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Momentum-shell recursion relations, anisotropic spins, and liquid crystals in $2+\epsilon$ dimensions

David R. Nelson* and Robert A. Pelcovits
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
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We describe in detail how to construct momentum-shell recursion relations for classical fixed-length spins in $d=2+\epsilon$ dimensions. The theory is then applied to anisotropic spin systems and to a model of nematic liquid crystals. We also develop a trajectory-integral formalism, which is used to produce the free energy, magnetization, and susceptibilities of isotropic spin systems to first order in $\epsilon=d-2$.

I. INTRODUCTION

It has recently become possible to study critical phenomena in fixed-length n-component classical spin systems near two dimensions. The original work on this problem by Polyakov[1] and Migdal[2] was generalized and extended to $O(\epsilon^2)$ ($\epsilon=d-2$) by Brézin and Zinn-Justin.[3] Subsequently, we made use of these ideas to study bicritical points and cubic symmetry-breaking fields near $d=2$.[4]

Although Brézin and Zinn-Justin used a field-theoretical approach,[3] we have found it convenient to construct recursion relations using the momentum-shell technique of Wilson and Kogut.[5] Momentum-shell recursion techniques in $2+\epsilon$ dimensions will be described in detail here, in the hope that this approach will be useful to investigators who have used similar methods near $d=4$. We shall illustrate these methods by applying them to anisotropic spin systems,[4] and to a model of nematic liquid crystals in two dimensions. Our conclusion for the liquid-crystal system is that deviations from an isotropic XY model are irrelevant variables at low temperatures, and that the isotropic XY behavior should consequently be accessible to experiment.

With a momentum-shell renormalization group at hand, we shall then use a trajectory integral "matching" formalism to produce closed-form expressions for the thermodynamic functions which characterize isotropic spin systems. A similar technique[6] has been rather useful in determining the thermodynamic functions which characterize complicated multicritical phenomena near $d=4$.[7] Although Brézin and Zinn-Justin have calculated similar quantities near $d=2$,[3] some of our results appear to be new.

We wish to stress at the beginning that the calculations described here are only carried to first order in $\epsilon=d-2$. In Ref. 3, it is explained how to go to $O(\epsilon^2)$ using field-theoretical methods. In practical calculations near dimension *four*, one typically only uses a momentum-shell renormalization group to first order in $4-d$. Once the fixed-point structure of a particular problem is understood, it is straightforward to apply the Feynman graph approach of Wilson[8] to extend the calculations to $O[(4-d)^2]$.[9] We expect that similar considerations will apply in $2+\epsilon$ dimensions.

The organization of this paper is as follows: In Sec. II, we derive recursion relations for isotropic spin systems with $n\geq 2$ components by integrating out the short-wavelength components of the spin fields. In Sec. III,[10] we describe the renormalization of a quadratic symmetry-breaking perturbation, and the resulting bicritical phase diagrams near two dimensions. A nematic liquid crystal in precisely two dimensions is studied in Sec. IV. Deviations from the one coupling-constant approximation are shown to be irrelevant variables at low temperatures, so the critical behavior should be that of the two-dimensional XY model. Finally, in Sec. V, we derive expressions for the free energy, magnetization, and susceptibilities of isotropic spin systems near $d=2$.

II. ISOTROPIC RECURSION RELATIONS

Following Polyakov,[1] Migdal,[2] and Ref. 3, we consider an n-component isotropic spin model (with $n\geq 2$) in $d=2+\epsilon$ dimensions with reduced Hamiltonian

$$\mathcal{H}_I\equiv-\frac{1}{2T}\int d^dx\,[\partial_\mu\vec{s}(\vec{x})]^2 \tag{2.1a}$$

with the restriction

$$|\vec{s}(\vec{x})|^2=1\,. \tag{2.1b}$$

A convenient continuum notation has been used in (2.1), although one actually expects the spins $\vec{s}(\vec{x})$ to populate a lattice. When transforming (2.1) into momentum space, we shall reinstate the lattice by the standard trick[5] of restricting the Fourier integrals to a Brillouin zone of unit radius. In the field-theoretic literature, (2.1) is called the nonlinear σ model, the nonlinearity arising from the constraint (2.1b). We shall follow standard conventions,[3] and denote the $\vec{S}$ field by $\vec{s} \equiv (\sigma, \vec{\pi})$, where $\vec{\pi}$ is an $(n-1)$-component vector, and $\sigma^2 + |\vec{\pi}|^2 = 1$.

Near four dimensions, one is used to doing calculations for (2.1a) with additional rs^2 and us^4 interactions, and in the absence of the constraint (2.1b). However, as Wilson has pointed out,[11] (2.1) is a special case of this familiar Landau-Ginzburg model, provided $r \to -\infty$ and $u \to +\infty$ with the ratio $-r/u$ fixed. Amit and Ma[12] have shown that deviations from this limit are irrelevant variables near $d=2$. Indeed, we expect that the nontrivial fixed point found by Wilson and Fisher[13] near four dimensions migrates toward this limit as $d \to 2$, provided the spins have more than two components.

