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Calculation of the dielectric function for an electron liquid

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A theory for calculating the frequency- and wave-number-dependent dielectric function of an electron liquid is presented by solving the equations of motion for the double-time retarded commutator of the charge-density fluctuation operators. It is based on a decoupling of the higher-order Green's functions, which has been achieved by demanding certain proportionality between the higher-order Green's functions and the lower order. The proportionality coefficient is determined by conserving the various frequency moments. It is shown that by conserving the first frequency moment only, we reproduce the Toigo-Woodruff result for the dielectric function. However, the present theory has the advantage of conserving frequency moments to an infinite order. The dielectric function obtained in this paper is a functional of the function $G(\vec{k},\omega)$ which is considered to account for the short-range correlations arising from both the exchange and Coulomb effects. This turns out to be the same as the one derived by Rajagopal who solved variationally the integral equation for the irreducible vertex function containing only linear exchange processes. Numerical evaluation of the function $G(\vec{k},\omega)$ has been made for various values of k in the limit $\omega = 0$, and the results are compared with those of the earlier theories. In contrast to all the earlier theories, we find a very sharp peak in the value of our $G(k)$ around $k = 2k_F$. It is further interesting to note that the present theory satisfies exactly the compressibility sum rule. An important result of this theory is that the value of $G(k)$, obtained in this paper in the limit $k \to \infty$, happens to be 1/3, which is in complete agreement with the value of $G(\infty)$ in the Hartree-Fock approximation. This we consider to be a great success over the other existing theories.

I. INTRODUCTION

It is well-known that many important properties of metals can be related to a model in which the conduction electrons form a system of degenerate Fermi gas with the ions replaced by a uniform positive background. The dielectric formulation of the many-electron system has been found to be very useful in studying various metallic properties like the density-fluctation excitation spectrum, the correlation energy and those related to transport phenomena. Properties such as the interionic potential and the screening of defects can also be studied provided one assumes that the dielectric function is not essentially altered by the discrete nature of the ions of the lattice. All these properties depend strongly on the electron-electron interactions. It is therefore of great importance to have a dielectric function, in the range of electron densities encountered in metals, which should have all the corrections that are due to the electron-electron interactions.

There have been many efforts at calculating the frequency- and wave-number-dependent dielectric function for the many-electron system. The dielectric function first given by Lindhard[1] is the one which corresponds to the random-phase approximation (RPA).[2,3] Though it provides a good description of the plasmon excitation modes and of the long-wavelength screening phenomena its validity is limited to high electron densities ($r_s < 1$) only. The inadequacy of the RPA dielectric function becomes manifest from the fact that the pair-distribution function, which should be positive definite, becomes negative[4,5] for small interparticle separations over the entire range of metallic densities ($1.8 \leq r_s \leq 6$). This arises due to the failure of the RPA to take into account the short-range correlation effects. An approximate procedure to improve upon the RPA dielectric function was first proposed by Hubbard[6] who could approximately sum an infinite number of ladder-bubble diagrams. Hubbard's approximation yields a modified expression for the dielectric function $\epsilon(\vec{k},\omega)$ which is given by

$$1 - \frac{1}{\epsilon(\vec{k},\omega)} = \frac{Q_0(\vec{k},\omega)}{1 + [1 - G(\vec{k})]Q_0(\vec{k},\omega)}, \tag{1.1}$$

where

$$Q_0(\vec{k},\omega) = -v(\vec{k})x_0(\vec{k},\omega), \tag{1.2}$$

$x_0(\vec{k},\omega)$ being the usual free-electron polarizability, and $v(\vec{k}) = 4\pi e^2/k^2$. The function $G(\vec{k})$ appearing in (1.1) takes into account the exchange efforts. Hubbard's $G(\vec{k})$ is of the form

$$G(\vec{k}) = \tfrac{1}{2}k^2/(k^2 + k_F^2). \tag{1.3}$$

It is known that the Hubbard scheme is an approximate solution to the integral equation obeyed by the irreducible vertex function which incorporates all the exchange processes in the lowest order. Many other forms of $G(\vec{k})$ have since then been proposed,[7–13] but we shall refer particularly to the ones suggested by Singwi *et al.* through a series of papers,[9,10] the one by Vashishta and Singwi (VS),[11] by Toigo and Woodruff,[12] and by Rajagopal.[13] Singwi *et al.* in their first paper (hereafter referred to as STLS) have arrived at an expression for the dielectric function, formally equivalent to (1.1), by solving the equation of motion for the one-particle distribution function in the presence of an external potential. By using an ansatz they have been able to relate $G(\vec{k})$, $S(\vec{k})$ (the static structure factor), and $\epsilon(\vec{k}, \omega)$ self-consistently. This theory yielded a physically acceptable pair correlation function, but the compressibility sum rule was not satisfied. In order to rectify this deficiency these authors, in the later version of their work (hereafter referred to as SSTL), screened the Coulomb potential entering into the local-field correction term of their paper. By doing this, the compressibility was improved, but was not still very satisfactory. A further modification of the STLS theory was made by Vashishta and Singwi by accounting for the change in the pair correlation function with respect to the external potential, which was ignored in the two earlier theories. With this, the compressibility sum rule was satisfied almost exactly and satisfactory results for the pair correlation function at small interparticle separations were obtained for densities up to $r_s = 2$.

Though the VS theory gives a very good dielectric function, still it is not devoid of the unphysical feature of having negative values of the pair correlation function for metallic densities in the range $2 < r_s \leq 6$. Besides, a first-principle justification of this theory and also of the earlier ones is not yet understood. A method of calculating a dielectric function beyond the RPA starting from first principles and using the equation-of-motion approach was first given by Toigo and Woodruff (TW).[12] The TW method is based on decoupling and solving the equation of motion for the double-time retarded commutator of density-fluctuation operators, using the moment-conserving method suggested by Tahir-Kheli and Jarrett.[14] The dielectric function obtained by Toigo and Woodruff closely resembles the form written in (1.1); but it is more general in the sense that the function $G(k)$ is now a function of ω also. An interesting feature of the TW dielectric function is that it satisfies the compressibility sum rule[15] justifying that the small-momentum behavior of the TW dielectric function is quite good. From the work of Shaw[8] it is well-known that calculation of metallic properties depend strongly on $G(\vec{k})$. Numerical results obtained by Toigo and Woodruff for $G(\vec{k})$ show that there is a peak in their value around $k = 2k_F$. This is in contrast with the results of Singwi *et al.* However, in the asymptotic limit, the function $G(\vec{k})$ should, according to Shaw, satisfy the inequality

$$\tfrac{1}{2} \leq G(\infty) \leq 1\,, \tag{1.4}$$

if the ansatz of STLS is valid. This relation is satisfied in the work of STLS for values of $r_s \leq 5$, and in that of SSTL for smaller values of r_s. [The $G(k)$ determined by the self-consistent procedure depends on r_s.] In the work of Vashishta and Singwi the relation (1.4) is satisfied only for $r_s \leq 3$. The pair correlation function calculated by Toigo and Woodruff[16] is almost the same as Ref. 11, although for larger r_s, it becomes more negative.

The major drawback in both the TW theory and VS theory lies in the value of $G(k)$ in the asymptotic limit. As pointed out by Geldart and Taylor,[17] the value of $G(\infty)$ in the Hartree-Fock (HF) approximation should be equal to $\frac{1}{3}$. Since the value of $G^{HF}(\infty)$ obtained by Vashishta and Singwi is $\frac{1}{2}$ and the value of $G(\infty)$ of the TW theory is $\frac{2}{3}$ (instead of being 0.762 which is due to the inaccuracy of their numerical computation[18]), this shows that both these theories do not reproduce the HF value of $G(k)$ in the large k limit. According to the remark made by Vashishta and Singwi,[11] negative values of their pair correlation function $g(r)$ at $r = 0$ for densities $r_s > 2$, may very well be due to this defect.

In the present paper we have made an attempt to improve upon the TW dielectric function by generalizing their theory so as to conserve the frequency moments to an infinite order in a certain sense to be explained later. We have determined the dielectric function by solving the equation of motion of the double-time retarded commutator of the density-fluctuation operators by using a decoupling procedure according to which the higher-order Green's functions are set equal to the lower-order ones through a certain proportionality constant. The proportionality constant is determined by conserving frequency moments to an infinite order. This is different from the TW approach in the sense that TW method is based on the moment-conserving scheme suggested by Tahir-Kheli and Jarrett.[14] In our theory, by conserving the first frequency moment, we obtain the TW expression for the dielectric function. This is what one should expect.

