Hydrodynamic theory of spin waves in spin glasses and other systems with noncollinear spin orientations

B. I. Halperin
Department of Physics, Harvard University, † Cambridge, Massachusetts 02138*
and Bell Laboratories, Murray Hill, New Jersey 07974

W. M. Saslow
Department of Physics, Texas A&M University, College Station, Texas 77843
(Received 28 March 1977)

The hydrodynamic theory of spin waves is extended to a magnetic system such as a spin glass or a crystal with helical spin order, in which there are equilibrium magnetizations on different sites, with spin directions that are not collinear. If the total magnetization is zero and if the interactions are assumed to be isotropic in spin space, one predicts well-defined spin waves at small wave vectors $\vec{k}$, with a linear dispersion relation, and *three* polarizations for each value of $\vec{k}$. The hydrodynamic assumption of a finite spin-stiffness constant may be questionable, however, for the spin glass. Nonzero magnetization is discussed briefly.

I. INTRODUCTION

A number of years ago, Halperin and Hohenberg developed a hydrodynamic theory of spin waves for certain simple examples of magnetically ordered systems.[1] The hydrodynamic spectrum was found to agree in form with the spectrum obtained from microscopic theories in the long-wavelength limit. In the Heisenberg ferromagnet, where the order parameter (the magnetization) is itself a constant of the motion, it was found that there is a single propagating spin-wave mode with quadratic dispersion. In the Heisenberg antiferromagnet, where the order parameter (staggered magnetization) is not a constant of the motion, it was found that there are two polarizations of propagating spin waves with linear dispersion. In addition to the results for the spin-wave frequencies, it was predicted that the spin-wave damping is proportional to the square of the frequency in the hydrodynamic regime.[2] Hohenberg and Swift have applied similar reasoning in the case of the single-triplet model.[3]

In the present work, we shall extend the analysis of Ref. 1 to more-complex systems in which the equilibrium orientations of the spins on various sites are noncollinear. Well known examples of this situation are helical spin arrangements, such as the structures observed in the rare-earth metals Tb, Dy, and Ho (Refs. 4 and 5) (See Sec. VI). An example of considerable current interest is the proposed "spin-glass state"[6] for a dilute system of magnetic impurities in a nonmagnetic metallic hose such as Mn in Cu.[7] We shall limit our considerations here to models in which the Hamiltonian is assumed to be *isotropic in spin space* (i.e., invariant under the simultaneous rotation of all spins), and in which the equilibrium state has *zero net magnetization*. (Systems with *nonzero* magnetization will be discussed briefly in Sec. VII.)

As our principal result, we find three polarizations of spin waves, each with a linear spectrum. The spin-wave velocities are inversely proportional to the square root of the magnetic susceptibility, and directly proportional to the square root of the ratio of a spin-stiffness constant ρ_s, which we have *assumed* to be nonzero. The damping predicted by the hydrodynamic theory is proportional to k^2. It must be emphasized, however, that the hydrodynamic theory has some major limitations, as will be discussed in Sec. III.

The prediction of a linear spectrum for the spin glass has also been made by Edwards and Anderson.[8] In their paper, however, they consider a model in which the spins are confined to the x-y plane, which gives them a single polarization of spin wave, analogous to the planar ferromagnet considered in Ref. 1. Also, in the Edwards-Anderson paper, the microscopic "spins" are more like electric dipoles, having a finite moment of inertia, but no intrinsic angular momentum. The excitations are more properly described as "phonons" in that case.

Huber and Ching[9] have obtained a linear spin-wave spectrum at $T=0$ for a Heisenberg spin glass, by using an approximate decoupling procedure in the equation of motion. Their approximation also permits them to estimate the spin-wave velocity for several models. However, their calculations lead them to predict an overdamped, diffusive behavior at long wavelengths for $T\neq 0$, in contrast to the linear propagating mode predicted by our hydrodynamic analysis.

Hydrodynamic analyses have been used, of

course, to discuss the mode structure of many other systems with broken symmetry, including superfluid ^{4}He, liquid crystals, and the various superfluid phases of ^{3}He.[10,11] The three spin-wave modes found in the present paper are related particularly closely to the spin-wave modes of ^{3}He-B, when the dipolar interaction is neglected.[11]

In Sec. II we present a brief overview of the systems of interest, and a heuristic derivation of the three spin-wave modes. A more-detailed discussion of our assumptions for the equilibrium state of the spin-glass system, and a more detailed hydrodynamic derivation are given in Secs. IV and V. Application to helical spin structures is discussed in Sec. VI, and results for the case of nonzero magnetization are given in Sec. VIII. In the Appendix, we present a derivation of the Larmor precession theorem, in the form needed in the text.

II. OVERVIEW AND HEURISTIC DERIVATION

In an ordinary Heisenberg antiferromagnet, where the spins in an equilibrium state G are aligned parallel or antiparallel to a single direction $\hat{x}$, the equilibrium state is invariant under the group O(1) of rotations about the $\hat{x}$ axis. Since the Heisenberg Hamiltonian is invariant under the larger group O(3) of uniform spin rotations, rotation of G by a small angle (θ_y, θ_z) about any axis in the $\hat{y}$-$\hat{z}$ plane, produces a new state G', distinguishible from G, but degenerate in energy. The two polarizations of antiferromagnetic spin waves derived in Ref. 1 were obtained by considering the time evolution of a state in which a rotation angle $\theta_y(\vec{r})$ or $\theta_z(\vec{r})$, varying slowly from point to point, is coupled dynamically to a slowly varying magnetization density $m_y(\vec{r})$ or $m_z(\vec{r})$.