Consider the partition function associated with (2.1), namely

$$Z = \int \mathfrak{D}\sigma(x) \int \mathfrak{D}\vec{\pi}(x) \prod_x \delta[\sigma^2(x) + |\vec{\pi}(x)|^2 - 1] \exp\left(-\frac{1}{2T} \int d^dx [(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2]\right) . \tag{2.2}$$

Following Ref. 3, we integrate out the σ field, taking into account contributions coming from the product of δ-function constraints:

$$Z = \int \mathfrak{D}\pi \exp\left[\int d^dx \left(-\frac{1}{2T} [\partial_\mu (1-\pi^2)^{1/2}]^2 - \frac{1}{2T} (\partial_\mu \vec{\pi})^2 - \tfrac{1}{2}\rho \ln(1-\pi^2) \right) \right] . \tag{2.3}$$

The final term $-\frac{1}{2}\rho \ln(1-\pi^2)$ results from integrating over the δ functions, where $\rho = N/V$ is just the number of degrees of freedom per unit volume associated with our spherical Brillouin zone:

$$\rho = \frac{1}{(2\pi)^d} \int_{\text{zone}} d^dq = \frac{S_d}{(2\pi)^d} \int_0^1 q^{d-1} dq = \frac{S_d}{d(2\pi)^d} . \tag{2.4}$$

The factor S_d is the surface area of a d-dimensional sphere. The density of degrees of freedom ρ is written as the integral of unity over the Brillouin zone to suggest how it can be incorporated into a momentum-shell recursion scheme.

A standard way of obtaining results from the Hamiltonian (2.1) at low temperatures is spin-wave theory. However, it is well known that serious infrared divergences invalidate such a direct approach in precisely two dimensions. The crucial observation of Polyakov[1] was that a spin-wave approach can nevertheless be used to construct *recursion relations*, which are well behaved even in $d=2$. With spin-wave ideas in mind, we take σ along the direction of mean magnetization, and assume fluctuations about this direction are small. If necessary, a small uniform magnetic field can be imposed to insure that this direction is well defined. Nonlinearities such as $(1-\pi^2)^{1/2}$ and $\ln(1-\pi^2)$ in (2.3) are then expanded in π^2 to produce a systematic low-temperature theory. The range of integration of the π variables is extended to $\pm\infty$ in such an approach.

Carrying out this expansion in (2.3), we obtain

$$\bar{\mathcal{H}}_I = -\frac{1}{2T} \int d^dx [(\partial_\mu \vec{\pi})^2 + (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 + \vec{\pi}^2 (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 - T\rho\vec{\pi}^2 - \tfrac{1}{2} T\rho\vec{\pi}^4 + \cdots] . \tag{2.5}$$

After transforming (2.5) into momentum space, we are ready to construct recursion relations in the standard way,[5] treating all terms except the first in perturbation theory. The meaning of the various vertices entering such an expansion is summarized in Fig. 1. The importance of a particular Feynman graph at low temperatures can be determined by noting that each propagator carries a factor of T.

Following Wilson and Kogut,[5] we decompose the Fourier-transformed spin field $\vec{\pi}(\vec{q})$,

$$\vec{\pi}(\vec{q}) \equiv \begin{cases} \vec{\pi}_<(\vec{q}), & 0 < |\vec{q}| < e^{-l} \\ \vec{\pi}_>(\vec{q}), & e^{-l} < |\vec{q}| < 1 , \end{cases} \tag{2.6}$$

and integrate out $\vec{\pi}_>(\vec{q})$. Polyakov[1] proceeded in a similar fashion, using an otherwise rather different approach. Upon rescaling momenta by $e^l \equiv b$ and the spins $\vec{\pi}_<(\vec{q})$ by ζ, we obtain a Hamiltonian of the form (2.5) with new temperature prefactor multiplying $(\partial_\mu \pi)^2$, namely,

$$T' = \zeta^{-2} b^{d+2} [T - (1/2\pi) T^2 \ln b] . \tag{2.7}$$

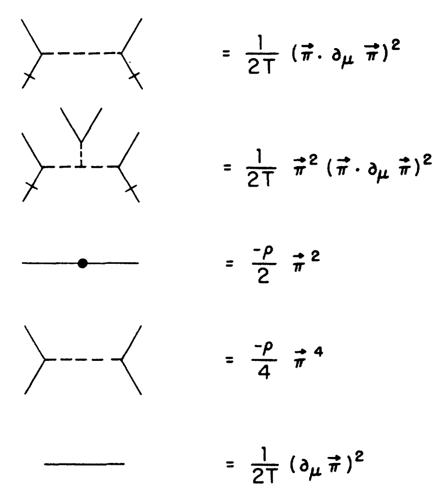

FIG. 1. Vertices entering the perturbation expansion of the partition function (2.3). Slashes on lines indicate derivatives. Dashed lines separate pairs of spins with common indices.

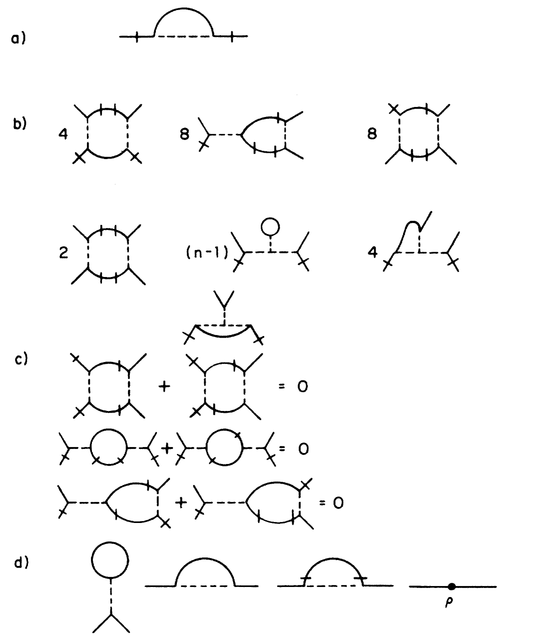

FIG. 2. (a) Graph contributing to the recursion relation (2.7) for the two-point function $(\partial_\mu \vec{\pi})^2$. (b) Graphs which renormalize the four-point interaction $(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2$. (c) Pairs of graphs which cancel identically. (d) Graphs relevant to the recursion relation (2.11) for the magnetic field. The first two graphs are derived from a four-point self-interaction due to the magnetic field rather than from the four-point coupling shown in Fig. 1.