Conserving higher frequency moments appears to be extremely difficult in the TW approach. In the present theory we have been able to bypass this difficulty. Doing so has led us to terms which form a geometric series and as such, they have been summed. The dielectric function obtained in this way turns out to be identical to the one obtained by Rajagopal[13] who solved variationally the integral equation for the irreducible vertex function. It is seen that our dielectric function has the merit of satisfying the compressibility sum rule. The asymptotic value of the function $G(\vec{k})$, as obtained in this paper, happens to be $\frac{1}{3}$, meaning thereby that the large k limit of our $G(k)$ is the HF value. Obviously this violates the condition given in (1.4). An interesting feature of this theory is that there is a very sharp peak in the value of $G(\vec{k})$ at $k=1.97k_F$, which is in contrast to the theory of Singwi *et al*. It has been pointed out by Geldart and Taylor[19] that such a peak is very much expected around $k=2k_F$ if one assumes the dielectric function to be of the form shown in (1.1). There is, of course, the indication for the appearance of a peak in the TW result at $k=1.95k_F$, but it is not so sharp as it is in our case. We find that the numerical values of our $G(\vec{k})$ differ drastically from those of Toigo and Woodruff in the momentum region $k>1.6k_F$. Comparing our $G(\vec{k})$ values (which are same for all r_s) with those of Vashishta and Singwi [whose $G(\vec{k})$ values change with r_s] we find that for certain values of k around $k=2k_F$, our results, like those of these authors, exceed unity. For larger k values our results for $G(\vec{k})$ start falling with k, whereas in the VS theory they go on increasing with k till they attain a saturation limit.

II. GENERAL THEORY

According to the theory of linear dissipative process[20] the dielectric response function $\epsilon(\vec{k},\omega)$ is given by the relation

$$1-\frac{1}{\epsilon(\vec{k},\omega)}=v(\vec{k})\,\langle\langle\rho_{\vec{k}}(t);\rho^{\dagger}_{\vec{k}}(0)\rangle\rangle^{\text{ret}}_{E=\omega}=v(\vec{k})\,\mathcal{G}(\vec{k},\omega)\,, \tag{2.1}$$

where the symbol $\langle\langle\rho_{\vec{k}}(t);\rho^{\dagger}_{\vec{k}}(0)\rangle\rangle^{\text{ret}}_{E=\omega}$ stands for the Fourier transform with respect to time of the double-time retarded Green's function $\mathcal{G}_r(\vec{k},t-t')$:

$$\mathcal{G}_r(\vec{k},t-t')=i\theta(t-t')\langle|[\rho_{\vec{k}}(t),\rho^{\dagger}_{\vec{k}}(t')]|\rangle\,, \tag{2.2}$$

$\langle\ \rangle$ being the average over the ground state of the fermion system. The $\rho_{\vec{k}}$'s are written as

$$\rho_{\vec{k}}(t)=\sum_{\vec{q},\sigma}a^{\dagger}_{\vec{q},\sigma}(t)a_{\vec{k}+\vec{q},\sigma}(t)\,, \tag{2.3}$$

where $a^{\dagger}$ and a are the creation and annihilation operators for the electrons in the Heisenberg representation, and the symbol σ denotes the spin index. Using (2.3), one may write (2.1) as

$$1-\frac{1}{\epsilon(\vec{k},\omega)}=v(\vec{k})\sum_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}\langle\langle a^{\dagger}_{\vec{q}_1,\sigma_1}(t)a_{\vec{k}+\vec{q}_1,\sigma_1}(t);a^{\dagger}_{\vec{k}+\vec{q}_2,\sigma_2}(0)a_{\vec{q}_2,\sigma_2}(0)\rangle\rangle^{\text{ret}}_{E=\omega}=v(\vec{k})\sum_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}F_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},\omega)\,, \tag{2.4}$$

where $F_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},\omega)$ is the Fourier transform with respect to time of

$$F_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},t)=\langle\langle a^{\dagger}_{\vec{q}_1,\sigma_1}(t)a_{\vec{k}+\vec{q}_1,\sigma_1}(t);a^{\dagger}_{\vec{k}+\vec{q}_2,\sigma_2}(0)a_{\vec{q}_2,\sigma_2}(0)\rangle\rangle\,. \tag{2.5}$$

The Hamiltonian of the many-electron system at zero temperature is

$$H=\sum_{\vec{p}\sigma}\mathcal{E}_{\vec{p}}a^{\dagger}_{\vec{p},\sigma}a_{\vec{p},\sigma}+\frac{1}{2}\sum_{\substack{\vec{k}'\neq 0\\ \vec{s}\sigma,\vec{s}'\sigma'}}v(\vec{k}')a^{\dagger}_{\vec{s}+\vec{k}',\sigma}a^{\dagger}_{\vec{s}'-\vec{k}',\sigma'}a_{\vec{s}',\sigma'}a_{\vec{s},\sigma}=H_0+H_I\,, \tag{2.6}$$

where $\mathcal{E}_{\vec{p}}=p^2/2m$ in the unit $\hbar=1$. Differentiating now (2.5) with respect to time, we write down the equation of motion for $F_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},t)$ as

$$i\frac{d}{dt}F_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},t)=-\delta(t)\langle|[a^{\dagger}_{\vec{q}_1,\sigma_1}(t)a_{\vec{k}+\vec{q}_1,\sigma_1}(t),a^{\dagger}_{\vec{k}+\vec{q}_2,\sigma_2}(0)a_{\vec{q}_2,\sigma_2}(0)]|\rangle-\langle\langle[H,a^{\dagger}_{\vec{q}_1,\sigma_1}(t)a_{\vec{k}+\vec{q}_1,\sigma_1}(t)];a^{\dagger}_{\vec{k}+\vec{q}_2,\sigma_2}(0)a_{\vec{q}_2,\sigma_2}(0)\rangle\rangle^{\text{ret}}\,. \tag{2.7}$$

Evaluating these commutators using the anticommutation relations for the fermion creation and annihilation operators, we obtain

$$\begin{aligned}i\frac{d}{dt}F_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},t)=&\ \delta(t)(n_{\vec{k}+\vec{q}_1,\sigma_1}-n_{\vec{q}_1,\sigma_1})\delta_{\sigma_1,\sigma_2}\delta_{\vec{q}_1,\vec{q}_2}+\omega(\vec{q}_1,\vec{k})F_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},t)\\&+\sum_{\vec{k}',\vec{s}\sigma}v(\vec{k}')\langle\langle[a^{\dagger}_{\vec{q}_1,\sigma_1}(t)a^{\dagger}_{\vec{s}+\vec{k}',\sigma}(t)a_{\vec{s},\sigma}(t)a_{\vec{q}_1+\vec{k}+\vec{k}',\sigma_1}(t)\\&-a^{\dagger}_{\vec{q}_1-\vec{k}',\sigma_1}(t)a^{\dagger}_{\vec{s}+\vec{k}',\sigma}(t)a_{\vec{s},\sigma}(t)a_{\vec{q}_1+\vec{k},\sigma_1}(t)];a^{\dagger}_{\vec{k}+\vec{q}_2,\sigma_2}(0)a_{\vec{q}_2,\sigma_2}(0)\rangle\rangle^{\text{ret}}\,,\end{aligned} \tag{2.8}$$

where $\omega(\vec{q}_1,\vec{k})=(\mathcal{E}_{\vec{q}_1+\vec{k}}-\mathcal{E}_{\vec{q}_1})$ and $n_{\vec{q}_1}$ is the Fermi distribution function:

$$n_{\vec{q}_1}=\begin{cases}1, & |\vec{q}_1|<k_F\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Taking the Fourier transform on both sides of (2.8) and assuming that the interaction is switched on adiabatically, we obtain

$$[\omega-\omega(\vec{q}_1,\vec{k})+i\delta]F_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},\omega)=(n_{\vec{k}+\vec{q}_1,\sigma_1}-n_{\vec{q}_1,\sigma_1})\delta_{\sigma_1,\sigma_2}\delta_{\vec{q}_1,\vec{q}_2}+\sum_{\vec{k}',\vec{s}\,\sigma}F^{(1)}_{\vec{k}',\vec{s}\,\sigma,\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},\omega)\,, \tag{2.9}$$

where $F^{(1)}(\vec{k},\omega)$ denotes the Fourier transform of the second term on the right-hand side of (2.8). It is seen that we can decouple (2.9) by putting $\vec{k}'=-\vec{k}$ and then pairing off the operators with equal momentum by writing $\langle a^{\dagger}_{\vec{q}_1,\sigma_1}a_{\vec{q}_1,\sigma_1}\rangle=n_{\vec{q}_1,\sigma_1}$. With this we obtain

$$[\omega-\omega(\vec{q}_1,\vec{k})+i\delta]F_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},\omega)=(n_{\vec{k}+\vec{q}_1,\sigma_1}-n_{\vec{q}_1,\sigma_1})\delta_{\sigma_1,\sigma_2}\delta_{\vec{q}_1,\vec{q}_2}-v(\vec{k})(n_{\vec{k}+\vec{q}_1,\sigma_1}-n_{\vec{q}_1,\sigma_1})\sum_{\vec{s},\sigma}F_{\vec{s}\sigma,\vec{q}_2\sigma_2}(\vec{k},\omega)\,, \tag{2.10}$$

which leads directly to the RPA result. Since we will be interested in going beyond the RPA we shall try to use a better method of decoupling of the higher-order Green's function denoted by $F^{(1)}(\vec{k},\omega)$. This would, of course, yield better results than can be obtained with the RPA. One such improved decoupling scheme is due to the method proposed by Tahir-Kheli and Jarrett.[14] This is the one which has been adopted by Toigo and Woodruff.[12] We shall not try to proceed in the way Toigo and Woodruff have done to evaluate the coefficients appearing in the expansion of the higher-order Green's function in terms of the lower-order ones. Our approach will be a different one.