In the present paper we consider systems where the symmetry group O(3) is *completely* broken in the equilibrium state G, and rotation about *any* axis leads to a new state G'. In the nonequilibrium hydrodynamic states we shall keep track of three rotation angles $\theta_\alpha(\vec{r})$, $\alpha = 1, 2, 3$, which will be coupled dynamically to the three components of the magnetization density $m_\alpha(\vec{r})$, to yield *three* polarizations of spin waves. For a nonequilibrium state in which there is a small magnetization density $\vec{m}(\vec{r})$, as well as a slowly varying rotation angle $\vec{\theta}(\vec{r})$, we expect an excess free energy of the form

$$\Delta F[\vec{m}, \vec{\theta}] = \tfrac{1}{2}\sum_{\alpha=1}^{3}\int d^3r\,(m_\alpha^2\chi^{-1} + \rho_s|\vec{\nabla}\theta_\alpha|^2)\,. \quad (2.1)$$

Here we have assumed that the equilibrium state is macroscopically isotropic, so that χ and ρ_s may be treated as scalar constants; in the most general case, χ and ρ_s are tensors in spin space, and ρ_s may be a tensor in real space as well. We shall derive below equations of motion for m_α and θ_α which, in the long-wavelngth limit, have the form

$$\frac{\partial\theta_\alpha(\vec{r})}{\partial t} = \gamma m_\alpha(\vec{r})\chi^{-1} = \gamma\frac{\delta(\Delta F)}{\delta m_\alpha(\vec{r})}\,, \quad (2.2)$$

$$\frac{\partial m_\alpha(\vec{r})}{\partial t} = \gamma\rho_s\nabla^2\theta(\vec{r}) = -\gamma\frac{\delta(\Delta F)}{\delta\theta_\alpha(\vec{r})}\,, \quad (2.3)$$

where $\gamma = g\mu_B/\hbar$ is the gyromagnetic ratio. At a given wave vector $\vec{k}$, these coupled equations have solutions at frequencies

$$\omega = \pm ck\,, \quad (2.4)$$

$$c = \gamma(\rho_s/\chi)^{1/2}\,, \quad (2.5)$$

corresponding to the positive and negative frequency parts of a spin wave with polarization α. (In the more-general case where χ and ρ_s are tensors, spin waves of different polarization may have different velocities, which may also depend on the direction of propagation.) Additional terms in the equations of motion lead to a spin-wave damping $\propto k^2$.

Equations (2.2) and (2.3) may be derived heuristically, in analogy to Anderson's discussion of the dynamics of superfluid helium,[12] by noting that $m_\alpha(\vec{r})$ and $\theta_\alpha(\vec{r})$ are canonically conjugate, in the sense

$$[\theta_\alpha(\vec{r}), m_\beta(\vec{r}')] = i\gamma\hbar\delta_{\alpha\beta}\delta(\vec{r}-\vec{r}') \quad (2.6)$$

[see Eq. (4.11)]. Equation (2.6), in turn, is a consequence of the transformation properties of θ_α under rotations, and the fact that the total spin operator is the infinitesimal generator of rotations in the system.

III. LIMITATIONS OF THE HYDRODYNAMIC THEORY

It is the fundamental assumption of the hydrodynamic theory that the state of a system with small long-wavelength deviations from equilibrium is essentially determined by the long-wavelength variation of the various conserved densities, together with any additional variables necessary to describe the degrees of freedom associated with any continuous broken symmetry of the Hamiltonian. In the present situation the conserved densities are the magentization density $\vec{m}(\vec{r})$ and the energy density $\epsilon(\vec{r})$, while the variables associated with the broken symmetry are $\theta_\alpha(\vec{r})$. The hydrodynamic assumption is valid, in particular, if the frequency of the long-wavelength mode is small compared to the relaxation rates of all microscopic degrees of freedom of the system. In practice, the situation is rarely unambiguous; there is usually a continuum of internal relaxation rates, extending down to very low frequencies, and the important question is the extent of the coupling between the hydrody-

namic mode and any short-wavelength degrees of freedom with comparable relaxation times.

In a periodic magnetic insulator, at low temperatures, a long characteristic time is the collision time τ_c for thermal magnons. The hydrodynamic formula for the damping of spin waves is certainly not valid unless $\omega\tau_c \ll 1$. The hydrodynamic formula for the real part of the spin-wave frequency remains valid for $\omega\tau_c \gg 1$, however, because as $T \to 0$, the number of excitations goes to zero, and it makes little difference to the spin-wave frequency whether the thermal magnons follow the long-wavelength motion or not.

The situation in the spin glass is not clear. The large linear specific heat observed experimentally in spin glasses, raises the possibility of a large density of low-energy excitation, that could couple strongly enough to change the frequency of the spin-wave mode, or even to overdamp the mode entirely. Another threat to the validity of hydrodynamics is the possibility that the spin stiffness constant ρ_s is zero in the spin-glass state. Indeed, these possibilities are suggested in a recent computer simulation, by Walker and Walstedt,[13] of randomly distributed spins, interacting via a Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction, with parameters appropriate to Mn in Cu. Qualitatively, Walker and Walstedt have observed that rather large variations of the relative orientations of various groups of spins can occur with very little cost of energy, which suggests that the spin-wave stiffness may indeed be very small or vanishing in the spin glass. Furthermore, upon solving the linearized equations of motion for small deviations from the equilibrium spin orientations, Walker and Walstedt found a large density of low-lying modes, delocalized in space, and no evidence for a density of modes vanishing as ω^2, which would be expected if the excitations were well-defined spin waves with a linear spectrum. In fact the density of excitations was such that it would account quantitatively for the large linear specific heat observed experimentally in these systems, if the excitations were treated simply as noninteracting bosons. It is interesting to note that in ordinary glasses, the phonon mean free path l is generally found to be linear in the wavelength λ of the phonon (with $l/\lambda \gtrsim 10^2$), for frequencies of the order of $k_B T/\hbar$, and $T \lesssim 1$ K.[14] These anomalously short mean free paths have been attributed to resonant scattering from localized excitations, which are present in sufficient numbers to give a specific heat large compared to that of the phonons at low temperatures. Despite this anomalous damping, the phonon mean free paths in the glass are still large compared to the wavelength, and the phonon frequencies remain very close to those expected from the macroscopic properties of the glass.

An additional limitation of the hydrodynamic theory arises because spin-anisotropy terms must occur in the Hamiltonian of any real system, and spin anisotropy always destroys the validity of the hydrodynamic predictions at sufficiently long wavelengths. The assumption of isotropy tends to be best for the case of S-state ions, in a relatively symmetric crystal environment, and tends to be rather poor in the rare-earth metals, where the helical structures are found.