The single graph contributing to this recursion relation is shown in Fig. 2(a).

To determine the spin rescaling factor ζ, we consider the renormalization of the four-point interaction $(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2$ which also carries a temperature prefactor. The relevant Feynman graphs are displayed in Fig. 2(b), while pairs of graphs one might expect to contribute but which actually cancel identically are shown in Fig. 2(c). The temperature-recursion relation obtained in this way is

$$T' = \zeta^{-4} b^{3d+2}[T - (n/2\pi)T^2 \ln b] \,. \tag{2.8}$$

By requiring that (2.8) and (2.7) agree, we determine ζ,

$$\zeta = b^d[1 - (1/4\pi)(n-1)T\ln b] \,. \tag{2.9}$$

A less tedious route to this result is obtained by adding a magnetic field term to (2.1):

$$\begin{aligned}\overline{\mathcal{H}}_I \to \overline{\mathcal{H}} &= \overline{\mathcal{H}}_I + \frac{h}{T}\int d^d x\, \sigma(x)\\ &= \overline{\mathcal{H}}_I + \frac{h}{T}\int d^d x (1-\pi^2)^{1/2}\\ &= \overline{\mathcal{H}}_I + \frac{h}{T}\int d^d x (1 - \tfrac{1}{2}\pi^2 - \tfrac{1}{8}\pi^4 + \cdots) \,.\end{aligned} \tag{2.10}$$

This term is readily incorporated into the above analysis by including the $\frac{1}{2}h\vec{\pi}^2/T$ part of (2.10) in the propagator, and treating $\frac{1}{8}h\vec{\pi}^4/T$ as a perturbation. The very simple graphs which renormalize h/T are shown in Fig. 2(d) and lead to the result

$$\left(\frac{h'}{T'}\right) = \zeta^2 b^{-d}\left(\frac{h}{T} + \frac{1}{4\pi}(n-1)\frac{h\ln b}{1+h}\right). \tag{2.11}$$

However, since h/T really represents a magnetic field, it must renormalize trivially under a momentum-shell renormalization group,[14]

$$h'/T' = \zeta h/T \,. \tag{2.12}$$

Equation (2.12) can only be consistent with (2.11) if

$$\zeta = b^d\left(1 - \frac{1}{4\pi}(n-1)\frac{T\ln b}{1+h}\right), \tag{2.13}$$

which is a generalization of (2.9). The analogous generalization of (2.7) to include a finite magnetic field is

$$T' = \zeta^{-2} b^{d+2}\left(T - \frac{1}{2\pi}\frac{T^2 \ln b}{1+h}\right) \,. \tag{2.14}$$

In deriving (2.11), we have been careful to consider the $\frac{1}{2}T\rho\vec{\pi}^2$ part of (2.5), writing

$$\rho = \frac{S_d}{(2\pi)^d}\int_0^{e^{-l}} q^{d-1}dq + \frac{S_d}{(2\pi)^d}\int_{e^{-l}}^1 q^{d-1}dq \,, \tag{2.15}$$

and incorporating the shell integral into the renormalized magnetic field. Feynman graphs have been evaluated in precisely $d=2$ throughout.

With the results (2.11), (2.13), and (2.14) at hand, we can readily derive differential equations for the "dressed" temperature $T(l)$ and magnetic field $h(l)$ by taking the limit $b\to 1$:

$$\frac{dT(l)}{dl} = -\epsilon T(l) + \frac{n-2}{2\pi}\frac{T^2(l)}{1+h(l)} \,, \tag{2.16a}$$

$$\frac{dh(l)}{dl} = 2h(l) + \frac{n-3}{4\pi}\frac{h(l)T(l)}{1+h(l)} \,. \tag{2.16b}$$

These equations are generalizations to finite (positive) magnetic field of the results of Polyakov,[1] and are momentum-shell versions of the lowest-order equations of Brézin and Zinn-Justin.[3] The approach taken here is easily extended to more complicated situations, as we shall illustrate in Secs. III and IV.

III. ANISOTROPIC SPINS

Anisotropic quadratic and cubic perturbations to (2.1) were originally considered in Ref. 4. Brézin, Zinn-Justin, and Le Guillou[15] subsequently calculated the eigenvalues of arbitrary relevant perturbations near two dimensions to $O(\epsilon^2)$. In this section, we present a recursion relation treatment of quadratic symmetry breaking, described by the reduced Hamiltonian

$$\bar{\mathcal{H}} = -\frac{1}{2T}\int d^dx[(\partial_\mu\vec{\pi})^2 + (\partial_\mu\sigma)^2 + g\sigma^2] . \qquad (3.1)$$

Near *four* dimensions, such a model is believed to describe spin-flopping uniaxial antiferromagnets The parameter g is then a function of the applied magnetic field along the direction of uniaxial symmetry. For g positive, (3.1) is expected to display $(n-1)$-component isotropic critical behavior, while an Ising transition should occur for g negative. As g is adjusted to zero, the $(n-1)$-Heisenberg and Ising critical lines should meet at a bicritical point,[16] as shown in Fig. 3(a).

Unfortunately, the simple description (3.1) of spin-flopping antiferromagnets is only expected to hold provided higher-order perturbations of, say, cubic or hexagonal symmetry can be neglected. Although this is almost certainly the case for most spin systems in three dimensions,[17] we have found[4] that cubic perturbations, for example, are strongly *relevant* perturbations near $d=2$. Consequently, we expect that bicritical phenomena near (and including) $d=2$ will actually be controlled by a fixed point with the discrete symmetry of the underlying lattice. Such fixed points are not easily analyzed by the techniques of Sec. II.