We concentrate on the evaluation of the function $\mathcal{G}(\vec{k},\omega)$, where

$$\mathcal{G}(\vec{k},\omega)=\sum_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}F_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},\omega)\,. \tag{2.11}$$

To determine $\mathcal{G}$ we sum both sides of (2.9) over $\vec{q}_2,\sigma_2$ to obtain

$$[\omega-\omega(\vec{q}_1,\vec{k}+i\delta]\mathcal{F}_{\vec{q}_1\sigma_1}(\vec{k},\omega)=(n_{\vec{q}_1+\vec{k},\sigma_1}-n_{\vec{q}_1,\sigma_1})+\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},\omega)\,, \tag{2.12}$$

where

$$\mathcal{F}_{\vec{q}_1\sigma_1}(\vec{k},\omega)=\sum_{\vec{q}_2\sigma_2}F_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},\omega)$$

and

$$\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},\omega)=\sum_{\vec{k}',\vec{s}\,\sigma,\vec{q}_2\sigma_2}F^{(1)}_{\vec{k}',\vec{s}\,\sigma,\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},\omega)\,. \tag{2.13}$$

In the RPA decoupling one expresses $\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},\omega)$ as a linear combination of all the $\mathcal{F}$'s, i.e.,

$$\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},\omega)=A_{\vec{q}_1\sigma_1}(\vec{k})\sum_{m\sigma}\mathcal{F}_{\vec{m}\sigma}(\vec{k},\omega)\,. \tag{2.14}$$

We shall try to express the $\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},\omega)$ appearing in (2.12) in the same way shown in (2.14), the only difference being that the $A_{\vec{q}_1\sigma_1}(\vec{k})$ is now to be considered as a function of both k and ω. This is, in this sense, different from the $A_{\vec{q}_1\sigma_1}(\vec{k})$ of Toigo and Woodruff's paper, where it was taken to be a function of k only. Later on we shall show how the $A_{\vec{q}_1\sigma_1}(\vec{k},\omega)$ would be determined. Substituting (2.14) into (2.12) and using the fact that $\sum_{\vec{m}\sigma}\mathcal{F}_{\vec{m}\sigma}(\vec{k},\omega)=\mathcal{G}(\vec{k},\omega)$, we obtain

$$\mathcal{F}_{\vec{q}_1\sigma_1}(\vec{k},\omega)=\frac{(n_{\vec{k}+\vec{q}_1,\sigma_1}-n_{\vec{q}_1,\sigma_1})+A_{\vec{q}_1\sigma_1}(\vec{k},\omega)\mathcal{G}(\vec{k},\omega)}{\omega-\omega(\vec{q}_1,\vec{k})+i\delta}\,. \tag{2.15}$$

Summing both sides of (2.15) over $\vec{q}_1$, σ_1, and solving for $\mathcal{G}(\vec{k},\omega)$, we get

$$\mathcal{G}(\vec{k},\omega)=\sum_{\vec{q}_1\sigma_1}\frac{n_{\vec{k}+\vec{q}_1,\sigma_1}-n_{\vec{q}_1,\sigma_1}}{\omega-\omega(\vec{q}_1,\vec{k})+i\delta}\Bigg/\left(1-\sum_{\vec{q}_1\sigma_1}\frac{A_{\vec{q}_1\sigma_1}(\vec{k},\omega)}{\omega-\omega(\vec{q}_1,\vec{k})+i\delta}\right). \tag{2.16}$$

With the help of (2.16) one can write down the expression for the dielectric function using the formula (2.1).

III. DERIVATION OF $A_{\vec{q}_1\sigma_1}(\vec{k},\omega)$

Following (2.14), $A_{\vec{q}_1\sigma_1}(\vec{k},\omega)$ may be written as

$$A_{\vec{q}_1\sigma_1}(\vec{k},\omega)=\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},\omega)/\mathcal{G}(\vec{k},\omega)\,, \tag{3.1}$$

which, with the help of Fourier transform, can be written in the form

$$A_{\vec{q}_1\sigma_1}(\vec{k},\omega)=\int_{-\infty}^{\infty}\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},t)e^{i\omega t}dt\Bigg/\int_{-\infty}^{\infty}\mathcal{G}(\vec{k},t)^{i\omega t}dt\,, \tag{3.2}$$

where

$$\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},t)=\sum_{\vec{k}',\vec{s}\,\sigma,\vec{q}_2\sigma_2}F_{\vec{k}',\vec{s}\,\sigma,\vec{q}_1\sigma_1,\vec{q}_2\sigma_2}(\vec{k},t)=i\theta(t)\sum_{\vec{k}',\vec{s}\,\sigma,\vec{q}_2\sigma_2}v(\vec{k}')\langle|[\mathfrak{A}(t),\mathfrak{B}(0)]|\rangle\,, \tag{3.3}$$

having

$$\mathcal{A}(t) = a^\dagger_{\vec{q}_1\sigma_1}(t)\, a^\dagger_{\vec{k}'+\vec{s},\sigma}(t)\, a_{\vec{s},\sigma}(t)\, a_{\vec{q}_1+\vec{k}+\vec{k}',\sigma_1}(t) - a^\dagger_{\vec{q}_1-\vec{k}',\sigma_1}(t)\, a^\dagger_{\vec{k}'+\vec{s},\sigma}(t)\, a_{\vec{s},\sigma}(t)\, a_{\vec{q}_1+\vec{k},\sigma_1}(t)\,, \tag{3.4}$$

and

$$\mathcal{B}(0) = a^\dagger_{\vec{k}+\vec{q}_2,\sigma_2}(0)\, a_{\vec{q}_2,\sigma_2}(0). \tag{3.5}$$

The commutator in (3.3) can be expanded into a power series using the well-known expansion of the Heisenberg operator

$$O(t) = e^{iHt} O e^{-iHt} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}[H,[H,\ldots[H,0]\ldots]]\,. \tag{3.6}$$

Following (3.6), we find

$$\langle\,|[\mathcal{A}(t),\mathcal{B}(0)]\,|\rangle = \langle\,|[\mathcal{A}(0),\mathcal{B}(0)]\,|\rangle + it\langle\,|[[H,\mathcal{A}(0)],\mathcal{B}(0)]\,|\rangle + \frac{(it)^2}{2!}\langle\,|[[H,[H,\mathcal{A}(0)]],\mathcal{B}(0)]\,|\rangle + \cdots\,. \tag{3.7}$$

As it is seen from (3.3) $F^{(1)}(\vec{k},\omega)$ depends linearly on the potential v. Since here our aim is to account for the effects of the potential to first order only, we will replace all the H in (3.7) by H_0. By doing this we are able to evaluate the commutators in (3.7) to all orders in time. The detailed evaluation of these commutators is shown in Appendix A. It is interesting to see that the results of these commutators form a geometric series. Thus we are able to write

$$\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},t) = i\theta(t)\sum_{\vec{q}_2\sigma_2}(n_{\vec{k}+\vec{q}_2,\sigma_2} - n_{\vec{q}_2,\sigma_2})\Big(\sum_{\vec{k}'} v(\vec{k}')(n_{\vec{q}_1+\vec{k}-\vec{k}',\sigma_1} - n_{\vec{q}_1+\vec{k}',\sigma_1})\,\delta_{\sigma_1,\sigma_2}\delta_{\vec{q}_1\vec{q}_2} + v(\vec{k})(n_{\vec{q}_1+\vec{k},\sigma_1} - n_{\vec{q}_1,\sigma_1}) + v(\vec{q}_1-\vec{q}_2)(n_{\vec{q}_1,\sigma_1} - n_{\vec{q}_1+\vec{k},\sigma_1})\delta_{\vec{\sigma}_1,\vec{\sigma}_2}\Big)e^{-i\omega(\vec{q}_2,\vec{k})t}\,. \tag{3.8}$$

Using (2.5) and (3.6) one can now write down the expression for $\mathcal{G}(\vec{k},t)$. As shown in Appendix A, we have

$$\mathcal{G}(\vec{k},t) = i\theta(t)\sum_{\vec{q}_2,\sigma_2}(n_{\vec{q}_2,\sigma_2} - n_{\vec{q}_2+\vec{k},\sigma_2})e^{-i\omega(\vec{q}_2,\vec{k})t}\,. \tag{3.9}$$

Now, if we define

$$A_{\vec{q}_1\sigma_1}(\vec{k}) = \mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},t)/\mathcal{G}(\vec{k},t) \tag{3.10}$$

and use (3.8) and (3.9) it can be easily shown that by keeping up to the terms proportional to it only, which is equivalent to conserving the first frequency moment, we find

$$A_{\vec{q}_1\sigma_1}(\vec{k}) = \sum_{\vec{q}_2\sigma_2}(n_{\vec{k}+\vec{q}_2,\sigma_2} - n_{\vec{q}_2,\sigma_2}) \times\Big(\sum_{\vec{k}'} v(\vec{k}')\omega(\vec{q}_2,\vec{k})(n_{\vec{q}_1+\vec{k}-\vec{k}',\sigma_1} - n_{\vec{q}_1+\vec{k}',\sigma_1})\delta_{\sigma_1,\sigma_2}\delta_{\vec{q}_1,\vec{q}_2} + v(\vec{k})\omega(\vec{q}_2,\vec{k})(n_{\vec{k}+\vec{q}_1,\sigma_1} - n_{\vec{q}_1,\sigma_1}) + v(\vec{q}_1-\vec{q}_2)\omega(\vec{q}_2,\vec{k})(n_{\vec{q}_1,\sigma_1} - n_{\vec{k}+\vec{q}_1,\sigma_1})\delta_{\sigma_1,\sigma_2}\Big)\Big/\sum_{\vec{q}_2\sigma_2}\omega(\vec{q}_2,\vec{k})(n_{\vec{q}_2,\sigma_2} - n_{\vec{q}_2+\vec{k},\sigma_2})\,. \tag{3.11}$$