IV. PROPERTIES OF THE SPIN-GLASS STATE

The models we shall consider are described by a Hamiltonian of the Heisenberg form

$$\mathcal{H}_0 = -\frac{1}{2}\sum_{i,j} J_{ij}\vec{S}_i \cdot \vec{S}_j, \tag{4.1}$$

where the $\vec{S}_i$ are spins on a lattice, which is not necessarily periodic, and the J_{ij} are the coupling constants between the sites. Note that $\mathcal{H}_0$ is invariant under simultaneous rotation of spins, about any axis.

For the spin-glass case, the magnetic sites are assumed to be randomly substituted on the nonmagnetic host lattice, and the J_{ij} are given by the RKKY interaction. Because the sign of J_{ij} is a rapidly oscillating function of the separation between the sites i and j, the coupling J_{ij} is often treated as a random variable, with zero mean. The long-range $1/r^3$ behavior of the RKKY interaction is not believed to play an essential role in the spin-glass state, and the considerations of the present paper should apply equally well to a nearest-neighbor Heisenberg model with couplings of random sign, as to the original RKKY model with random site positions.

As a consequence of the competing nature of the coupling constants between different sites, it is impossible to minimize all the bond energies simultaneously. It appears that the best compromise is not any simple ferromagnetic or antiferromagnetic configuration, but a more complicated arrangement in which the spins on various sites point in various directions of the unit sphere.

For the purposes of the present paper we shall assume a number of properties for the spin-glass state. If the system is cooled to a temperature T below the transition temperature T_c, the system will be found in one of a large number of possible thermodynamic "equilibrium states" which all have similar macroscopic properties, but which would be distinguishable in principle, if one could measure the time-averaged orientation of the spins on individual impurity sites. Let us pick out one of the many possible equilibrium states, and de-

note it by G. The state G is represented by a density matrix ρ_G that is constant in time. The expectation value $\langle \vec{S}_i \rangle_G$ of any given spin in the state G is nonzero; however, the net magnetization

$$\vec{M}_0 \equiv g\mu_B \sum_{i=1}^{n} \vec{S}_i \tag{4.2}$$

is assumed to have zero expectation value in G.

More exactly, let $\vec{m}(\vec{r})$ be the magnetization density in a region R, of volume v, containing many spins, centered at point $\vec{r}$:

$$\vec{m}(r) = v^{-1} g\mu_B \sum_{i \in R} \vec{S}_i \,. \tag{4.3}$$

Then the expectation value $\langle \vec{m}(\vec{r}) \rangle_G$ is assumed to be small, of order $n^{-1/2}$, where n is the number of spins in the volume v. The distribution of expectation values $\langle \vec{S}_i \rangle_G$ is assumed to have the additional property

$$n^{-1} \sum_{i \in R} \langle S_i^{\alpha} \rangle_G \langle S_i^{\beta} \rangle_G = q\delta_{\alpha\beta} + O(n^{-1/2}) \,, \tag{4.4}$$

where α, β denote Cartesian components of the vector $\vec{S}_i$, and q is a positive constant.

The parameter q, which is taken to be the "order parameter" of the spin-glass by Edwards and Anderson,[6] is generally believed to go to zero continuously, as T approaches the spin-glass transition temperature from below. In the present paper, we do not make any particular assumptions with regard to the temperature dependence of q, nor do we utilize the concept of an "order parameter."

We shall next consider the action of uniform rotations on G. Let $U(\vec{\theta})$ be the operator

$$U(\theta) \equiv \exp(i\vec{\theta} \cdot \vec{S}) \,, \tag{4.5a}$$

$$\vec{S} \equiv \sum_i \vec{S}_i = \frac{\vec{M}}{g\mu_B} \,, \tag{4.5b}$$

where θ_α are c numbers.

We may then define a rotated ground state G', whose density matrix is given by

$$\rho_{G'} = U(\vec{\theta})^{-1} \rho_G U(\vec{\theta}) \,. \tag{4.6}$$

It is clear that the free energy of the state G' is identical to that of G.

Let us define a (one-body) operator

$$t_{\alpha\beta}(\vec{r}) = n^{-1} \sum_{i \in R} \langle S_i^{\alpha} \rangle_G S_i^{\beta} \,, \tag{4.7}$$

where R, as before, is a region containing a large number n spins, centered about point $\vec{r}$. For general rotation angles $\vec{\theta}$, $\langle t_{\alpha\beta}(\vec{r}) \rangle_{G'}$ equals $q\Gamma_{\alpha\beta}(\vec{\theta})$, where $\Gamma_{\alpha\beta}$ is the rotation matrix corresponding to $\vec{\theta}$. If the rotation angle $\vec{\theta}$ in (4.6) is small, we will have

$$\langle t_{\alpha\beta}(\vec{r}) \rangle_{G'} = q[\delta_{\alpha\beta} + \epsilon_{\alpha\beta\gamma}\theta_\gamma + O(\theta^2)] \,, \tag{4.8}$$

where $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric unit tensor.

We shall also wish to consider nonequilibrium states in which $\langle t_{\alpha\beta}(\vec{r}) \rangle$ varies slowly as a function of $\vec{r}$. We define an operator $\vec{\theta}(\vec{r})$ for the spin glass, which measures the local rotation angle, as

$$\theta_\gamma(\vec{r}) = (1/2q)\epsilon_{\alpha\beta\gamma}[t_{\alpha\beta}(\vec{r}) - q\delta_{\alpha\beta}] \,. \tag{4.9}$$

{This definition is sensible only when the difference $[\langle t_{\alpha\beta}(\vec{r}) \rangle - q\delta_{\alpha\beta}]$ is small.}

Using the basic commutation relation $[S_j^{\alpha}, S_{j'}^{\beta}] = i\delta_{jj'}\epsilon_{\alpha\beta\gamma}S_j^{\gamma}$, one may readily verify the following general relation:

$$[\vec{m}_\alpha(\vec{r}), t_{\delta\beta}(\vec{r})] = ig\mu_B v^{-1} \epsilon_{\alpha\beta\gamma} t_{\delta\gamma}(\vec{r}) \,. \tag{4.10}$$

From this it follows, for the spin glass, that

$$[\vec{m}_\alpha(\vec{r}), \theta_\beta(\vec{r})] = \tfrac{1}{2} ig\mu_B v^{-1} q^{-1} \times [t_{\alpha\beta}(r) - \delta_{\alpha\beta} t_{\gamma\gamma}(r)] \,. \tag{4.11}$$

For small deviations from equilibrium, we may replace the right-hand side of (4.11) by $-ig\mu_B v^{-1}\delta_{\alpha\beta}$, its expectation value in the state G. [This constitutes a derivation of Eq. (2.6), quoted above.]