The Hamiltonian (3.1) is, nevertheless, of considerably theoretical interest in its own right in

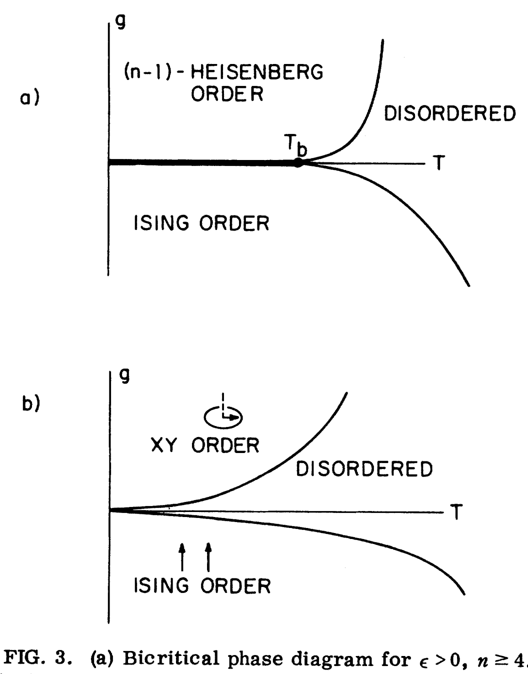

FIG. 3. (a) Bicritical phase diagram for $\epsilon > 0$, $n \geq 4$. The bicritical point is located at $T = T_b$, $g = 0$. (b) Bicritical phase diagram for $d=2$, $n=3$. Note the unusually sharp cusp at the bicritical point, $T=g=0$, in contrast to (a).

precisely $d=2$. Furthermore, ϵ-expansion results for (3.1) above dimension two should have some validity when evaluated at $\epsilon=1$, where higher-order perturbations are indeed irrelevant. In contrast to previous work[4,15] near $d=2$, we shall study the complete crossover from a fixed point of $O(n)$ symmetry to one describing the $O(n-1)$-symmetric critical line [see Fig. 4(a)].

Momentum-shell recursion relations will be derived for (3.1) with *g positive*, so we expect the spontaneous magnetization to lie in the plane perpendicular to the σ direction. Similar results are easily produced for $g<0$, as well as for other symmetry-breaking perturbations. Regarding (3.1) as a phenomenological description of a real crystal, we would expect an additional term like $\bar{g}(\partial_\mu\sigma)^2$. This term was shown to be an irrelevant variable in Ref. 4, so we shall set it equal to zero from the start.

Suppose the magnetization is along π_1 direction. It is then natural to expand the reduced Hamiltonian (3.1) in powers of σ and π_i, $i=2,\ldots,n-1$, using the δ-function constraint to eliminate the π_1 field. In this way, one readily obtains the Hamiltonian

$$\bar{\mathcal{H}} = -\frac{1}{2T}\int d^dx\,[(\partial_\mu\vec{S})^2 + gS_1^2 + (\vec{S}\cdot\partial_\mu\vec{S})^2 + \cdots] , \qquad (3.2)$$

where $\vec{S}$ is an $(n-1)$-component vector defined by

$$\vec{S} \equiv (\sigma, \pi_i) , \; i=2,\ldots,n-1 . \qquad (3.3)$$

Of course, we must be careful to include a term proportional to $\ln(1-S^2)$ in (3.2), similar to the one appearing in Eq. (2.3).

Considering now the renormalization of $(\partial_\mu\pi_i)^2$, $(\partial_\mu S_1)^2, S_1^2$, and $(\pi_i\,\partial_\mu\pi_i)^2$, respectively, in (3.2), a straightforward application of the methods of Sec. II gives us the following recursion equations:

$$T' = \zeta_{\pi_i}^{-2} b^{d+2}[(T - (1/2\pi)T^2\ln b] , \qquad (3.4a)$$

$$T' = \zeta_{S_1}^{-2} b^{d+2}\left(T - \frac{1}{2\pi}\frac{T^2}{1+g}\ln b\right) , \qquad (3.4b)$$

$$\frac{g'}{T'} = \zeta_{S_1}^{2} b^{-d}\left(\frac{g}{T} - \frac{1}{2\pi}\frac{g}{1+g}\ln b\right), \qquad (3.4c)$$

$$T' = \zeta_{\pi_i}^{-4} b^{3d+2}\left(T - \frac{1}{2\pi}\frac{T^2[(n-1)g+n]}{1+g}\ln b\right) . \qquad (3.4d)$$

We have introduced separate spin rescalings ζ_{π_i} and ζ_{S_1} for S_1 and $\pi_i (i=2,\ldots,n-2)$ spin fields. These spin rescalings are fixed by demanding that (3.4a), (3.4b), and (3.4d) lead to identical recursion relations. These, in turn, produce our final results for the renormalization of partially dressed couplings $T(l)$ and $g(l)$, namely,

$$\frac{dT}{dl} = -\epsilon T + \frac{1}{2\pi}\left(T^2 \frac{[(n-3)(1+g)+1]}{1+g}\right), \qquad (3.5)$$

$$\frac{dg}{dl} = 2g - \frac{1}{\pi}\frac{Tg}{1+g}. \qquad (3.6)$$

An *arbitrary* symmetry-breaking interaction can, in principle, couple to an infinite spectrum of relevant eigenoperators in $2+\epsilon$ dimensions.[4,15] This does not happen for quadratic symmetry-breaking fields: Brézin *et al.* have shown[15] that the eigenoperators associated with the symmetric model are just Gegenbauer polynomials in σ, $C_l^{n/2-1}(\sigma)$. The quadratic perturbation considered here corresponds to $C_l^{n/2-1}(\sigma) \sim \sigma^2 - 1/n$; the constant term proportional to $1/n$ has been surpressed in Eq. (3.1). Since we have considered an eigenperturbation about the symmetric model, no additional relevant operators are generated by the renormalization procedure to linear order in g about the isotropic $g=0$ behavior. It is straightforward to check that the *nonlinear* description of crossover implied by (3.5) and (3.6) is correct to first order in ϵ.

