in the analytic solution (22).

Equation (19) can now be written for $x>0$, and *a fortiori* for $y>0$,

$$\frac{df}{dx}+f(x)\frac{d\phi(x)}{dx}=f(y)\frac{d\phi(y)}{dy}\frac{dy}{dx}. \tag{26}$$

We formally integrate this equation from $x=X\geq 0$ to some large value $x=L$ using $e^{-\phi(x)}$ as an integrating factor. This gives the integral equation

$$f(X)e^{\phi(X)}-f(L)e^{\phi(L)}=-\int_X^L e^{\phi(x)}f(y)\frac{d\phi(y)}{dy}\frac{dy}{dx}dx.$$

In the integral over x we may change the integration variation to y using the functional relation between these variables. Hence, for $X\geq 0$, we have

$$f(X)=Ce^{-\phi(X)}-e^{-\phi(X)}\int_{y(X)}^{y(L)}e^{\phi(x)}f(y)\frac{d\phi(y)}{dy}dy, \tag{27}$$

where $C=f(L)e^{\phi(L)}$ is an undetermined constant. In the integral it is now understood that x is related to y via Eq. (25).

In the lower-range $-1<x<0$ below threshold, only the gain term (second term) is present on the right-hand side of Eq. (26) so that

$$\frac{df}{dx}=f(y)\frac{d\phi(y)}{dy}\frac{dy}{dx}.$$

Integrating this equation from $x=-1$ to $x=X\leq 0$ and noting that the distribution function vanishes at $x=-1$, corresponding to the low-velocity limit v'', we obtain

$$f(X)=\int_0^{y(X)}f(y)\frac{d\phi(y)}{dy}dy, \tag{28}$$

since $y=0$ at $x=-1$. To ensure continuity at $X=0$ we equate the values of $f(X)$ given by Eqs. (27) and (28). This shows that the constant C must be determined self-consistently from the condition

$$C=\int_0^{y(0)}f(y)\frac{d\phi(y)}{dy}dy+\int_{y(0)}^{y(L)}e^{\phi(x)}f(y)\frac{d\phi(y)}{dy}dy. \tag{29}$$

The numerical solution of the integral equation commences by taking $f(x)=e^{-\phi(x)}$ as a first approximation. Evaluating the constant C from Eq. (29) and the integral in Eq. (27) gives a second approximation to $f(X)$ in the range $X\geq 0$. This sequence of steps is repeated. The iteration procedure is found to converge very rapidly for small values of the field parameter $b \lesssim 0.1$, when the trial function $e^{-\phi(x)}$ is in fact a close approximation to the final solution. However even for higher fields corresponding to $b \sim 0.3$ only ten iterative steps are needed. When the distribution function has been obtained to sufficient accuracy in the range $x > 0$ it may be calculated for lower range $-1 < x < 0$ directly from Eq. (28). The average velocity can then be computed taking into account that the calculated distribution function is not normalized. The number of iterative steps was defined by requiring an accuracy of 0.01% in the calculated average velocity. Choice of the upper limit L is not crucial. For a given value of b the distribution function decays rapidly at sufficiently high velocities. It is only necessary that L be so large, and therefore $f(L)$ so small, that the range $x > L$ should contribute negligibly to the average velocity. For electric fields such that $b \sim 0.3$ this was well satisfied by $L \sim 25$ and smaller values were chosen for weaker fields.

To illustrate the main features of the solutions, we have plotted two examples of calculated velocity distributions in Fig. 5. The first example for $b = 0.03$ shows all the characteristics of the analytic solution and corresponds to the small excess drift velocity $\bar{v} - v_L = 0.21(2p_0/m)$. The second example for $b = 0.19$ is shifted to higher velocities but becomes a broader and more symmetric curve about the mean velocity which in this case corresponds to $\bar{v} - v_L = 1.72(2p_0/m)$.

We would comment that the integral equation derived above is not the simplest, nor perhaps the most obvious, alternative form for the Boltzmann equation. Although the original differential Eq. (19) immediately suggests the apparently simpler form

$$\frac{eE}{m} f(v) = \int_{w=v}^{w=u(v)} R(w) f(w) dw \ ,$$

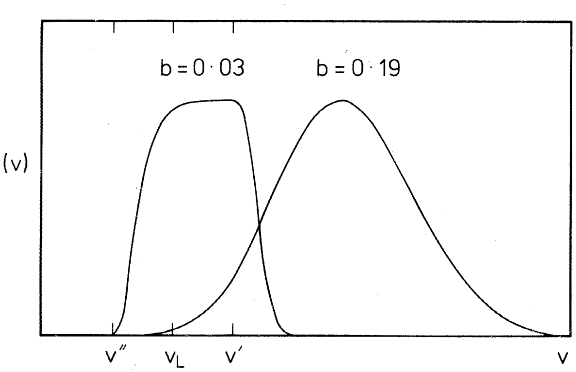

FIG. 5. Examples of computed velocity distributions (unnormalized).

a numerical solution based on this equation did not converge after 50 iterations. We did not investigate the properties of this equation in detail, rather we sought an alternative formulation which took advantage of the known asymptotic behavior $e^{-\phi(x)}$ of the distribution function.