It should be emphasized that there may be many other equilibrium states G'', having the same free energy as G, but which cannot be derived from G by a rotation, and for which $\vec{\theta}(\vec{r})$ cannot be reasonably defined. A simple example is the time reverse of G or of any rotated state G', for which $\langle t_{\alpha\beta}(\vec{r}) \rangle_{G''} = -q\Gamma_{\alpha\beta}$. There may be many other equilibrium states in which $\langle t_{\alpha\beta}(\vec{r}) \rangle_{G''} = O(n^{-1/2})$. By hypothesis, however, each of these states is separated from G by a large enough free energy barrier or a large enough distance in phase space, so that a state initially in G will have negligible probability of making a transition to these other states.

Having defined the variable $\vec{\theta}(\vec{r})$, we now turn to the dependence of the free energy on $\vec{\theta}$. If $\vec{\theta}(\vec{r})$ varies in space, the free energy is expected to be higher than in the equilibrium state. If one considers the state of lowest possible free energy, consistent with a given long-wavelength variation in $\vec{\theta}$, it is natural to suppose that the free energy will be raised by an amount proportional to $(\vec{\nabla}\vec{\theta})^2$, or

$$F = F_G + \tfrac{1}{2}\rho_s \int d^3r \, |\vec{\nabla}\vec{\theta}|^2 \,, \tag{4.12}$$

where ρ_s is a finite stiffness constant and

$$|\vec{\nabla}\vec{\theta}|^2 = \sum_\alpha |\vec{\nabla}\theta_\alpha|^2 \,. \tag{4.13}$$

As a guess, one might expect ρ_s to be roughly proportional to q, disappearing at high temperatures when the magnetic order disappears.

It is easy to establish an upper bound to the free energy rise, of form (4.12). Consider a state K,

defined by

$$\rho_K = U^{-1}\rho_G U, \tag{4.14}$$

$$U \equiv \exp\left(i\sum_j \vec{\theta}(\vec{r}_j)\cdot\vec{S}_j\right), \tag{4.15}$$

$$\vec{\theta}(\vec{r}) \equiv \vec{\theta}_0 \cos\vec{k}\cdot\vec{r}, \tag{4.16}$$

where $\vec{\theta}_0$ and $\vec{k}$ are small. The orientation of each spin $\vec{S}_j$ in K is rotated by the angle $\vec{\theta}(\vec{r}_j)$ relative to its orientation in G. The entropy of K is the same as that of G, so that the increase in free energy is given by the difference in energies, $\langle\mathcal{H}_0\rangle_K - \langle\mathcal{H}_0\rangle_G$. This difference is readily computed to give

$$\Delta F = \tfrac{1}{2}V\rho_s^0|\vec{\theta}_0|^2k^2, \tag{4.17}$$

$$\rho_s^0 = \tfrac{1}{18}V^{-1}\sum_{i,j}|\vec{r}_i - \vec{r}_j|^2 J_{ij}\langle\vec{S}_i\cdot\vec{S}_j\rangle_G, \tag{4.18}$$

where V is the total volume of the system.[15] Equation (4.18) gives an upper bound to the stiffness constant ρ_s, provided that the sum on the right-hand side converges, for large separations $|\vec{r}_i - \vec{r}_j|$.

In order to compute the true stiffness constant ρ_s, one must remove the restriction that each spin be rotated by the precise amount $\vec{\theta}_0\cos\vec{k}\cdot\vec{r}_i$. Instead, we should require that

$$\langle\vec{S}_j\rangle - \langle\vec{S}_j\rangle_G = \vec{\theta}_0\times\langle\vec{S}_j\rangle_G\cos\vec{k}\cdot\vec{r}_j + \vec{\tau}_j + O(\theta^2), \tag{4.19}$$

where $\vec{\tau}_j$ may vary rapidly from one site to the next, subject only to the constraints that $\vec{\tau}_j$ be small on every site, and that

$$\sum_j \tau_j^\alpha\langle S_j^\beta\rangle_G\cos\vec{k}\cdot\vec{r}_j = 0 \tag{4.20}$$

for all α, β. [Equation (4.20) may be thought of as a constraint that some suitable weighted averages of $\vec{\tau}_j$ vanish for the system, which is obviously much weaker than the constraint that $\vec{\tau}_j$ vanish on every site.] One must choose the deviations $\vec{\tau}_j$ in such a manner as to minimize the free energy, subject to the above constraints on $\vec{\tau}_j$. In general, $\vec{\tau}_j$ will be of order $k\theta_0$, in the limit $k\to 0$. Nevertheless, the stiffness constant ρ_s may be reduced considerably below the upper bound ρ_s^0. The deviations $\vec{\tau}_j$ permit the gradient of the rotation angle $\theta(\vec{r})$ to be redistributed so that the gradient between strongly coupled spin pairs is reduced while the gradient between weakly coupled pairs is increased. Indeed, we cannot rule out the possibility that $\rho_s = 0$, for the spin-glass state, although we shall assume $\rho_s \neq 0$ below.

Next we consider states where the net magnetization density $\vec{m}(\vec{r})$ varies from one region to another.