The flows induced by (3.5) and (3.6) in $2+\epsilon$ dimensions are shown in Fig. 4(a) for $n \geq 4$. Two trivial zero-temperature fixed points appear at $g=0$ and $g=\infty$. The $O(n)$ symmetric fixed point discovered by Polyakov[1] and Migdal[2] appears at

$$g^* = 0, \quad T^* = T_c = 2\pi\epsilon/(n-2), \qquad (3.7)$$

while a fixed point with $O(n-1)$ symmetry is located at

$$g^* = \infty, \quad T^* = T_c = 2\pi\epsilon/(n-3). \qquad (3.8)$$

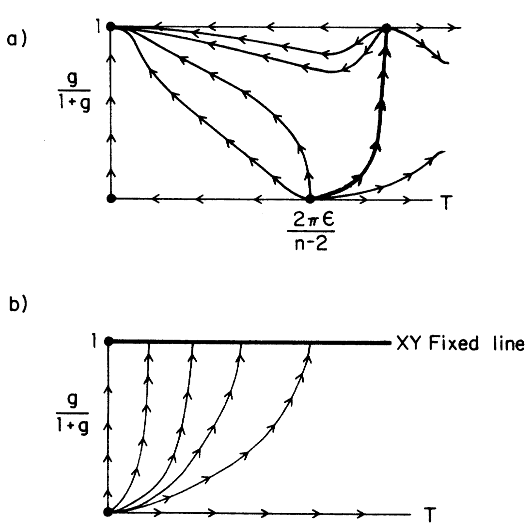

FIG. 4. (a) Renormalization-group flows induced by (3.5) and (3.6) for $d \gtrsim 2$. The arrows indicate the direction of the flow of the effective Hamiltonian under iteration. (b) Renormalization-group flows for $d=2$, $n=3$.

The fixed point (3.7) should describe the critical behavior of (3.1) for $g=0$, while (3.8) controls the critical properties for all $g>0$. The bold line connecting these two fixed points corresponds to the $O(n-1)$-symmetric phase boundary shown in Fig. 3(a).

The crossover exponent entering a scaling description of the fixed point (3.7) is given in terms of the eigenvalue of g,

$$\lambda_g = 2 - \frac{2\epsilon}{n-2} + O(\epsilon^2). \qquad (3.9)$$

The actual crossover exponent is[4]

$$\phi^{-1} = \frac{1}{\nu\lambda_g} = \tfrac{1}{2}\epsilon + \frac{\epsilon^2}{n-2} + O(\epsilon^3), \qquad (3.10)$$

where we have made use of an expression for the correlation-length critical exponent ν derived in Ref. 3. The exponent (3.10) should enter scaling expressions for thermodynamic functions such as the susceptibility,

$$\chi(t,g) = t^{-\gamma}\,\Phi(g/t^{\phi}), \qquad (3.11)$$

where[1-3]

$$\gamma^{-1} = \tfrac{1}{2}\epsilon + O(\epsilon^2), \quad t = \left(T - \frac{2\pi\epsilon}{n-2}\right)\Big/\frac{2\pi\epsilon}{n-2}. \qquad (3.12)$$

A more general homogeneity expression can be derived by techniques developed in Sec. V, namely,[4]

$$\chi(T,g) = \exp\left((2+\epsilon)l - \frac{1}{2\pi}(n-1)\int_0^l T(l')\,dl'\right) \times \chi[T(l),g(l)], \qquad (3.13)$$

where $T(l)$ and $g(l)$ are the solutions of (3.5) and (3.6). This expression is useful in deriving a prediction for the interesting case $\epsilon=0$, $n=3$, specifically[4]

$$\chi(T,g) \sim e^{4\pi/T}\Phi(ge^{4\pi/T}) \qquad (3.14)$$

for small T and g. The corresponding Hamiltonian flows are shown in Fig. 4(b). For $g>0$, all flow lines terminate in a fixed line at $g=\infty$. Although we cannot demonstrate it in the context of this theory, this $XY(n=2)$ line of fixed points is believed to terminate[19,20] at some finite T_c of order unity. The phase diagram corresponding to this picture is shown in Fig. 3(b), where we have also displayed the Ising critical line for $g<0$. According to phenomenological crossover scaling theories, the shape of the Ising and XY bicritical lines in Fig. 3(b) should be given by

$$ge^{4\pi/T} = \text{const.} \qquad (3.15)$$

This conclusion has also been reached by Khokhlachev,[21] and is consistent with Monte Carlo work by Binder and Landau.[22]

IV. NEMATIC LIQUID CRYSTALS

The renormalization-group ideas presented in Sec. II are not limited to magnetic systems, and can, for example, easily be applied to nematic liquid crystals. In an ideal nematic liquid crystal below the nematic → isotropic transition, the molecules are aligned along a preferred axis $\vec{n}$. However, even in equilibrium this direction may vary from point to point. If the variations are small on an intermolecular scale, we may describe the liquid crystal by a continuum elastic theory.[23]

The elastic theory yields the following expression for the free-energy density of a two-dimensional liquid crystal[23]:

$$F(\vec{r}) = \tfrac{1}{2}K_1[\vec{\nabla}\cdot\vec{n}(\vec{r})]^2 + \tfrac{1}{2}K_3[\vec{n}(\vec{r})\times(\vec{\nabla}\times n(\vec{r}))]^2 . \quad (4.1)$$

The K_2 Franck coefficient corresponding to "twist" is absent in two dimensions.