COMPARISON WITH EXPERIMENT

From the computed solutions to the Boltzmann equation the variation of reduced drift velocity $\bar{x}$ with the electric-field parameter b was found for various values of the reduced recoil velocity a. For $b < 0.1$, corresponding to low fields, a plot of $a(\bar{x} + \frac{1}{2}) = (\bar{v} - v_L)/v_L$ vs b reveals a straight line, which corresponds to the $E^{1/3}$ dependence as predicted by the kinetic approach or the approximate analytic solution of the Boltzmann equation. Thus by plotting the experimentally measured values of $\bar{v}$ against $E^{1/3}$ we can find the value of v_L from the intercept and from the slope we obtain the scale factor $b/E^{1/3} = (e/Kmv_L)^{1/3}$. The value of v_L obtained in this way from experimental measurements[5,6] at 0.35 K with a drift space of length 10 mm is 46.3 m sec^{-1}. Bearing in mind that there is a 3% experimental uncertainty in the length of the the drift space this is in excellent accord with the value $v_L = 45.6$ mm sec^{-1} deduced from the roton parameters at 25 bar given by neutron-scattering data.[9] The recent high-field measurements[6] were made with drift spaces of 2.5 and 1 mm in which there are correspondingly greater experimental uncertainties. Accordingly the relative values of the drift lengths were scaled so as to give as close agreement as possible between the three sets of data in the velocity ranges of overlap. Further, for convenience in comparing the field dependence of the velocity data with theory, we have taken the absolute values of the drift lengths such that the intercept corresponds precisely to the value $v_L = 45.6$ m sec^{-1}. The slope of the linear part of the $\bar{v}$ vs $E^{1/3}$ then gives

$$E^{1/3}/b = (Kmv_L/e)^{1/3} = 500 \pm 10 \text{ V}^{1/3}\text{m}^{-1/3} \ .$$

In this region, the behavior is almost independent of the magnitude of the recoil momentum, which serves only to define the range of validity of the $E^{1/3}$ law.

The point at which the experimental drift velocity curve begins to deviate from the $E^{1/3}$ behavior corresponds to the changeover at $\bar{v} = v_L + (2p_0/m)$ from Eq. (5) to Eq. (6) in the kinetic model and is clearly visible in Fig. 6 at $\bar{v} \sim 53$–55 m sec^{-1}. This gives a rough estimate for the recoil velocity. A more precise value is found by determining the value of $a = 2p_0/mv_L$ which gives the closest fit between the theoretical curve and experimental data. This

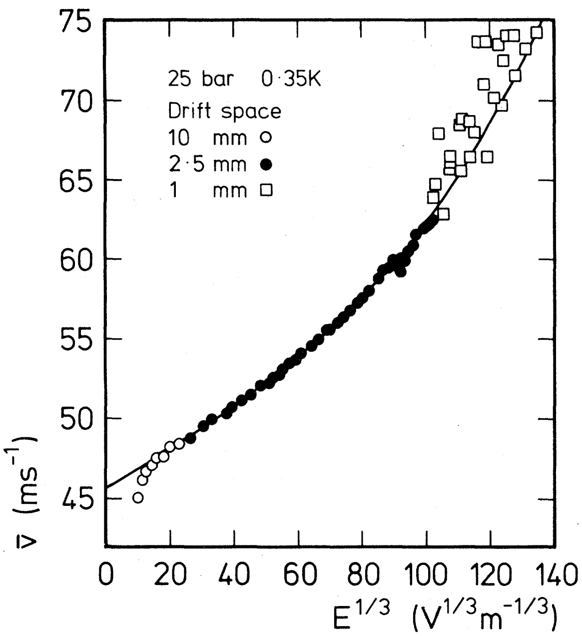

FIG. 6. Experimental drift-velocity data and theoretical curve drawn for parameters given in the text. Many data points have been omitted in the regions of overlap. Experimental results have been taken from Ref. 6 and McClintock (private communication).

occurs for $a = 0.20 \pm 0.01$ which gives for recoil velocity and effective mass of the ion

$$2p_0/m = 9.1 \text{ m sec}^{-1}, \quad m = 71 m_4,$$

where m_4 is the mass of a ^{4}He atom. We have taken $k_0 = p_0/\hbar = 2.04 \times 10^{10}$ m^{-1} from the neutron scattering[9] data at 25 bar. The ionic mass is in remarkably good agreement with the hydrodynamic effective mass $m^* = \frac{2}{3}\pi\rho r^3$. In fact, taking the radius of the negative ion at 25 bar to be $r = 11$ Å from Ostermeier's analysis[11] of experimental data and the liquid-helium density $\rho = 0.172$ g cm^{-3}, gives $m^* = 72 m_4$. Thus if we take values for the ionic radius r and liquid-^{4}He parameters v_L, p_0, and ρ from independent experiments our theory only requires one further parameter to describe the supercritical drift-velocity data. This is the emission-rate constant which is found to be