This is what Toigo and Woodruff have obtained. As it is seen from (3.11) the coefficient $A_{\vec{q}_1\sigma_1}(\vec{k})$ is independent of ω. To determine $A_{\vec{q}_1\sigma_1}$ we will use (3.2) instead of (3.10), because it is not possible to conserve higher moments if one uses (3.10). Using the expressions for $\mathcal{F}^{(1)}(\vec{k},t)$ and $\mathcal{G}(\vec{k},t)$ and performing the integration over time, we now obtain

$$A_{\vec{q}_1\sigma_1}(\vec{k},\omega) = v(\vec{k})(n_{\vec{q}_1\sigma_1} - n_{\vec{k}+\vec{q}_1,\sigma_1}) + \frac{n_{\vec{q}_1,\sigma_1} - n_{\vec{k}+\vec{q}_1,\sigma_1}}{\omega-\omega(\vec{q}_1,\vec{k})+i\delta}\sum_{\vec{k}} v(\vec{k}')(n_{\vec{q}_1+\vec{k}',\sigma_1} - n_{\vec{q}_1+\vec{k}-\vec{k}',\sigma_1})\Big/\sum_{\vec{q}_2\sigma_2}\frac{n_{\vec{q}_2,\sigma_2} - n_{\vec{k}+\vec{q}_2,\sigma_2}}{\omega-\omega(\vec{q}_2,\vec{k})+i\delta} - (n_{\vec{q}_1,\sigma_1} - n_{\vec{k}+\vec{q}_1,\sigma_1})\sum_{\vec{q}_2}\frac{v(\vec{q}_1-\vec{q}_2)(n_{\vec{q}_2,\sigma_1} - n_{\vec{k}+\vec{q}_2,\sigma_1})}{\omega-\omega(\vec{q}_2,\vec{k})+i\delta}\Big/\sum_{\vec{q}_2\sigma_2}\frac{n_{\vec{q}_2,\sigma_2} - n_{\vec{k}+\vec{q}_2,\sigma_2}}{\omega-\omega(\vec{q}_2,\vec{k})+i\delta}\,. \tag{3.12}$$

From (3.12) it is seen that $A_{\vec{q}_1\sigma_1}$ depends explicitly on ω. In the second term in (3.12) we make the change of variables $\vec{k}'\to\vec{q}_1-\vec{q}_2$. Dividing $A_{\vec{q}_1\sigma_1}$ by $\omega-\omega(\vec{q}_1,\vec{k})+i\delta$ and summing over $\vec{q}_1,\sigma_1$ we can write

$$-\sum_{\vec{q}_1\sigma_1} \frac{A_{\vec{q}_1\sigma_1}(\vec{k},\omega)}{\omega-\omega(\vec{q}_1,\vec{k})+i\delta} = Q_0(\vec{k},\omega)\left(1-\frac{P_0(\vec{k},\omega)}{Q_0(\vec{k},\omega)}\right), \tag{3.13}$$

where

$$Q_0(\vec{k},\omega) = -v(\vec{k})\sum_{\vec{q}_1\sigma_1} \frac{n_{\vec{q}_1\sigma_1}-n_{\vec{k}+\vec{q}_1,\sigma_1}}{\omega-\omega(\vec{q}_1,\vec{k})+i\delta} = -v(\vec{k})\chi_0(\vec{k},\omega) \tag{3.14}$$

and

$$P_0(\vec{k},0) = -\sum_{\vec{q}_2,\vec{q}_1\sigma_1} v(\vec{q}_1-\vec{q}_2)\frac{1}{\omega-\omega(\vec{q}_1,\vec{k})+i\delta}\left(\frac{1}{\omega-\omega(\vec{q}_2,\vec{k})+i\delta}-\frac{1}{\omega-\omega(\vec{q}_1,\vec{k})+i\delta}\right)$$
$$\times(n_{\vec{q}_1,\sigma_1}-n_{\vec{k}+\vec{q}_1,\sigma_1})(n_{\vec{q}_2,\sigma_1}-n_{\vec{k}+\vec{q}_2,\sigma_1})/\chi_0(\vec{k},\omega). \tag{3.15}$$

The function $P_0(\vec{k},\omega)$ in this case accounts for the non-RPA contribution; hence it should include the exchange and correlation effects. Using (3.13) the expression for $\mathcal{G}(\vec{k},\omega)$ is written from (2.16) as

$$\mathcal{G}(\vec{k},\omega) = \frac{-\chi_0(\vec{k},\omega)}{1+Q_0(\vec{k},\omega)[1-P_0(\vec{k},\omega)/Q_0(\vec{k},\omega)]}. \tag{3.16}$$

The dielectric function is obtained using the relation (2.1):

$$1-\frac{1}{\epsilon(\vec{k},\omega)} = \frac{Q_0(\vec{k},\omega)}{1+Q_0(\vec{k},\omega)[1-P_0(\vec{k},\omega)/Q_0(\vec{k},\omega)]}. \tag{3.17}$$

Comparing (3.17) with (1.1) we identify the function G to be

$$G(\vec{k},\omega) = P_0(\vec{k},\omega)/Q_0(\vec{k},\omega), \tag{3.18}$$

which shows that in our case G depends explicitly on ω.

IV. CALCULATION OF $G(\vec{k},\omega)$ IN THE STATIC LIMIT ($\omega = 0$)

A. Compressibility

In this paper we shall study the properties of the dielectric function given by (3.17), using the static value of $G(\vec{k},\omega)$. In this approximation the dielectric function assumes the form

$$\epsilon(\vec{k},\omega) = 1+\frac{Q_0(\vec{k},\omega)}{1-G(\vec{k},0)Q_0(\vec{k},\omega)}. \tag{4.1}$$

Since we will be interested in the study of the compressibility of the system of electrons we shall look into the form $\epsilon(\vec{k},\omega)$ in the limit $k\to 0$ at $\omega=0$. It is well-known that the dielectric function $\epsilon(\vec{k},0)$ in the limit $k\to 0$ is connected to the compressibility through the relation[21]

$$\lim_{\vec{k}\to 0}\epsilon(\vec{k},0) = 1+(k_{FT}/k)^2\mathcal{K}/\mathcal{K}_0, \tag{4.2}$$

where $\mathcal{K}_0$ and $\mathcal{K}$ are, respectively, the compressibility corresponding to the free and the interacting electron gas and k_{FT} is the Thomas-Fermi screening wave vector. The behavior of $Q_0(\vec{k},0)$ in the limit $k\to 0$ being known, we shall here investigate the form of $P_0(\vec{k},0)$ in the limit $k\to 0$. Putting $\omega=0$ in (3.15), we find after doing the summation over spin,

$$P_0(\vec{k},0) = -2\sum_{\vec{q}_1,\vec{q}_2} v(\vec{q}_1-\vec{q}_2)\frac{1}{\omega(\vec{q}_1,\vec{k})}\times\left(\frac{1}{\omega(\vec{q}_2,\vec{k})}-\frac{1}{\omega(\vec{q}_1,\vec{k})}\right)\times(n_{\vec{q}_1}-n_{\vec{k}+\vec{q}_1})(n_{\vec{q}_2}-n_{\vec{k}+\vec{q}_2})/\chi_0(\vec{k},0), \tag{4.3}$$

where $\chi_0(\vec{k},0)$ is given by

$$\chi_0(\vec{k},0) = -\frac{mk_F}{2\pi^2}\left[1+\frac{k_F}{k}\left(1-\frac{k^2}{4k_F^2}\right)\ln\left|\frac{k+2k_F}{k-2k_F}\right|\right]. \tag{4.4}$$

In order to extract the small-k behavior of $P_0(\vec{k},0)$, we shall expand the θ functions in (4.3) for small k. The detailed calculations are shown in Appendix B. Using this result, we have

$$\lim_{\vec{k}\to 0} G(\vec{k},0) = \gamma k^2/k_F^2, \tag{4.5}$$

where $\gamma=\frac{1}{4}$. Following (4.1) the dielectric function in the limit $k\to 0$ and $\omega=0$ is, therefore, written as

$$\lim_{\vec{k}\to 0}\epsilon(\vec{k},0) = 1+\frac{(k_{FT}/k)^2}{1-\gamma(k_{FT}/k_F)^2}. \tag{4.6}$$

Comparing (4.6) with (4.2), we find

$$(\mathcal{K}_0/\mathcal{K}) = 1-\gamma(k_{FT}/k_F)^2, \tag{4.7}$$

with $\gamma=\frac{1}{4}$. This value of γ gives compressibilities which are in close agreement with the values tabulated by Rice[22] following a calculation of the second derivative of the ground-state energy for the electron gas. This also agrees very well with the value obtained by Hedin and Lundqvist.[23] The fact that our calculation leads to $\gamma=\frac{1}{4}$ suggests that the small-k behavior of our dielectric function is well represented, that is, the exchange effects have been well taken care of for small k. It may be mentioned that Toigo and Woodruff have also obtained this value of γ from $k\to 0$ limit of their $G(k)$.