Upon making the substitution

$$\vec{n}(\vec{r}) = \cos\theta(\vec{r})\hat{x} + \sin\theta(\vec{r})\hat{y} , \quad (4.2)$$

we may rewrite (4.1) in two equivalent forms:

$$F(\vec{r}) = \tfrac{1}{2} K_3 \sum_{i=1}^{2} [\partial_i\theta(\vec{r})]^2 + \frac{K_1-K_3}{2} [\sin\theta(\vec{r})\partial_1\theta(\vec{r}) - \cos\theta(\vec{r})\partial_2\theta(\vec{r})]^2 , \quad (4.3a)$$

$$F(\vec{r}) = \tfrac{1}{2} K_1 \sum_{i=1}^{2} [\partial_i\theta(\vec{r})]^2 + \frac{K_3-K_1}{2} [\cos\theta(\vec{r})\partial_1\theta(\vec{r}) - \sin\theta(\vec{r})\partial_2\theta(\vec{r})]^2 , \quad (4.3b)$$

where $\partial_1 \equiv \partial_x$ and $\partial_2 \equiv \partial_y$. Typically, one considers the one-constant approximation, i.e., $K_1 = K_3$ and the two forms, (4.3a) and (4.3b) are equivalent to the ferromagnetic classical XY model. Within this approximation, one may calculate, e.g., the correlations and light scattering in a nematic film floating on the surface of a fluid.[24] Using the renormalization-group methods outlined in Sec. II, we may now study the general case $K_1 \neq K_3$, treating the second terms in (4.3a) and (4.3b) as small perturbations. To insure that we expand about a stable Hamiltonian, we will use (4.3a) when $K_1 > K_3$, and (4.3b) when $K_3 > K_1$.

We consider the low-temperature nematic phase where the fluctuations of the director $\vec{n}(\vec{r})$ about the locally preferred direction are small. For convenience we choose this direction to be along $\hat{x}$ and thus $\theta(\vec{r}) \ll 1$. Expanding the reduced Hamiltonian $\bar{H} = (1/k_B T)\int F(\vec{r})\, d^2\vec{r}$ in powers of $\theta(\vec{r})$, we obtain from (4.3a),

$$\bar{H} = -\frac{K_3}{2k_BT}\int d^2\vec{r}\,\{[\partial_1\theta(\vec{r})]^2 + (1+\Delta)[\partial_2\theta(\vec{r})]^2 + \Delta\theta^2(\vec{r})[\partial_1\theta(\vec{r})]^2 - \Delta\theta^2(\vec{r})[\partial_2\theta(\vec{r})]^2 - \tfrac{1}{3}\Delta\theta^4(\vec{r})[\partial_1\theta(\vec{r})]^2$$
$$+ \tfrac{1}{3}\Delta\theta^4(r)[\partial_2\theta(\vec{r})]^2 - 2\Delta\theta(\vec{r})[\partial_1\theta(\vec{r})][\partial_2\theta(\vec{r})] + \tfrac{4}{3}\Delta\theta^3(\vec{r})[\partial_1\theta(\vec{r})][\partial_2\theta(\vec{r})] + O(\theta^6)\} , \quad (4.4)$$

where $\Delta = (K_1 - K_3)/K_3$. A similar expression can be obtained from (4.3b) for the case $K_3 > K_1$.

By considering the renormalization of the $[\partial_1\theta(\vec{r})]^2$, $[\partial_2\theta(\vec{r})]^2$, and $\theta(\vec{r})[\partial_1\theta(\vec{r})][\partial_2\theta(\vec{r})]$ terms, respectively, we obtain the following equations:

$$t' = \zeta^{-2} b^{+4}\left(t - \Delta\,\frac{\ln b}{2\pi}\,\frac{t^2}{(1+\Delta)^{1/2}}\right) , \quad (4.5)$$

$$\frac{1+\Delta'}{t'} = \zeta^2 b^{-4}\left(\frac{1+\Delta}{t} - \Delta\,\frac{\ln b}{2\pi}\,\frac{1}{(1+\Delta)^{1/2}}\right) , \quad (4.6)$$

$$\frac{\Delta'}{t'} = \zeta^3 b^{-6}\left(\frac{\Delta}{t} - \frac{\ln b}{2\pi}\,\frac{2\Delta}{(1+\Delta)^{1/2}}\right) , \quad (4.7)$$

where $t = k_BT/K_3$.

After determining ζ, as usual, from a self-consistency requirement, we find recursion relations for t and Δ, namely,

$$\left.\begin{aligned} \frac{dt}{dl} &= -\frac{\Delta t^2}{2\pi}\,\frac{1}{(1+\Delta)^{1/2}} && (4.8a)\\ \frac{d\Delta}{dl} &= -\frac{\Delta t}{2_\pi}\,\frac{\Delta+2}{(1+\Delta)^{1/2}} && (4.8b)\end{aligned}\right\}\; K_1 > K_3 , \quad \Delta > 0 .$$

For the case $K_3 > K_1$ we obtain

$$\left.\begin{aligned} \frac{d\bar{t}}{dl} &= \frac{\bar{\Delta}\bar{t}^2}{2\pi}\,\frac{1}{(1-\Delta)^{1/2}} && (4.9a)\\ \frac{d\bar{\Delta}}{dl} &= \frac{\bar{\Delta}\bar{t}}{2\pi}\,\frac{(-2+\Delta)}{(1-\Delta)^{1/2}} && (4.9b)\end{aligned}\right\}\; K_3 > K_1 , \quad \bar{\Delta} < 0 ,$$

where $\bar{t} = k_BT/K_1$ and $\bar{\Delta} = (K_1 - K_3)/K_1 < 0$.

From the recursion relations (4.8b) and (4.9b) for Δ and $\bar{\Delta}$ we see that $K_1 - K_3$ is an irrelevant parameter. Thus, at low temperatures, the one-constant approximation is correct in describing the long-range or critical properties of a nematic liquid crystal (e.g., the long-wavelength scattering intensity), even when $K_3 \neq K_1$.