For small amplitude disturbances, $\Delta F[\vec{m}]$ is quadratic in m and given by

$$\Delta F[\vec{m}] = \frac{1}{2}\int|\vec{m}|^2\chi^{-1}d^3r, \tag{4.21}$$

where χ is the macroscopic magnetic susceptibility. (As a practical matter, we are concerned here with the "reversible susceptibility," measured with a weak magnetic field, at a frequency sufficiently low that anisotropic perturbations will cause the magnetization to follow the field. This is not necessarily the same as the thermodynamic susceptibility one would obtain by including all states, accessible or not, in the partition function.) When both $\vec{m}$ and $\vec{\nabla}\vec{\theta}$ are present, the free energy increase, to quadratic order, is given by the sum of the contributions

$$\Delta F[\vec{m},\vec{\theta}] = \frac{1}{2}\int[|\vec{m}|^2\chi^{-1} + \rho_s|\vec{\nabla}\vec{\theta}|^2]d^3r, \tag{4.22}$$

which was the result used in Sec. II. Terms proportional to $m_\alpha\vec{\nabla}\theta_\alpha$ are absent in the above, because they would require a preferred direction in real space, contrary to our assumptions about the macroscopic properties of the spin glass. A term proportional to $\vec{m}\cdot\vec{\theta}$ is also impossible; such a term would cause the equilibrium free energy (after minimization with respect to $\vec{M}$) to depend on $\vec{\theta}$, which would violate the rotational invariance of $\mathcal{H}_0$.

V. HYDRODYNAMIC DERIVATION

The remainder of the hydrodynamic derivation is similar in spirit, but different in detail, from that in Ref. 1. We begin with the relation between the differentials of the entropy density s, of the rotation gradient $\vec{\nabla}\theta_\alpha$, and of the conserved quantities ϵ (energy) and m_α, which we write in the form

$$d\epsilon = Tds + \mu_\alpha dm_\alpha + \vec{\phi}_\alpha\cdot d(\vec{\nabla}\theta_\alpha). \tag{5.1}$$

Here T is the local temperature, and (5.1) may be taken as the defining equation for μ_α and $\vec{\phi}_\alpha$. If we assume T constant, and consider only slow spatial variations in θ_α, comparison to (4.22) gives

$$\mu_\alpha = m_\alpha/\chi, \tag{5.2}$$

$$\vec{\phi}_\alpha = \rho_s\vec{\nabla}\theta_\alpha. \tag{5.3}$$

We assume, as before, that there is no applied magnetic field.

The conservation laws may be written in the form

$$\frac{\partial\epsilon}{\partial t} + \vec{\nabla}\cdot\vec{j}^\epsilon = 0, \tag{5.4}$$

$$\frac{\partial m_\alpha}{\partial t} + \vec{\nabla}\cdot\vec{j}^\alpha = 0, \tag{5.5}$$

where $\vec{j}^\epsilon$ is the energy current and $\vec{j}^\alpha$ is the current of m_α. (Both $\vec{j}^\epsilon$ and $\vec{j}^\alpha$ must be determined.) We

also write the Larmor precession theorem (see Appendix) in the form

$$\frac{\partial\theta_\alpha}{\partial t}=\gamma(\mu_\alpha+h'_\alpha), \tag{5.6}$$

where $\gamma=g\mu_B/\hbar$ is the gyromagnetic ratio, and h'_α, which must be determined, vanishes in the uniform state.

Since the total entropy is a nondecreasing function of time, we may define an entropy current $\vec{j}^s$ such that the entropy production

$$R\equiv\frac{\partial s}{\partial t}+\vec{\nabla}\cdot\vec{j}^s\geq 0 \tag{5.7}$$

at all points. Combining Eqs. (5.1)–(5.4) gives

$$TR=-\vec{\nabla}\cdot(\vec{j}^\epsilon-T\vec{j}^s-\mu_\alpha\vec{j}^\alpha+\gamma h'_\alpha\vec{\phi}_\alpha)$$
$$-\vec{j}^s\cdot\vec{\nabla}T-(\vec{j}^\alpha+\gamma\vec{\phi}_\alpha)\cdot\vec{\nabla}\mu_\alpha+\gamma h'_\alpha\vec{\nabla}\cdot\vec{\phi}_\alpha. \tag{5.8}$$

The quantities $\vec{j}^\epsilon$, $\vec{j}^s$, $\vec{j}^\alpha$, and h'_α are unknown, and must be determined. The quantities T, μ_α, and $\vec{\nabla}\cdot\vec{\phi}_\alpha$ will be considered to be the independent variables, which may be arbitrarily specified.

Considerations of symmetry under space and time inversion for the reactive parts of $\vec{j}^s$, $\vec{j}^\alpha$, and h'_α, and the requirement that the total dissipation be ≥ 0, give

$$\vec{j}^{s(R)}=0, \tag{5.9}$$

$$\vec{j}^{\alpha(R)}=-\gamma\vec{\phi}_\alpha, \tag{5.10}$$

$$h'^{(R)}_\alpha=0. \tag{5.11}$$

Similar considerations for the dissipative terms (which must be proportional to the thermodynamic forces $\vec{\nabla}T$, $\vec{\nabla}\mu_\alpha$, and $\vec{\nabla}\cdot\vec{\phi}_\alpha$) give, when isotropy in spin space and cubic symmetry in real space are taken into account,

$$\vec{j}^{s(D)}=-T^{-1}\kappa\vec{\nabla}T, \tag{5.12}$$

$$\vec{j}^{\alpha(D)}=-K\vec{\nabla}\mu_\alpha, \tag{5.13}$$

$$h'^{(D)}_\alpha=\zeta\vec{\nabla}\cdot\vec{\phi}_\alpha. \tag{5.14}$$

Here κ, K, and ζ must be ≥ 0, so that the total dissipation is non-negative. Furthermore, the requirement that $R\geq 0$ at each point imposes

$$\vec{j}^\epsilon=T\vec{j}^s+\mu_\alpha\vec{j}^\alpha-\gamma h'_\alpha\vec{\phi}_\alpha\approx-\kappa\vec{\nabla}T, \tag{5.15}$$

where the last equality is correct to first order in the deviations from equilibrium.