B. Asymptotic limit

To determine the value of $G(\vec{k},0)$ in the limit $k\to\infty$ we have to look into the large-k behavior of $P_0(\vec{k},0)$ and $Q_0(\vec{k},0)$. By suitable change of variables, the expression for $P_0(\vec{k},0)$, as given in (4.3), can be written in the form

$$P_0(\vec{k},0)=-\frac{64\pi e^2m^2}{\chi_0(\vec{k},0)}\sum_{\vec{q}_1,\vec{q}_2}\left[\left(\frac{1}{|\vec{q}_1+\vec{q}_2+\vec{k}|^2}-\frac{1}{|\vec{q}_1-\vec{q}_2|^2}\right)\frac{1}{(k^2+2\vec{q}_1\cdot\vec{k})^2}+\left(\frac{1}{|\vec{q}_1+\vec{q}_2+\vec{k}|^2}+\frac{1}{|\vec{q}_1-\vec{q}_2|^2}\right)\frac{1}{(k^2+2\vec{q}_1\cdot\vec{k})(k^2+2\vec{q}_2\cdot\vec{k})}\right]n_{\vec{q}_1}n_{\vec{q}_2}\,. \tag{4.8}$$

Expanding the integrand in (4.8) for large k and using the value of $\chi_0(\vec{k},0)=-4mk_F^3/3\pi^2k^2$ as $k\to\infty$, we have

$$P_0(\vec{k},0)=\frac{96\pi^3e^2m}{(2\pi)^6k^4k_F^3}\iint d\vec{q}_1\,d\vec{q}_2\left(1+\frac{2}{k^2}\,\frac{(\vec{q}_1\cdot\vec{k})(\vec{q}_2\cdot\vec{k})-(\vec{q}_1\cdot\vec{k})^2}{|\vec{q}_1-\vec{q}_2|^2}\right)n_{\vec{q}_1}n_{\vec{q}_2}\quad\text{as } k\to\infty\,, \tag{4.9}$$

where the summation over $\vec{q}_1$ in (4.8) has been replaced by the integration $[1/(2\pi)^3]\int d\vec{q}_1$, and so on. To carry out the integrations in (4.9) we choose the coordinate system such that $\vec{q}_1$ is taken along the z axis. Let θ_1 and θ_2 be the angles between the vectors $(\vec{k},\vec{q}_1)$ and $(\vec{q}_1,\vec{q}_2)$, respectively. If Θ be the angle between $(\vec{k},\vec{q}_2)$, then we have

$$\cos\Theta=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi\,, \tag{4.10}$$

where ϕ is the azimuthal angle for the planes $[\vec{q}_2,\vec{q}_1]$ and $[\vec{q}_1,\vec{k}]$.

After the ϕ integration is done, (4.9) reduces to the form

$$P_0(\vec{k},0)=\frac{3\alpha r_s}{2\pi^3k^4}\left[(\tfrac{4}{3}\pi)^2+8\pi^2\int_0^1dq_1\int_0^1dq_2\int_{-1}^1dx_1\int_{-1}^1dx_2\,q_1^2q_2^2\left(\frac{q_1q_2x_1^2x_2-q_1^2x_1^2}{q_1^2+q_2^2-2q_1q_2x_1x_2}\right)\right]\quad\text{as } k\to\infty\,, \tag{4.11}$$

where x_1 and x_2 represent cosines of θ_1 and θ_2, respectively, and $\alpha=(4/9\pi)^{1/3}$. In writing (4.11) we have expressed all momenta in the unit of k_F. The integrations over x_1 and x_2 in (4.11) can be done trivially. We then have

$$P_0(\vec{k},0)=\frac{12\alpha r_s}{\pi k^4}\left[\frac{2}{9}-\frac{2}{3}\int_0^1dq_1\int_0^1dq_2\,q_1q_2\left(q_1q_2+\frac{q_1^2-q_2^2}{2}\ln\left|\frac{q_1+q_2}{q_1-q_2}\right|\right)\right]\quad\text{as } k\to\infty\,. \tag{4.12}$$

In (4.12) the integral involving the logarithmic term is zero, because the integrand is odd with respect to the interchange of the variables q_1 and q_2. The remaining term when integrated gives

$$P_0(\vec{k},0)=16\alpha r_s/9\pi k^4\quad\text{as } k\to\infty\,. \tag{4.13}$$

Following (3.14), the large-k limit for $Q_0(\vec{k},0)$ is given by

$$Q_0(\vec{k},0)=16\alpha r_s/3\pi k^4\quad\text{as } k\to\infty\,. \tag{4.14}$$

From (4.13) and (4.14) the asymptotic limit of $G(k)$ follows as

$$G(\vec{k},0)=\tfrac{1}{3}\quad\text{as } k\to\infty\,. \tag{4.15}$$

The fact that $G(\vec{k})=\frac{1}{3}$ as $k\to\infty$, justifies that the value of our $G(\vec{k})$ in the large-k limit is the HF value. This suggests that our dielectric function takes into account the exchange effects for large-k values. The TW approach does not reproduce the large-k limit of the exchange correction to the static dielectric function. This has been discussed in detail by Geldart, Richard, and Resolt.[18] The theories of STLS, SSTL, and Vashishta and Singwi have also this deficiency.

C. General method of calculation of $G(\vec{k},0)$

Here we discuss the general method for calculating $G(\vec{k},0)$ numerically, for various values of k. For this, we need the function $Q_0(\vec{k},0)$, where

$$Q_0(\vec{k},0)=\frac{2mk_Fe^2}{\pi k^2}\left[1+\frac{k_F}{k}\left(1-\frac{k^2}{4k_F^2}\right)\ln\left|\frac{k+2k_F}{k-2k_F}\right|\right]. \tag{4.16}$$

In order to proceed to evaluate $P_0(\vec{k},0)$ from (4.3), we break up the integral into two parts by writing

$$(n_{\vec{q}}-n_{\vec{k}+\vec{q}_1})(n_{\vec{q}_2}-n_{\vec{k}+\vec{q}_2})=(n_{\vec{q}_1}-n_{\vec{k}+\vec{q}_1})n_{\vec{q}_2}-(n_{\vec{q}_1}-n_{\vec{k}+\vec{q}_1})n_{\vec{k}+\vec{q}_2}. \tag{4.17}$$

Using (4.17) in (4.3) if we now make the change of variables $\vec{q}_1+\vec{k}\to-\vec{q}_1$ and $\vec{q}_2+\vec{k}\to-\vec{q}_2$, in the second term, we have

$$P_0(\vec{k},0)=\frac{64\pi e^2m^2}{\chi_0(\vec{k},0)}\sum_{\vec{q}_1,\vec{q}_2}\frac{1}{|\vec{q}_1-\vec{q}_2|^2}\left(\frac{1}{(k^2+2\vec{q}_1\cdot\vec{k})^2}-\frac{1}{(k^2+2\vec{q}_1\cdot\vec{k})(k^2+2\vec{q}_2\cdot\vec{k})}\right)(n_{\vec{q}_1}-n_{\vec{k}+\vec{q}_1})n_{\vec{q}_2}. \tag{4.18}$$

Further introducing the transformation $\vec{q}_1\to\vec{q}_1-\frac{1}{2}\vec{k}$ and $\vec{q}_2\to\vec{q}_2-\frac{1}{2}\vec{k}$, we write (4.17) as

$$P_0(\vec{k},0)=\frac{16\pi e^2m^2}{\chi_0(\vec{k},0)}\sum_{\vec{q}_1,\vec{q}_2}\frac{1}{|\vec{q}_1-\vec{q}_2|^2}\frac{(\vec{q}_2-\vec{q}_1)\cdot\vec{k}}{(\vec{q}_1\cdot\vec{k})^2(\vec{q}_2\cdot\vec{k})}(n_{\vec{q}_1-\vec{k}/2}n_{\vec{q}_2-\vec{k}/2}-n_{\vec{q}_1+\vec{k}/2}n_{\vec{q}_2-\vec{k}/2}). \tag{4.19}$$

In the second integral in (4.19) we change $\vec{q}_1\to-\vec{q}_1$, and write

$$P_0(\vec{k},0)=-\frac{16\pi e^2m^2}{(2\pi)^6\chi_0(\vec{k},0)}[I^++I^-], \tag{4.20}$$

where the summations over $\vec{q}_1$ and $\vec{q}_2$ in (4.19) have been replaced by corresponding integrations. The symbol $I^\pm$ in (4.20) denotes

$$I^\pm=\iint d\vec{q}_1d\vec{q}_2\frac{(\vec{q}_1\pm\vec{q}_2)\cdot\vec{k}}{|\vec{q}_1\pm\vec{q}_2|^2}\frac{1}{(\vec{q}_1\cdot\vec{k})^2(\vec{q}_2\cdot\vec{k})}n_{\vec{q}_1-\vec{k}/2}n_{\vec{q}_2-\vec{k}/2}. \tag{4.21}$$

At this stage we express all the momentum variables in units of k_F. With this we find

$$P_0(\vec{k},0)=[\alpha r_s/2\pi^3F(k)][I^++I^-], \tag{4.22}$$

where

$$F(k)=1+\frac{1}{k}(1-\tfrac{1}{4}k^2)\ln\left|\frac{k+2}{k-2}\right|. \tag{4.23}$$