$$K = 0.93 \times 10^{12} \text{ sec}^{-1}.$$

We see from Fig. 6 that for electric fields between 2×10^3 and 2×10^6 V m^{-1} there is near perfect agreement between theory and experiment. At lower fields the rapid decrease of the drift velocity is due to scattering processes which produce dissipation at subthreshold velocities. The substantial linear part of the curve is itself a convincing demonstration that the supercritical dissipation is due to two-roton emission since the $E^{1/3}$ law depends solely on the form of the available density of two-roton states. But for drift velocities $\bar{v} > v_L + 2p_0/m \cong 55$ m sec^{-1}, the deviation from the $E^{1/3}$ law depends more critically on the details of the dynamical processes. Bearing in mind the approximations made in the calculation of the emission rate and formulation of the transport equation the agreement between theory and experiment at these high velocities may be regarded as a justification of the assumption of a constant interaction matrix element. The two-roton matrix element must therefore be quite insensitive to the roton energies and relative angle of emission over the substantial ranges of these variables indicated earlier.

For fields higher than 10^6 V m^{-1} corresponding to drift velocities $\bar{v} > 65$ m sec^{-1}, the theoretical curve follows the general trend of the experimental results though the random errors are much larger for the measurements made with the shortest drift space of 1 mm. However, as mentioned earlier, our calculation of the two-roton emission rate is accurate to at least 10% at ion velocities $v \sim 70$ m sec^{-1} but becomes less reliable at higher velocities since the emitted excitations no longer all lie on the rotonlike part of the liquid-^{4}He dispersion curve. The range of ion velocities in the distribution is roughly $\bar{v} \pm p_0/m$ and $p_0/m \sim 5$ m sec^{-1}, so that in the range above $\bar{v} = 65$ m sec^{-1} we are not justified in expecting such close quantitative agreement between theory and experiment as is found for the lower range.

CONCLUSION

The theory of the supercritical drift velocities of negative ions in liquid ^{4}He presented in this paper can account for the experimental results over a wide range of electric fields on the basis of two-roton spontaneous emission processes. One-roton processes are apparently completely absent although the threshold velocity for single-quantum emission $v_1' = v_L + (p_0/2m)$ is less than that for two-quantum emission.

Of course it is possible that the matrix element is so small that one-roton processes are unimportant. We have shown previously that the one-roton emission rate is of the form

$$R_1(v) \cong K_1 \left(\frac{v - v_1'}{v_L}\right)^{1/2} \theta(v - v_1'),$$

which leads to a field dependence of the drift mobility[4]

$$\bar{v} - v_L \cong v_L \left(\frac{3eE}{2K_1 m v_L}\right)^{2/3} \quad (\bar{v} - v_L < p_0/m), \qquad (30)$$

in the kinetic approach. It is possible that such an $E^{2/3}$ dependence might be revealed at low fields $< 10^3$ V m^{-1} provided scattering of the ions by thermal excitations were reduced by going to lower temperatures and scattering by ^{3}He impurity atoms reduced by purification of the liquid helium. However, using a quantum-hydrodynamical theory, Takken has estimated[12] the emission rate constant $K_1 = 5\times 10^{17}$ sec^{-1} which is so large that the one-quantum process should completely dominate. Since this is manifestly not so a quantitative theory of the ion-roton interaction is urgently needed. It is interesting to observe that in neutron scattering experiments[13] single-roton emission processes are clearly observed but that two-roton emission cannot be unambiguously identified in the continuous background intensity arising from the scattering by multiphonon processes.

The close fit between theory and experiment for drift velocities up to 65 m sec^{-1} also seems to preclude the appearance of three-roton processes, at the appropriate threshold velocity $v_2' \cong 52.4$ m sec^{-1}, with any significant rate. More accurate measurements at still higher drift velocities would be required to confirm this. But their interpretation even on the two-quantum hypothesis presents difficulties, since the emitted excitations are no longer confined to the rotonlike part of the liquid-^{4}He dispersion curve.

Finally we mention that the theory presented here can be generalized to take account of vortex nucleation. At a pressure of 25 bar the critical velocity for vortex nucleation v_v is appreciably larger than v_L and only that small fraction of ions which attain speeds greater than v_v are able to nucleate and become trapped on vortices. The consequent perturbation of the velocity distribution is small since it happens that the vortex nucleation rate[14] is $\sim 10^5$ sec^{-1} which is much less than the mean roton emission rate $\sim 10^{11}$ sec^{-1}. But, at low pressures, v_v is significantly less than v_L. The bare-ion signal is then severely attenuated since the entire velocity distribution is above the critical speed v_v and all ions may decay into the trapped vortex state. However, measurements of the field dependence of the drift velocity should be possible with a sufficiently small drift space to allow detection of the much reduced bare-ion pulse. The two-roton emission theory can therefore be tested over a wide range of pressures for negative ions, and the pressure dependence of the parameters may provide some clue as to the nature of the ion-roton interaction.

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[8] The Landau critical velocity v_L is strictly given by the minimum value of the ratio $\epsilon k/\hbar k$. This value occurs so close to the minimum of the liquid-helium dispersion curve, that the approximate expression $v_L \cong \Delta/p_0$ is accurate to better than 1%. See Ref. 5.

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