The coupled equations for m_α and θ_α can now be written

$$\frac{\partial\theta_\alpha}{\partial t}=\gamma m_\alpha\chi^{-1}+\gamma\zeta\rho_s\nabla^2\theta_\alpha, \tag{5.16a}$$

$$\frac{\partial m_\alpha}{\partial t}=\gamma\rho_s\nabla^2\theta_\alpha+K\chi^{-1}\nabla^2 m_\alpha. \tag{5.16b}$$

The first terms on the right-hand sides of (5.16) will be recognized as the terms quoted in (2.2) and (2.3) above, which lead to a real spin-wave frequency, proportional to k. The second terms on the right-hand sides of (5.16) yield an imaginary contribution to the spin-wave frequency, of order k^2, so that

$$\omega_k=\pm ck-\tfrac{1}{2}iDk^2, \tag{5.17}$$

$$c=\gamma(\rho_s/\chi)^{1/2}, \tag{5.18}$$

$$D=K\chi^{-1}+\gamma\rho_s\zeta, \tag{5.19}$$

as for the ordinary Heisenberg antiferromagnet.[1]

For completeness, we note that the energy density ϵ does not couple linearly to m_α or θ_α, so that its behavior is described by

$$\frac{\partial\epsilon}{\partial t}=\kappa\nabla^2T=\kappa C^{-1}\nabla^2\epsilon, \tag{5.20}$$

where κ is the thermal conductivity and C is the specific heat per unit volume. Hence energy diffuses, just as for a nonmagnetic material.

Since the structure of these equations is the same as for the usual Heisenberg antiferromagnet, we may immediately write down the hydrodynamic contribution to the correlation functions by analogy to that system[1]

$$C_{m_\alpha m_\alpha}(\vec{k},\omega)=\frac{2\chi k_BT[c^2Dk^4+\chi^{-1}\kappa k^2(\omega^2-c^2k^2)]}{[(\omega-ck)^2+(\frac{1}{2}Dk^2)^2][(\omega+ck)^2+(\frac{1}{2}Dk^2)^2]}, \tag{5.21}$$

$$C_{\epsilon\epsilon}(\vec{k},\omega)=2Ck_BT^2D_Tk^2/[\omega^2+(D_Tk^2)^2], \tag{5.22}$$

where $D_T=\kappa C^{-1}$. Inelastic neutron scattering experiments have cross sections which are proportional to $C_{m_\alpha m_\alpha}(k,\omega)$. In the spin glass, there will be additional *elastic* scattering at all wave vectors, due to the frozen-in magnetic and nonmagnetic disorder.

Huber and Ching[9] have estimated the spin-wave velocity in a spin glass by using a sum rule for the frequency-dependent magnetic susceptibility[15]:

$$\langle\omega_k^2\rangle\equiv\int_{-\infty}^{\infty}\omega\chi''(k,\omega)\,d\omega\Big/\int_{-\infty}^{\infty}\omega^{-1}\chi''(k,\omega)\,d\omega$$
$$=\frac{\gamma^2\rho_s^0k^2}{\chi}, \tag{5.23}$$

where ρ_s^0 is the bare stiffness constant, given by (4.18). In view of the discussion in Sec. IV, we see that (5.23) is an overestimate of c^2k^2, even when the spin-wave spectrum is sharply peaked at $\omega=ck$, in the limit of long wavelengths. The reason is that the high-frequency tails of the spin-wave spectrum, which give a negligible contribution to the integral in the denominator of (5.23) in the limit $k\to 0$, can nevertheless give a contribution to the numerator comparable to that of the spin-wave peaks.

VI. HELICAL SPIN ORDERING

The classic cases of helical spin ordering occur in the rare-earth metals Tb, Dy, and Ho.[4,5] In the helical phases, the spins are aligned ferromagnetically within each hexagonal plane, but the orientation changes from one plant to the next. Anisotropic terms in the Hamiltonian cause the spins to lie perpendicular to the hexagonal axis. If we treat the anistropy energy as very small, then the system may be subjected to a hydrodynamic analysis, very similar to that outlined above for the spin glass. The principal change is that the spin orientations are now hightly anisotropic. We now have, in the ground state G,

$$\langle t_{\alpha\beta}(\vec{r})\rangle_G = q(\delta_{\alpha\beta} - \delta_{\alpha 3}\delta_{\beta 3}), \tag{6.1}$$

where we have chosen the three axis as the hexagonal axis. In the rotated system we have

$$t_{\alpha\beta}(\vec{r}) = \langle t_{\alpha\gamma}(r)\rangle_G[\delta_{\gamma\beta} + \epsilon_{\gamma\beta\delta}\theta_\delta(\vec{r})], \tag{6.2}$$

$$\theta_\gamma(\vec{r}) = (1/2q)(2\epsilon_{\gamma\alpha\beta} - \delta_{\gamma 3}\epsilon_{3\alpha\beta})t_{\alpha\beta}(\vec{r}). \tag{6.3}$$

Note that *three* components of θ_γ may still be defined. The susceptibility χ and the stiffness constant ρ_s are now tensors:

$$\chi_{\alpha\beta} = \chi(\alpha)\delta_{\alpha\beta}, \tag{6.4}$$

$$\rho^{ij}_{s\alpha\beta} = \rho_s(i, \alpha)\delta_{\alpha\beta}\delta_{ij}, \tag{6.5}$$

with

$$\begin{aligned} &\chi(1) = \chi(2) \neq \chi(3), \\ &\rho_s(i, 1) = \rho_s(i, 2) \neq \rho_s(i, 3), \\ &\rho_s(1, \alpha) = \rho_s(2, \alpha) \neq \rho_s(3, \alpha). \end{aligned} \tag{6.6}$$

For any given direction in k space, the three polarizations of spin waves have frequencies

$$\omega = \pm c_\alpha k, \tag{6.7}$$

$$c_\alpha = \gamma\chi(\alpha)^{-1/2}\left(\sum_{i=1}^{3}\frac{k_i^2\rho_s(i, \alpha)}{k^2}\right)^{1/2}. \tag{6.8}$$

Note that $c_1 = c_2 \neq c_3$.