Following (4.16) we have

$$Q_0(\vec{k},0)=\frac{2\alpha r_s}{\pi k^2}F(k). \tag{4.24}$$

Using the results of (4.22) and (4.24) the explicit form of $G(\vec{k},0)$ can now be written as

$$G(\vec{k},0)=\frac{k^2}{4\pi^2}\frac{I^++I^-}{[F(k)]^2}. \tag{4.25}$$

Evaluation of $I^\pm$ is done by choosing a coordinate system in which the vector $\vec{k}$ is taken along the z axis. If the coordinates of the vectors $\vec{q}_1$ and $\vec{q}_2$ are denoted by (q_1,θ_1,ϕ_1) and (q_2,θ_2), respectively, and Θ is the angle between the vectors $(\vec{q}_1,\vec{q}_2)$, then $\cos\Theta=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi_1$. With this choice of coordinates, the ϕ_1 integration is done analytically, and one gets

$$I^\pm=\frac{4\pi^2}{k^2}\int_0^\infty dq_1\int_0^\infty dq_2\int_{-1}^1dx_1\int_{-1}^1dx_2\frac{q_1}{x_2}\left(\frac{q_2}{x_1}\pm\frac{q_1}{x_2}\right)\frac{1}{(a_1x_1^2\mp b_1x_1+c_1)^{1/2}}n_{\vec{q}_1-\vec{k}/2}n_{\vec{q}_2-\vec{k}/2}, \tag{4.26}$$

where x_1 and x_2 are the cosines of angles θ_1 and θ_2, respectively, and

$$a_1=4q_1^2q_2^2,\quad b_1=-4q_1q_2x_2(q_1^2+q_2^2),\quad c_1=4q_1^2q_2^2x_2^2+(q_1^2-q_2^2)^2. \tag{4.27}$$

To perform the x_1 and x_2 integrations we have to analyze for the angular restrictions imposed by the Θ functions associated with $I^\pm$. From the fact that

$$n_{\vec{q}_1-\vec{k}/2}=\begin{cases}1 & \text{for } |\vec{q}_1-\frac{1}{2}\vec{k}|<1,\\ 0 & \text{otherwise},\end{cases}$$

we find

$$\left.\begin{array}{ll}\text{for} & 0<q_1<1-\tfrac{1}{2}k \\ & -1<x_1<1\,, \\ \text{for} & 1-\tfrac{1}{2}k<q_1<1+\tfrac{1}{2}k \\ & \gamma_1<x_1<1\,,\end{array}\right\} k\leqslant 2\,,$$

and

$$\left.\begin{array}{ll}\text{for} & \tfrac{1}{2}k-1<q_1<\tfrac{1}{2}k+1 \\ & \gamma_1<x_1<1\,,\end{array}\right\} k\geqslant 2\,, \tag{4.28}$$

where $\gamma_1=(q_1^2+\frac{1}{4}k^2-1)/q_1k$. From $n_{\vec{q}_2-\vec{k}/2}$, a similar set of restrictions follow. With the help of (4.28) we write

$$I=I^++I^-=\frac{4\pi^2}{k^2}\int_{1-k/2}^{1+k/2}dq_2\int_{1-k/2}^{1+k/2}dq_1\int_{\gamma_2}^{1}dx_2\int_{\gamma_1}^{1}dx_1\,F(q_2,q_1,x_2,x_1)\quad\text{for } k\leqslant 2 \tag{4.29}$$

and

$$I=\frac{4\pi^2}{k^2}\int_{k/2-1}^{k/2+1}dq_2\int_{k/2-1}^{k/2+1}dq_1\int_{\gamma_2}^{1}dx_2\int_{\gamma_1}^{1}dx_1\,F(q_2,q_1,x_2,x_1)\quad\text{for } k\geqslant 2\,, \tag{4.30}$$

where

$$F(q_2,q_1,x_2,x_1)=\frac{q_1}{x_2}\left(\frac{q_2}{x_1}+\frac{q_1}{x_2}\right)\frac{1}{(a_1x_1^2-b_1x_1+c_1)^{1/2}}+\frac{q_2}{x_2}\left(\frac{q_2}{x_1}-\frac{q_1}{x_2}\right)\frac{1}{(a_1x_1^2+b_1x_1+c_1)^{1/2}}\,, \tag{4.31}$$

and

$$\gamma_2=(q_2^2+\tfrac{1}{4}k^2-1)/q_2k\,.$$

As it can be seen from (4.21), the integral in both I^+ and I^- contains singularities which are due to the energy denominators and the bare Coulomb interactions. While performing the integrations over x_1 in (4.29) and (4.30), one will have to take care of these singularities. This has been done here by grouping the integrand in the fashion as shown in (4.31). After doing the x_1 integration in (4.29) and (4.30), we obtain

$$I=\frac{4\pi^2}{k^2}\int_{1-k/2}^{1+k/2}dq_2\int_{1-k/2}^{1+k/2}dq_1\int_{\gamma_2}^{1}dx_2\,R(q_2,q_1,x_2)\quad\text{for } k\leqslant 2 \tag{4.32}$$

and

$$I=\frac{4\pi^2}{k^2}\int_{k/2-1}^{k/2+1}dq_2\int_{k/2-1}^{k/2+1}dq_1\int_{\gamma_2}^{1}dx_2\,R(q_2,q_1,x_2)\quad\text{for } k\geqslant 2\,, \tag{4.33}$$

where

$$\begin{aligned}R(q_2,q_1,x_2)=\frac{q_1q_2}{x_2c_1^{1/2}}&\left(\ln\left|\frac{2[c_1(a_1\gamma_1^2+b_1\gamma_1+c_1)]^{1/2}+2c_1+b_1\gamma_1}{2[c_1(a_1+b_1+c_1)]^{1/2}+2c_1+b_1}\right|\right.\\&\left.+\ln\left|\frac{2[c_1(a_1\gamma_1^2-b_1\gamma_1+c_1)]^{1/2}+2c_1-b_1\gamma_1}{2[c_1(a_1-b_1+c_1)]^{1/2}+2c_1-b_1}\right|-2\ln|\gamma_1|\right)\\&+\frac{q_1^2}{x_2^2a_1^{1/2}}\left(\ln\left|\frac{2[a_1(a_1\gamma_1^2+b_1\gamma_1+c_1)]^{1/2}+2a_1\gamma_1+b_1}{2[a_1(a_1+b_1+c_1)]^{1/2}+2a_1+b_1}\right|\right.\\&\left.-\ln\left|\frac{2[a_1(a_1\gamma_1^2-b_1\gamma_1+c_1)]^{1/2}+2a_1\gamma_1-b_1}{2[a_1(a_1-b_1+c_1)]^{1/2}+2a_1-b_1}\right|\right)\quad\text{for } q_1\neq q_2,\ x_2\neq 1\,,\end{aligned} \tag{4.34a}$$

$$R(q_2,q_1,x_2)=\frac{q_1q_2}{c_1^{1/2}}\left(\ln\left|\frac{b_1\gamma_1+2c_1}{b_1+2c_1}\right|+\ln\left|\frac{b_1\gamma_1-2c_1}{b_1-2c_1}\right|-2\ln|\gamma_1|\right)-\frac{q_1^2}{a_1^{1/2}}\left(\ln\left|\frac{2a_1\gamma_1+b_1}{2a_1+b_1}\right|+\ln\left|\frac{2a_1\gamma_1-b_1}{2a_1-b_1}\right|\right)$$

$$\text{for } q_1\neq q_2\,,\ x_2=1\,, \tag{4.34b}$$

$$R(q_2, q_1, x_2) = (1/x_2^2)\ln|x_2/\gamma_1| \quad \text{for } q_1 = q_2,\ x_2 \neq 1, \tag{4.34c}$$

and

$$R(q_2, q_1, x_2) = -\ln|\gamma_1| \quad \text{for } q_1 = q_2,\ x_2 = 1. \tag{4.34d}$$

With those above conditions in mind one can safely go for numerical computation to evaluate the remaining integrals in (4.32) and (4.33). We have done this by applying the Gaussian quadrature formula for multiple integrals.