The irrelevancy of the parameter $K_1 - K_3$ is indicated graphically in Fig. 5, where the renormalization-group trajectories are plotted. If we begin with a system where $K_1 \neq K_3$, successive iterations will transform the initial system to one exhibiting the fixed-line behavior of an isotropic XY model.

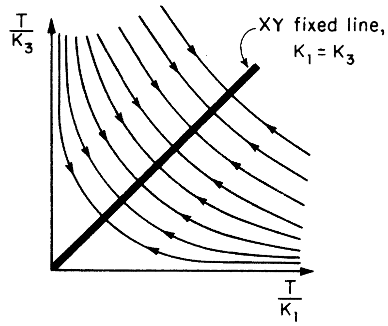

FIG. 5. Renormalization-group flows predicted by (4.8). These flows terminate in a fixed line characterized by $K_1 = K_3$.

V. THERMODYNAMIC FUNCTIONS NEAR TWO DIMENSIONS

Momentum-shell recursion relations, such as those derived in Sec. II, lead straightforwardly to the susceptibility, free energy, and magnetization of fixed-length spin systems in $2+\epsilon$ dimensions. We shall use a trajectory integral "matching" formalism,[6,7] which has been rather useful in calculations near $d=4$.

Hamiltonian flows in temperature and magnetic field generated by Eqs. (2.16) are shown in Fig. 6. Direct calculations of quantities such as the magnetization, susceptibility, etc., are not difficult at low temperatures provided the magnetic field is sufficiently large. The propagator entering a graphical perturbation series in T is

$$G_0(h,T,q) = T/(q^2+h)\,. \tag{5.1}$$

The magnetic field h is a "mass" which provides an infrared cutoff for the Feynman graphs.

Of course, infrared logarithms in h spoil such a direct expansion in the interesting region near the coexistence curve in Fig. 6. We shall circumvent these small h difficulties by intergrating the re-

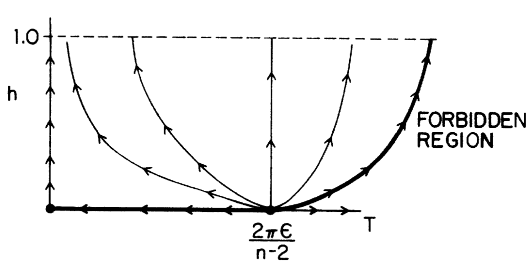

FIG. 6. Renormalization-group flows induced by (2.16). The coexistence curve is the flow line connecting the nontrivial fixed point $h=0$, $T=2\pi\epsilon/(n-2)$ and the zero-temperature fixed point. The bold trajectory terminated at $h(l)=1$ marks the border of a forbidden region which is not accessible to explicit calculation.

cursion relations (2.16) out of the troublesome coexistence-curve region until $h(l)$ is of order unity. At this point, any thermodynamic function can be calculated by ordinary perturbation theory. Because renormalization theory relates thermodynamic functions calculated with an effective Hamiltonian $\bar{\mathcal{H}}(l)$ to those calculated with the initial Hamiltonian (see below), we can use such calculations to produce quantities of interest close to the coexistence curve.

The limitations of such an approach in $2+\epsilon$ dimensions can be seen by solving the system (2.16). This is readily done using techniques described in Ref. 6(b). To solve (2.16a), we can neglect $h(l)$ entirely and obtain immediately,

$$T(l) = Te^{-\epsilon l}\Big/\left(1+\frac{(n-2)T}{2\pi\epsilon}\,(e^{-\epsilon l}-1)\right). \tag{5.2}$$

The solution of (2.16b) is only slightly more complicated, and is given in implicit form by

$$h(l) = \frac{he^{2l}}{\{1+[(n-2)T/2\pi\epsilon]\,(e^{-\epsilon l}-1)\}^{(n-3)/2(n-2)}} - (1/8\pi)(n-3)T(l)h(l)\ln[1+h(l)]\,. \tag{5.3}$$

The solutions (5.2) and (5.3) are correct to leading order in $T(l)$ and ϵ, provided $T(l)=O(\epsilon)$, and $h(l) \le O(1)$. Our plan is to integrate (5.3) until $l=l^*$ such that

$$h(l^*)=1\,. \tag{5.4}$$

However, in order that a perturbation theory in temperature be possible at this point, we require that $T(l^*)$ remain of order ϵ. Requiring for concreteness that $T(l^*)$ be less than twice T_c $[T(l^*)<4\pi\epsilon/(n-2)]$, we discover a forbidden region in Fig. 6 where we cannot do calculations. In particular, the zero-field high-temperature phase is entirely inaccessible.

Consider first the transverse susceptibility,

$$\chi_\perp \equiv \int d^dx \langle\vec{\pi}(0)\cdot\vec{\pi}(\vec{x})\rangle_{\bar{\mathcal{H}}}\,, \tag{5.5}$$

where the average is evaluated in the ensemble specified by (2.10). The transverse susceptibility after one iteration of the momentum-shell renormalization group of Sec. II is easily shown to be

$$\chi_\perp' = \zeta^{-2}b^d\chi_\perp\,. \tag{5.6}$$

By repeatedly iterating the differential version of this transformation, we determine $\chi_\perp$ in terms of the susceptibility $\chi_\perp(l)$ associated with the "dressed" Hamiltonian $\bar{H}(l)$,

$$\chi_\perp = \exp\left(dl - \frac{1}{2\pi}\,(n-1)\int_0^l \frac{T(l')\,dl'}{1+h(l')}\right)\chi_\perp(l)\,. \tag{5.7}$$