It is interesting to compare these results with the microscopic spin-wave spectrum of the helical system, as calculated by Cooper *et al.*[5] Their results may be written in the form

$$\begin{aligned} \omega_q \sim \{&[\mathcal{J}(\vec{Q}) - \tfrac{1}{2}\mathcal{J}(\vec{q}+\vec{Q}) - \tfrac{1}{2}\mathcal{J}(\vec{q}-\vec{Q})] \\ &\times [\mathcal{J}(\vec{Q}) - \mathcal{J}(\vec{q})]\}^{1/2}, \end{aligned} \tag{6.9}$$

$$\mathcal{J}(\vec{k}) = \sum_i J(\vec{R}_i)\exp(i\vec{k}\cdot\vec{R}_i), \tag{6.10}$$

where $J(\vec{R}_i)$ is the exchange constant for two sites separated by $\vec{R}_i$, $\vec{k} = \pm\vec{Q}$ defines the maximum value of $\mathcal{J}(\vec{k})$, and $\vec{q}$ is the wave vector of the spin wave, in a spiral coordinate system. Note that $\omega_q \sim |\vec{q}|$ as $|\vec{q}| \to 0$, and $\omega_q \sim |\vec{q} \pm \vec{Q}|$ as $|\vec{q} \pm \vec{Q}| \to 0$. When the spin coordinates are transformed back to the laboratory frame, the spin wave at $\vec{q}$ is found to have a component of the magnetization at $k = \vec{q} \pm Q$, as well as at $k = \vec{q}$. Thus the spin waves near $\vec{q} = 0$, $\vec{q} = \vec{Q}$, and $\vec{q} = -\vec{Q}$ may be interpreted as three polarizations of spin waves with wave vector near $k = 0$.

In the rare-earth systems, the anisotropy aligning the spins in the x-y plane is generally much stronger than the anisotropy within the plane. If one takes into account only the larger anisotropy, the microscopic theory predicts a gap in the spin-wave spectra for two of the three polarizations, while the third has a frequency that varies as k, for $k \to 0$. A single spin-wave mode, at $\omega = \pm ck$, is indeed what one obtains from the hydrodynamic theory in this case; the system now has the characteristics of the "planar ferromagnet" discussed in Ref. 1.

VII. SYSTEMS WITH FINITE MAGNETIZATION

For suitable choices of the coupling constants J_{ij}, the Heisenberg Hamiltonian (4.1) can have a ground state which has a noncollinear spin alignment, coexisting with a nonzero average magnetization $\vec{M}_0$.[16]

Applying the hydrodynamic theory in a straightforward manner to the components of $\vec{\theta}(r)$ and $\vec{m}(r)$ parallel to M_0, one finds a linear spin-wave mode identical in form to the case of zero magnetization

$$\omega = (\rho_{s\parallel}/\chi_\parallel)^{1/2}k. \tag{7.1}$$

Here $\rho_{s\parallel}$ is the spin-stiffness constant for rotations about the M_0 axis, and $\chi_\parallel$ is the susceptibility parallel to M_0. The spin-wave modes involving the components of $\vec{m}$ perpendicular to $\vec{M}_0$ will then have a behavior similar to the spin waves in a conventional ferrimagnet[17] or the singlet-triplet model considered in Ref. 3; one branch of the spin-wave spectrum has a frequency given by

$$\omega = \rho_{s\perp}k^2/M_0 \tag{7.2}$$

for small k, while the second branch has a finite frequency Δ at $k = 0$, with $\Delta \sim M_0\chi_\parallel^{-1}$.[18] [Although the finite frequency mode is technically not a hydrodynamic mode, it may be described by a continuum equation of motion in the limit where M_0 is small. The quadratic mode is a true hydrodynamic mode, and (7.2) coincides with the spin-wave formula for the Heisenberg ferromagnet.]

It must be noted, however, that there is a serious difficulty with formula (7.1) at finite temperatures. It is expected that $\chi_\parallel$ is divergent at finite temperatures in three dimensions, for a Heisenberg system with $M_0 \neq 0$, due to the effects of thermally excited long-wavelength spin waves.[19] Because of this singularity, one cannot expand the free energy in powers of $\vec{m}$, $\vec{\theta}$, and their gradi-

ents, and the hydrodynamic assumptions break down. At $T=0$, we expect that $\chi_{\parallel}$ is finite, however, and (7.1) should describe one branch of the excitation spectrum in this case.

In principle, (7.1) should apply at finite temperatures for spatial dimensionality greater than 4, since the susceptibility $\chi_{\parallel}$ will be finite in that case.

ACKNOWLEDGMENTS

We would like to thank P. W. Anderson, R. J. Birgeneau, B. R. Cooper, L. R. Walker, and R. E. Walstedt for helpful conservations. The hospitality of the Aspen Center for Physics, where this work was begun, is gratefully acknowledged.

APPENDIX: LARMOR PRECESSION THEOREM

Let the Hamiltonian be given by

$$\mathcal{H}=\mathcal{H}_0-h_z M_z, \tag{A1}$$

where $\mathcal{H}_0$ is invariant under rotations, and h_z is a magnetic field in the z direction. Assume that at $t=0$, the density matrix for the system may be written

$$\rho=Z^{-1}e^{-\beta W}, \quad Z=\mathrm{Tr}e^{-\beta W}, \quad W\equiv\mathcal{H}_0-\mu_z M_z-D, \tag{A2}$$

where μ_z plays the role of a chemical potential, and D is an infinitesimal symmetry-breaking field, introduced in order to pick out a particular one of the many degenerate equilibrium states, in the broken symmetry situation. A possible choice, designed to pick out state G, is

$$D=\lambda\int d^3r\, t_{\alpha\alpha}(\vec{r}), \tag{A3}$$

where $t_{\alpha\beta}$ is defined by (4.7), and $\lambda\to 0$ as the size of the system becomes large. If G is only a metastable state, so that there exists a state of lower free energy at the given temperature β^{-1}, then D must include an artificial potential barrier to eliminate the unwanted states. (The argument below may still be used in this case, because if G is metastable, the system will not enter the forbidden region of phase space in a finite length of time.)