V. RESULTS AND DISCUSSION

The integrals in (4.32) and (4.33) have been evaluated numerically and the results obtained for $G(k)$ are given in Table I. Our tabulated result shows that the asymptotic value of $G(k)$ is 0.333 65. This agrees with the values of $G(k)=\frac{1}{3}$, as $k\to\infty$ which has been obtained analytically. As we have maintained this asymptotic value of $G(k)$ in our theory is the Hartree-Fock value of $G(k)$ for large k. The asymptotic value of $G(k)$ obtained by Toigo and Woodruff is $\frac{2}{3}$ and that obtained by Vashishta and Singwi is ≈ 1 for $r_s = 6$. Since our theory satisfies the compressibility sum rule in the small-k limit like that of TW, it means that conserving the frequency moments up to an infinite order does not disturb the small-k behavior of the dielectric function. But it does disturb to a great extent the

TABLE I. Values of $G(k)$ for different values of k.

k	$G(k)$	k	$G(k)$	k	$G(k)$
0.10	0.00248	2.03	1.12736	3.90	0.38898
0.20	0.00995	2.04	1.06708	4.00	0.38566
0.30	0.02255	2.05	1.01855	5.00	0.36454
0.40	0.04048	2.06	0.97803	6.00	0.35427
0.50	0.06407	2.07	0.94335	7.00	0.34842
0.60	0.09372	2.08	0.91312	8.00	0.34476
0.70	0.13003	2.09	0.88639	9.00	0.34230
0.80	0.17371	2.10	0.86250	10.00	0.34058
0.90	0.22572	2.20	0.70973	11.00	0.33931
1.00	0.28727	2.30	0.62802	12.00	0.33836
1.10	0.35994	2.40	0.57537	13.00	0.33762
1.20	0.44576	2.50	0.53814	14.00	0.33704
1.30	0.54745	2.60	0.51025	15.00	0.33657
1.40	0.66862	2.70	0.48852	16.00	0.33619
1.50	0.81420	2.80	0.47109	17.00	0.33587
1.60	0.99096	2.90	0.45681	18.00	0.33661
1.70	1.20813	3.00	0.44489	19.00	0.33539
1.80	1.47685	3.10	0.43479	20.00	0.33520
1.90	1.79698	3.20	0.42614	25.00	0.33456
1.95	1.94114	3.30	0.41866	30.00	0.33422
1.97	1.96407	3.40	0.41212	45.00	0.33379
1.99	1.89920	3.50	0.40636	50.00	0.33371
2.00	1.75113	3.60	0.40126	90.00	0.33365
2.01	1.32512	3.70	0.39672		
2.02	1.20682	3.80	0.39265		

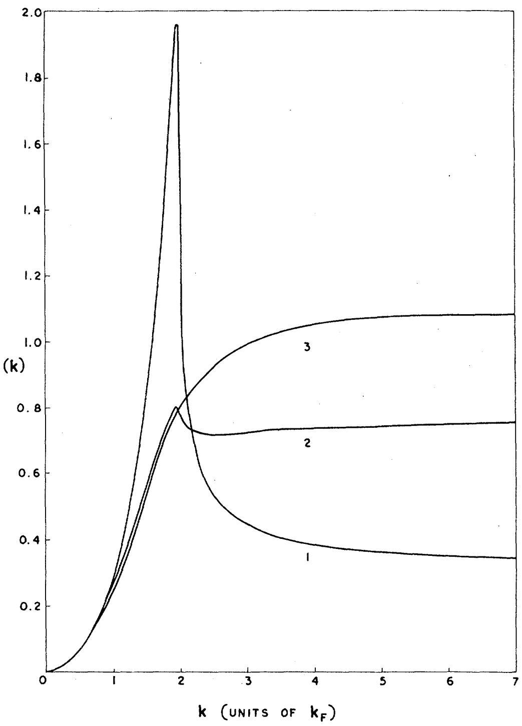

FIG. 1. $G(k)$ vs k/k_F. Curve 1 is the result of the present paper; curve 2 is the TW result; and curve 3 the VS result for $r_s = 6$.

large-k behavior of the dielectric function. This is seen from the fact that our $G(k)$ values for $k\to\infty$ are very much different from that of these authors. This, in other words, means that conserving frequency moments to infinite order makes a quite significant correction to the value of $G(k)$ for large-k values, that is, for $k > 2.3k_F$. Because of this the value of $G(k)$ in our case is reduced gradually and finally goes to $\frac{1}{3}$ at $k\to\infty$. Looking at the $G(k)$ values of TW we find that these values almost remain the same in the region $1.8 \leq k < \infty$. Such stabilization of their $G(k)$ values is most probably due to the lack of the cancellation effects that come from the conservation of frequency moments beyond the first order. We feel that this very argument may be true for the VS theory, because in the VS case the $G(k)$ values go on increasing with k and attain a saturation value of $G(k) \simeq 1.07$ for $r_s = 6$. We further observe that there is a peak in the value of our $G(k)$ around $k = 2k_F$. As it can be seen from Table I, this peak is very sharp in our case. For the sake of comparison, we have given plots of $G(k)$ vs k (Fig. 1) using the results of the present theory, those of the TW theory and of the theory of Vashishta and Singwi for $r_s = 6$.

It is gratifying to note that an entirely different analysis based on a different set of principles should yield the same result for the dielectric function as that of the linearized vertex function analysis of the same problem. Geldart and Taylor[19] calculated a perturbative solution to this integral equation approach. Recently Rau and Rajagopal[24] have developed a different but equivalent formulation of the problem not only to set up the integral equations including terms beyond the lowest-order exchange processes, but also provide a method of obtaining variational estimates of the irreducible polarizability as well as the spin susceptibility. This method yields certain integrals which are similar in structure to the perturbation answers of Geldart and Taylor. To deduce these higher-order terms in the present moment conserving scheme, one will have to include the interaction part of the Hamiltonian in calculating the moments. In principle, the exchange processes should involve a screened Coulomb potential (a result which follows naturally in the integral equation approach[13,24]) which again probably will involve extending our moment conserving techniques in some important ways.

In conclusion, we like to point out that our dielectric function is a good one both in the low and high momentum regions, whereas the TW dielectric function is good only for small k. From the results of the calculation of the pair correlation function $g(r)$ made by Toigo and Woodruff[15] in the range of metallic densities we get the feeling that with our $G(k)$ values we will get physically acceptable results for $g(r)$ in the entire density range. By this, we will succeed in the eliminating the unphysical features of the earlier theories. This will be reported in our forthcoming paper.

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APPENDIX A: DETERMINATION OF $\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},t)$ AND $\mathcal{G}(\vec{k},t)$

Using (3.7), we write $\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},t)$ as

$$\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k},t)=i\theta(t)\sum_{\vec{k}',\,\vec{s}\sigma,\,\vec{q}_2\sigma_2} v(\vec{k}')\left(\langle\|[\mathfrak{A}(0),\mathfrak{B}(0)]\|\rangle+it\langle\|[[H,\mathfrak{A}(0)],\mathfrak{B}(0)]\|\rangle+\frac{(it)^2}{2!}\langle\|[[H,[H,\mathfrak{A}(0)]],\mathfrak{B}(0)]\|\rangle+\cdots\right), \tag{A1}$$

where

$$H=H_0+H_I\,,$$

$$\mathfrak{A}(0)=a^\dagger_{\vec{q}_1,\sigma_1}(0)a^\dagger_{\vec{s}+\vec{k}',\sigma}(0)a_{\vec{s},\sigma}(0)a_{\vec{q}_1+\vec{k}+\vec{k}',\sigma_1}(0)-a^\dagger_{\vec{q}_1-\vec{k}',\sigma_1}(0)a^\dagger_{\vec{s}+\vec{k}',\sigma}(0)a_{\vec{s},\sigma}a_{\vec{q}_1+\vec{k},\sigma_1}(0)\,, \tag{A2}$$

and

$$\mathfrak{B}(0)=a^\dagger_{\vec{k}+\vec{q}_2,\sigma_2}(0)a_{\vec{q}_2,\sigma_2}(0)\,. \tag{A3}$$

Since we will take into account the potential energy effects to first order, we will replace all the H in (A1) by H_0. Considering the first term in (A1) and carrying out all the commutations involved therein, we now apply Wick's theorem to evaluate the product of four operators as follows:

$$\langle a_1^\dagger a_2^\dagger a_3 a_4\rangle=\langle a_1^\dagger a_4\rangle\langle a_2^\dagger a_3\rangle-\langle a_1^\dagger a_3\rangle\langle a_2^\dagger a_4\rangle\,. \tag{A4}$$

When this is done, we find

$$\sum_{\vec{k}',\vec{s}\sigma,\vec{q}_2\sigma_2} v(\vec{k}')\langle\|[\mathfrak{A}(0),\mathfrak{B}(0)]\|\rangle=\sum_{\vec{q}_2\sigma_2} S(\vec{q}_1\sigma_1,\vec{q}_2\sigma_2,\vec{k})\,, \tag{A5}$$

where

$$S(\vec{q}_1\sigma_1,\vec{q}_2\sigma_2,\vec{k})=(n_{\vec{k}+\vec{q}_2,\sigma_2}-n_{\vec{q}_2,\sigma_2})\Bigg(\sum_{\vec{k}'} v(\vec{k}')(n_{\vec{q}_1+\vec{k}-\vec{k}',\sigma_1}-n_{\vec{q}_1+\vec{k}',\sigma_1})\delta_{\sigma_1,\sigma_2}\delta_{\vec{q}_1,\vec{q}_2}$$
$$+v(\vec{k})(n_{\vec{q}_1+\vec{k},\sigma_1}-n_{\vec{q}_1,\sigma_1})+v(\vec{q}_1-\vec{q}_2)(n_{\vec{q}_1,\sigma_1}-n_{\vec{q}_1+\vec{k},\sigma_1})\delta_{\sigma_1,\sigma_2}\Bigg)\,. \tag{A6}$$