The graphs entering a direct calculation of $\chi_\perp(l)$ are just those of Fig. 2(d). Evaluating these graphs, we obtain

$$\chi_\perp(l)=\frac{h(l)}{T(l)}+\frac{1}{8\pi}(n-1)h(l)\ln[1+h(l)]-\frac{1}{8\pi}(n-1)h(l)\ln h(l)\,. \tag{5.8}$$

The exponential prefactor in (5.7) is readily determined using the methods of Ref. 6. In particular,

$$\frac{1}{2\pi}(n-1)\int_0^l \frac{T(l')\,dl'}{1+h(l')} \approx \frac{n-1}{n-2}\ln\left(1+\frac{(n-2)T}{2\pi\epsilon}(e^{-\epsilon l}-1)\right)+(1/8\pi)(n-1)T(l)\ln[1+h(l)]\,. \tag{5.9}$$

On combining (5.9), (5.8), and (5.7) (with $l=l^*$) and using the recursion-relation solutions (5.2) and (5.3) (5.3), we obtain, to leading order in ϵ and $T(l^*)$,

$$\chi_\perp=\frac{T}{h}\left(1+\frac{(n-2)T}{2\pi\epsilon}(e^{-\epsilon l^*}-1)\right)^{(n-1)/2(n-2)}\times[1-(1/8\pi)(n-1)T(l^*)\ln h(l^*)]\,. \tag{5.10}$$

After setting $l^*\simeq-\frac{1}{2}\ln h$, this becomes

$$\chi_\perp=\frac{T}{h}\left(1+\frac{(n-2)T}{2\pi\epsilon}(h^{\epsilon/2}-1)\right)^{(n-1)/2(n-2)}\,. \tag{5.11}$$

It is more conventional to absorb the overall factor of T in (5.11) into the definition of $\chi_\perp$.

The evaluation of the free energy is very similar. The "matching" relation analogous to (5.7), which relates the free energy of interest to one evaluated at $l=l^*$ is[25,26]

$$F(T,h)=\int_0^{l^*} e^{-dl}G_0(l)\,dl+e^{-dl^*}F[T(l^*),h(l^*)]\,. \tag{5.12}$$

The kernel $G_0(l)$ of the trajectory integral term in (5.12) arises from the differential contribution to the free energy generated by our momentum-shell renormalization group. For fixed-length spins in $2+\epsilon$ dimensions this is

$$G_0(l)=\frac{1}{4\pi}(n-1)\left(\ln[1+h(l)]-1+\frac{1}{4\pi}(n-1)\frac{T(l)}{1+h(l)}-\ln T(l)\right) \tag{5.13}$$

to lowest order in $T(l)$ and ϵ. In deriving (5.13), we have been careful to take into account the spin rescaling factor (2.13).

The free energy entering the right-hand side of (5.12) is, to lowest order,

$$F(l^*)=\frac{1}{4\pi}(n-1)\int_0^1 q\,dq\{\ln[q^2+h(l^*)]-\ln T(l^*)\}\,. \tag{5.14}$$

After some fairly tedious manipulations on the trajectory integral part of (5.12) [similar to those in Appendix A of Ref. 6(b)], we determine a contribution to the free energy which apparently depends on l^*, namely

$$\frac{1}{4\pi}(n-1)\int_0^{l^*} he^{-\epsilon l}dl\bigg/\left(1+\frac{n-2}{2\pi\epsilon}T(e^{-\epsilon l}-1)\right)^{(n-3)/2(n-2)}-\frac{1}{8\pi}(n-1)e^{-dl^*}h(l^*)\ln h(l^*)\,. \tag{5.15}$$

It is easy to check that (5.15) is, in fact, independent of the precise choice of l^* to lowest order, as we would expect. On setting $h(l^*)\equiv1$, evaluating the integral in (5.15), and surpressing various regular parts of the free energy, we obtain our final result for the singular part:

$$F_s(T,h)=-\frac{h}{T}\left(1+\frac{(n-2)T}{2\pi\epsilon}(h^{\epsilon/2}-1)\right)^{(n-1)/2(n-2)} \tag{5.16}$$

A more conventional definition of the free energy would absorb the overall factor $-T$ into F_s.

The magnetization follows immediatey by differentiating (5.26). Keeping only the leading term, we obtain

$$M(T,h)=-\frac{d}{dh}[TF(T,h)]=\left(1+\frac{(n-2)T}{2\pi\epsilon}(h^{\epsilon/2}-1)\right)^{(n-1)/2(n-2)} \tag{5.17}$$

which agrees with results by Brézin and Zinn-Justin.[3] Of course, Eq. (5.17) could have been deduced immediately from (5.11) by using the general Ward identity[27] $\chi_\perp=TM/h$. Another differentiation with respect to h gives us the longitudinal susceptibility,

$$\chi_\parallel=\frac{T}{8\pi}(n-1)h^{-1+\epsilon/2}\times\left[\left(1+\frac{(n-2)T}{2\pi\epsilon}(h^{\epsilon/2}-1)\right)^{(n-3)/2(n-2)}\right]^{-1} \tag{5.18}$$

It is instructive to evaluate our basic results (5.11), (5.16), (5.17), and (5.18) in various limits. For example, in the limit $\epsilon \to 0$, $n \to 2$, (5.16) becomes

$$F \sim h^{1+T/8\pi} \tag{5.19}$$

in agreement with the results of Berezinskii[28] for the two-dimensional XY model.

Note added in proof. Readers may be interested in other work on critical phenomena in $2+\epsilon$ dimensions. Two recent references are C. De Dominicis, S.-K. Ma, and L. Peliti [Phys. Rev. B 15, 4313 (1977)], who treat the dynamics of time-dependent Ginzburg-Landau models, and J. Sak [Phys. Rev. B 15, 4344 (1977)], who studies long-range interactions with a momentum-shell technique similar to ours.

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