Let A be any operator. Then by expressing $\mathcal{H}$ in terms of W, D, and M_z, we find, at $t=0$,

$$i\hbar\frac{d}{dt}\langle A\rangle=\langle[A,W]\rangle+\langle[A,D]\rangle+(\mu_z-h_z)\langle[A,M_z]\rangle. \tag{A4}$$

The first term on the right-hand side of (A4) vanishes identically, the second term vanishes because D is infinitesimal, and we are left with the third term. By choosing A as the various components of $\vec{S}_i$, we see that the expectation value of any spin must precess about the z axis at the rate $\gamma(\mu_z-h_z)$. Similarly, using (4.11) we see that, for small θ,

$$\frac{d\theta_z}{dt}=\gamma(\mu_z-h_z). \tag{A5}$$

As expected, $d\theta_z/dt=0$ when $\mu_z=h_z$.

Note added in proof. Professor I. E. Dzyaloshinskiĭ has called our attention to a paper by Andreev and Marchenko[20] which discusses the low-frequency spin-wave spectrum of general *crystalline* magnetic structures, including noncollinear ferrimagnets and antiferromagnets, and which seems to predict additional "anomalous spin-wave modes" under certain circumstances. The derivation of these modes and their relationship to the modes discussed in the present paper are not completely clear to us.

---

*Present address.

†Work at Harvard supported in part by the NSF through the Materials Research Laboratory Program and Grant No. DMR72-02977-A03.

[1] B. I. Halperin and P. C. Hohenberg, Phys. Rev. 188, 898 (1969).

[2] Experimental support for the form of the spin-wave damping has been found in the Heisenberg antiferromagnet $RbMnF_3$ by C. G. Windsor, H. D. Saunderson, and E. Schelder, Phys. Rev. Lett. 37, 855 (1976).

[3] P. C. Hohenberg and J. Swift, J. Phys. C 7, 4009 (1974).

[4] *Magnetic Properties of Rare Earth Metals*, edited by R. J. Elliott (Plenum, New York, 1972).

[5] B. R. Cooper, R. J. Elliott, S. J. Nettel, and H. Suhl, Phys. Rev. 127, 57 (1962).

[6] S. F. Edwards and P. W. Anderson, J. Phys. F 5, 965 (1975); D. J. Thouless, P. W. Anderson, and R. Palmer, Philos. Mag. (to be published); A. B. Harris, T. C. Lubensky, and J.-h. Chen, Phys. Rev. Lett. 36, 415 (1976); J.-h. Chen and T. C. Lubensky, Phys. Rev. B (to be published).

[7] For a review, see J. A. Mydosh, AIP Conf. Proc. 24, 131 (1975).

[8] S. F. Edwards and P. W. Anderson, J. Phys. F 6, 1927 (1976).

[9] D. L. Huber and W. Y. Ching, paper presented at the Symposium on Amorphous Magnetism, Troy, N.Y., 1976 (unpublished).

[10] See, for example, D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry and Correlation Functions* (Benjamin, Reading, Mass., 1975); R. Graham and H. Pleiner, Phys. Rev. Lett. 34, 792 (1975); C.-R. Hu and W. M. Saslow, *ibid.* 38, 605 (1977).

[11] A. J. Leggett, Rev. Mod. Phys. 47, 331 (1975).

[12] P. W. Anderson, Rev. Mod. Phys. 38, 298 (1966); *Quantum Fluids*, edited by D. F. Brewer (North-Holland, Amsterdam, 1966), p. 146.

[13] R. E. Walstedt and L. R. Walker, Phys. Rev. Lett. 38, 514 (1977); and private communication.

[14] See, for example, R. O. Pohl and G. L. Salinger, Ann. N. Y. Acad. Sci. 279, 150 (1976); B. Golding and J. E. Graebner, Phys. Rev. Lett. 37, 852 (1976).

[15]The bare stiffness constant ρ_s^0 is also related to the frequency-dependent magnetic susceptibility by

$$\rho_s^0 = \lim_{k\to 0} \lim_{\omega/k\to\infty} \frac{\omega^2}{\gamma^2 k^2}\, \chi(k,\omega)$$

$$= \lim_{k\to 0} \frac{1}{\gamma^2 k^2} \int_{-\infty}^{\infty} \omega' \chi''(k,\omega') \frac{d\omega'}{\pi} .$$

See, for example, W. Marshall and R. D. Lowde, Rept. Prog. Phys. 31, 705 (1968).

[16]As a theoretical example, we may consider a system with two kinds of atoms, A and B, on an NaCl lattice. Suppose that the A atoms are coupled to each other by a strong ferromagnetic interaction, and to the B atoms by a weak ferromagnetic interaction. If the B atoms have a strong antiferromagnetic interaction, they will tend to form a spiral structure, with spins oriented perpendicular to the A spins, but canted slightly along the A spin direction. We may note, also, that spin glasses can be prepared in a metastable state with a small remanent magnetization, if the material is cooled through the spin-freezing temperature in the presence of a magnetic field. On the other hand, the conical spin arrangments found in erbium, and in holmium below 19 K, do not properly fall under the description of this paper, because the structure is a direct result of strong anisotropy in the spin Hamiltonian (see Ref. 5). For similar reasons, we exclude from consideration the ferromagnetic amorphous metals in which variations in the spin direction result from random local anisotropy fields. [See, for example, *Amorphous Magnetism*, edited by H. O. Hooper and A. M. de Graaf (Plenum, New York, 1973).]

[17]W. P. Wolf, Rept. Prog. Phys. 24, 212 (1961).

[18]For wave vectors greater than $(M_0^2/\rho_s\chi_\parallel)^{1/2}$, all three branches of the spin-wave spectrum have a linear spectrum of the form (7.1).

[19]K. Kawasaki and H. Mori, Prog. Theor. Phys. 25, 1043 (1961); and 38, 1052 (1967); E. Brêzin, D. J. Wallace, and K. G.Wilson, Phys. Rev. B 7, 232 (1973); E. Brêzin and D. J. Wallace, *ibid.* 7, 1967 (1973); D. R. Nelson, *ibid.* 13, 2222 (1976).

[20]A.S. Andreev and V.I. Marchenko, Zh. Eksp. Teor. Fiz. 70, 1522 (1976) [Sov. Phys.-JETP 43, 794