Considering the second term in (A1) with H replaced by H_0, we find

$$[H_0, \mathfrak{A}(0)] = (\mathcal{E}_{\vec{q}_1} + \mathcal{E}_{\vec{k}'+\vec{s}} - \mathcal{E}_{\vec{s}} - \mathcal{E}_{\vec{q}_1+\vec{k}+\vec{k}'})(a^\dagger_{\vec{q}_1,\sigma_1} a^\dagger_{\vec{s}+\vec{k}',\sigma} a_{\vec{s},\sigma} a_{\vec{q}_1+\vec{k}+\vec{k}',\sigma_1})$$
$$- (\mathcal{E}_{\vec{q}_1-\vec{k}'} + \mathcal{E}_{\vec{k}'+\vec{s}} - \mathcal{E}_{\vec{s}} - \mathcal{E}_{\vec{q}_1+\vec{k}})(a^\dagger_{\vec{q}_1-\vec{k}',\sigma_1} a^\dagger_{\vec{s}+\vec{k},\sigma} a_{\vec{s},\sigma} a_{\vec{q}_1+\vec{k},\sigma_1}) . \quad \text{(A7)}$$

It is seen from (A7) that the commutator $[H_0, \mathfrak{A}(0)]$ gives two terms which involve some multiplication factor times the operators that are contained in $\mathfrak{A}(0)$. Taking the commutation of (A7) with $\mathfrak{B}(0)$, with the help of (A4), we get

$$\sum_{\vec{k}',\vec{s}\sigma,\vec{q}_2\sigma_2} v(\vec{k}')\langle|[[H_0, \mathfrak{A}(0)], \mathfrak{B}(0)]|\rangle = -\sum_{\vec{q}_2\sigma_2} \omega(\vec{q}_2, \vec{k}) S(\vec{q}_1\sigma_1, \vec{q}_2\sigma_2, \vec{k}) . \quad \text{(A8)}$$

Considering the third term in (A1) it can be similarly shown that

$$\sum_{\vec{k}',\vec{s}\sigma,\vec{q}_2\sigma_2} v(\vec{k}')\langle|[[H_0, [H_0, \mathfrak{A}(0)]], \mathfrak{B}(0)]|\rangle = \sum_{\vec{q}_2\sigma_2} [\omega(\vec{q}_2, \vec{k})]^2 S(\vec{q}_1\sigma_1, \vec{q}_2\sigma_2, \vec{k}) . \quad \text{(A9)}$$

Continuing in this way one can evaluate the commutator associated with the factor $(it)^n$. It is found that these terms form a geometric series, which when summed, gives rise to (3.8).

Following (2.11) one can write using (2.5),

$$\mathcal{G}(\vec{k}, t) = i\theta(t) \sum_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2} \langle|[a^\dagger_{\vec{q}_1\sigma_1}(t) a_{\vec{k}+\vec{q}_1,\sigma_1}(t), a^\dagger_{\vec{k}+\vec{q}_2,\sigma_2}(0) a_{\vec{q}_2,\sigma_2}(0)]|\rangle . \quad \text{(A10)}$$

According to (3.7) this becomes

$$\mathcal{G}(\vec{k}, t) = i\theta(t)\Bigg(\sum_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2} \Big\{\langle|[a^\dagger_{\vec{q}_1\sigma_1}(0) a_{\vec{k}+\vec{q}_1,\sigma_1}(0), a^\dagger_{\vec{k}+\vec{q}_2,\sigma_2}(0) a_{\vec{q}_2,\sigma_2}(0)]|\rangle$$
$$+ it\langle|[[H, a^\dagger_{\vec{q}_1,\sigma_1}(0) a_{\vec{k}+\vec{q}_1,\sigma_1}(0)], a^\dagger_{\vec{k}+\vec{q}_2,\sigma_2}(0) a_{\vec{q}_2,\sigma_2}(0)]|\rangle$$
$$+ \frac{(it)^2}{2!}\langle|[[H, [H, a^\dagger_{\vec{q}_1,\sigma_1}(0) a_{\vec{k}+\vec{q}_1,\sigma_1}(0)]], a^\dagger_{\vec{k}+\vec{q}_2,\sigma_2}(0) a_{\vec{q}_2,\sigma_2}(0)]|\rangle + \cdots\Big\}\Bigg) , \quad \text{(A11)}$$

where in this case one finds

$$\sum_{\vec{q}_1\sigma_1} [H_I, a^\dagger_{\vec{q}_1,\sigma_1}(0) a_{\vec{k}+\vec{q}_1,\sigma_1}(0)] = 0 . \quad \text{(A12)}$$

Following in the manner that we have used in evaluating $\mathcal{F}^{(1)}_{\vec{q}_1\sigma_1}(\vec{k}, t)$, we find

$$\sum_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2} \langle|[a^\dagger_{\vec{q}_1,\sigma_1}(0) a^\dagger_{\vec{k}+\vec{q}_1,\sigma_1}(0), a_{\vec{k}+\vec{q}_2,\sigma_2}(0) a_{\vec{q}_2,\sigma_2}(0)]|\rangle = \sum_{\vec{q}_2,\sigma_2} (n_{\vec{q}_2,\sigma_2} - n_{\vec{k}+\vec{q}_2,\sigma_2}) ,$$

$$\sum_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2} \langle|[[H, a^\dagger_{\vec{q}_1,\sigma_1}(0) a_{\vec{k}+\vec{q}_1,\sigma_1}(0)], a^\dagger_{\vec{k}+\vec{q}_2,\sigma_2}(0) a_{\vec{q}_2,\sigma_2}(0)]|\rangle = -\sum_{\vec{q}_2\sigma_2} w(\vec{q}_2, \vec{k})(n_{\vec{q}_2,\sigma_2} - n_{\vec{k}+\vec{q}_2,\sigma_2}) ,$$

$$\sum_{\vec{q}_1\sigma_1,\vec{q}_2\sigma_2} \langle|[[H, [H, a^\dagger_{\vec{q}_1,\sigma_1}(0) a_{\vec{k}+\vec{q}_1,\sigma_1}(0)]], a^\dagger_{\vec{k}+\vec{q}_2,\sigma_2}(0) a_{\vec{q}_2,\sigma_2}(0)]|\rangle = \sum_{\vec{q}_2,\sigma_2} [w(\vec{q}_2, \vec{k})]^2 (n_{\vec{q}_2,\sigma_2} - n_{\vec{k}+\vec{q}_2,\sigma_2}) , \quad \text{(A13)}$$

and so on.

As it has happened in the previous case, all the terms in (A11) form a geometric series. On summing this series, we get the expression for $\mathcal{Y}(\vec{k}, t)$ to be of the form given by (3.9).

APPENDIX B: CALCULATION OF $G(k)/k^2$ IN THE LIMIT $k \to 0$

To find the small-k behavior of $P_0(\vec{k}, 0)$ we expand the θ functions in (4.3) for small values of k. That is, we write

$$n_{\vec{q}_1+\vec{k}} \simeq n_{\vec{q}_1} - \frac{\vec{q}_1 \cdot \vec{k}}{|\vec{q}_1|} \delta(k_F - |\vec{q}_1|) . \quad \text{(B1)}$$

Similarly we expand $n_{\vec{q}_2+\vec{k}}$. Now expanding $1/w(\vec{q}_1, \vec{k})$ in (4.3) for small k, we have

$$\frac{1}{\omega(\vec{q}_1, \vec{k})} = \frac{m}{\vec{q}_1 \cdot \vec{k}} \left(1 - \frac{k^2}{2\vec{q}_1 \cdot \vec{k}} + \frac{k^4}{4(\vec{q}_1 \cdot \vec{k})^2} - \cdots\right) . \quad \text{(B2)}$$

From (4.4) the value of $\chi_0(\vec{k}, 0)$ for $k \to 0$, is obtained as

$$\lim_{\vec{k}\to 0} \chi_0(\vec{k}, 0) = -\, m k_F/\pi^2 . \tag{B3}$$

This gives

$$\lim_{\vec{k}\to 0} Q_0(\vec{k}, 0) = 4me^2 k_F/\pi k^2 . \tag{B4}$$

Using (B1), (B2), and (B3), we write

$$\lim_{\vec{k}\to 0} P_0(\vec{k}, 0) = \frac{8\pi^3 e^2 m}{(2\pi)^6 k_F} \times \int\int d\vec{q}_1 d\vec{q}_2 \frac{1}{|\vec{q}_1 - \vec{q}_2|^2} \times \frac{\vec{q}_2\cdot\vec{k}}{|\vec{q}_1|\cdot|\vec{q}_2|}\left(\frac{1}{\vec{q}_2\cdot\vec{k}} - \frac{1}{\vec{q}_1\cdot\vec{k}}\right) \times \delta(k_F - |\vec{q}_1|)\delta(k_F - |\vec{q}_2|). \tag{B5}$$

In evaluating (B5) we choose the same coordinate system as used by us while evaluating $I^{\pm}$. Thus we have

$$\lim_{\vec{k}\to 0} P_0(\vec{k}, 0) = \frac{16\pi^5 e^2 m}{(2\pi)^6 k_F} \int_{-1}^{1} dx_1 \int_{-1}^{1} dx_2 \left(1 - \frac{x_2}{x_1}\right) \frac{1}{|x_1 - x_2|} . \tag{B6}$$

The remaining integrations in (B6) are done without much difficulty, and we get

$$\lim_{\vec{k}\to 0} P_0(k, 0) = e^2 m/\pi k_F . \tag{B7}$$

Using (B4) and (B7), we have

$$\lim_{\vec{k}\to 0} G(\vec{k}, 0) = \lim_{\vec{k}\to 0} \frac{P_0(\vec{k}, 0)}{Q_0(\vec{k}, 0)} = \frac{k^2}{4k_F^2} . \tag{B8}$$

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